COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 52, nº 2 (1984), p. 221-229

<http://www.numdam.org/item?id=CM_1984__52_2_221_0>

Compositio Mathematica 52 (1984) 221–229

HOMOGENEOUS-RATIONAL MANIFOLDS AND UNIQUE FACTORIZATION

Manfred Steinsiek

Introduction. Statement of the results

All varieties occurring in this note are assumed to be defined over $\mathbb{C}$. We call an affine (resp. projective) variety *factorial* if its affine (resp. homogeneous) coordinate ring is a unique factorization domain. It seems that the question whether a given affine or projective variety is factorial, goes back to Felix Klein and Max Noether in the late 19th century (see e.g. [20, p. 32]).

Some well-known examples of factorial projective varieties are, besides the trivial example $\mathbb{P}_n$, the nonsingular quadric $Q_n \subset \mathbb{P}_{n+1}$ for $n \geqslant 3$ (Klein), and the Grassmann variety $G_{n,k}$ of k-planes in $\mathbb{P}_n$ considered as a projective variety in $\mathbb{P}_N$ via Plücker embedding, where $N = \binom{n+1}{k+1} - 1$ (the question whether $G_{n,k}$ is factorial was raised by Severi around 1915 and answered in the affirmative by Samuel in the early 1960's (cf. [1] and [20, pp. 37 ff.])).

Since the examples mentioned above are all *homogeneous-rational* manifolds, it is a quite natural question whether any homogeneous-rational manifold is factorial or, more realistically, to decide which of them are. Here, by a homogeneous-rational manifold we mean a compact homogeneous projective-rational complex manifold of positive dimension. Equivalently, a compact complex manifold X of positive dimension is homogeneous-rational if and only if either there is a connected semisimple complex Lie group G acting transitively on X such that $X = G/H$, where H is a proper parabolic subgroup of G, or X is homogeneous with vanishing first Betti number and nonvanishing Euler characteristic, or X is homogeneous and Kähler with $H^1(X, \mathcal{O}) = 0$ (see [2], [3], [8]).

Now, none of the above equivalent conditions for homogeneous-rationality involves an embedding of X into $\mathbb{P}_N$. Hence the question for factoriality of homogeneous-rational manifolds should be stated more precisely in the following form: *Given a holomorphic embedding f: $X \to \mathbb{P}_N$ of a homogeneous-rational manifold X, under which conditions (on X and f) is $f(X)$ factorial?*

We first define some rather special embeddings. For this purpose let X be a homogeneous-rational manifold, G a connected *simply-connected* semisimple complex Lie group acting transitively on X. A holomorphic embedding f: $X \to \mathbb{P}_N$ is called *homogeneously normal* if f is G-equivariant, i.e. if there is a holomorphic representation ϕ_f: $G \to \mathrm{SL}(N+1, \mathbb{C})$ such that $\phi_f(g)(f(x)) = f(g(x))$ for all $g \in G$ and $x \in X$. It is not difficult to see that this definition is independent of G, i.e. if G^* is another connected simply-connected semisimple complex Lie group acting likewise transitively on X, then a holomorphic embedding f: $X \to \mathbb{P}_N$ is G-equivariant if and only if f is G^*-equivariant (cf. [22, Kap. II, Sect. 2.3]). A holomorphic embedding f: $X \to \mathbb{P}_N$ is called *homogeneously minimal* if it is homogeneously normal and if N is minimal, i.e. $N \leqslant M$ for any homogeneously normal embedding f^*: $X \to \mathbb{P}_M$. Then we have the following result which is a special case of a theorem of Tits ([24, III.D]):

THEOREM T: *There exists a homogeneously minimal embedding of X, and this is unique up to an automorphism of the ambient projective space.*

Note that it is necessary to assume the group G to be simply-connected in order to obtain homogeneously minimal embeddings which everybody would expect (for instance, there is no $\mathrm{PGL}(2, \mathbb{C})$-equivariant embedding of $\mathbb{P}_1$ in $\mathbb{P}_1$). In the case $X = G_{n,k}$, the homogeneously minimal embedding of X is just the Plücker embedding. It should be pointed out that, in contrast to the name "minimal", this N is not necessarily so small: For instance, if $X = G/B$, where B is a Borel subgroup of G, then $N = 2^{\dim X} - 1$. On the other hand, it is well-known that any projective-algebraic manifold of dimension d admits an embedding into $\mathbb{P}_{2d+1}$ (cf. [9, p. 173]).

Let X be a homogeneous-rational manifold and G a connected semisimple complex Lie group acting transitively on X. Write $X = G/H$ with H a proper parabolic subgroup of G, and denote by H' the commutator group of H. We define the *rank* [(1)] of X, written $\mathrm{rk}(X)$, as the dimension of the complex Lie group H/H'. Equivalently, $\mathrm{rk}(X)$ is the number of maximal parabolic subgroups of G which contain H. Using this description of the rank and a theorem of Remmert-van de Ven ([19, Satz (2.2)]), it is easy to see that the rank of X depends only on X, but not on the group G. One can also show that $\mathrm{rk}(X) = b_2(X)$, where $b_2(X)$ denotes the 2[nd] Betti number of X (cf. [5, p. 245] and [23, Remark in § 3]). Obviously, $\mathrm{rk}(X) = 1$ if and only if H is a maximal parabolic subgroup of G and, by [18], this is equivalent to the condition that each holomorphic map h: $X \to Y$ of X into a complex space Y of dimension $< \dim X$ be

(1) This definition of the rank has nothing whatsoever to do with the rank of a symmetric (in particular hermitian symmetric) space in differential geometry. However, our notation seems to be rather familiar, see e.g. [25, p. 114].

constant. In particular, every rank 1-homogeneous-rational manifold is *irreducible*. (A homogeneous-rational manifold X is called irreducible if $\mathrm{Aut}(X)$ is a simple complex Lie group, reducible otherwise. Evidently, a homogeneous-rational manifold X is reducible if and only if there are homogeneous-rational manifolds X_1, X_2 such that $X \cong X_1 \times X_2$.)

Let us look at some examples: The homogeneous-rational manifolds of rank 1 are projective spaces, quadrics of dimension $\geqslant 3$, Grassmannians, "Grassmannians" of linear subspaces of $\mathbb{P}_{n+1}$ which lie on the quadric $Q_n \subset \mathbb{P}_{n+1}$ (cf. [24, II.C.7, II.C.11]), "Grassmannians" of linear subspaces of $\mathbb{P}_n$, $n \geqslant 5$ odd, which are totally isotropic with respect to a nullcorrelation (cf. [24, II.C.7, II.C.11]), and finally 24 pairwise non-isomorphic rank 1-homogeneous-rational manifolds, whose automorphism groups are exceptional simple complex Lie groups (cf. [23]). In higher rank, the probably best-known examples are, besides direct products of rank 1-homogeneous-rational manifolds, the flag manifolds of $\mathbb{P}_n$, $n \geqslant 2$, of rank n, the simplest being the rank 2-homogeneous-rational manifold $F_2 = \{(x, L) \in \mathbb{P}_2 \times \mathbb{P}_2^*;\ x \in L\}$ of dimension 3.

Now we are in position to state our main results.

THEOREM 1 (FACTORIALITY CRITERION): *The following statements about a holomorphic embedding $f\colon X \to \mathbb{P}_N$ of a homogeneous-rational manifold X are equivalent*:

(i) *$f(X)$ is factorial*;

(ii) *(a) $\mathrm{rk}(X) = 1$ and*

(*b*) *there is a linear k-plane $\mathbb{P}_k \subset \mathbb{P}_N$ such that $f(X) \subset \mathbb{P}_k$ and $f\colon X \to \mathbb{P}_k$ is homogeneously minimal.*

The proof of this theorem is carried out in Section 2 by inspecting the divisor class group $\mathrm{Cl}(X)$ of X and using a criterion of factoriality which is due to Samuel. Fundamental for the proof is the following Normality Criterion which is proved in Section 1.

THEOREM 2 (NORMALITY CRITERION): *The following statements about a holomorphic embedding $f\colon X \to \mathbb{P}_N$ of a homogeneous-rational manifold X are equivalent*:

(i) *$f(X)$ is projectively normal*;

(ii) *f is homogeneously normal.*

Recall that a projective variety is called *projectively normal* if its homogeneous coordinate ring is a normal domain. We suspect that (at least) part of the Normality Criterion is known, but we do not know any adequate reference (except for the case $X = G_{n,k}$, $f =$ Plücker embedding: Severi showed in 1915 that $f(X)$ is projectively normal (cf. [21, p. 100], see also [13])).

Let X be a homogeneous-rational manifold homogeneously minimally embedded in $\mathbb{P}_N$. We define an *affine kernel* X_a of X to be the complement of a general hyperplane section in X. Thus X_a is an affine variety, and we ask for the divisor class group of X_a. This question was raised by Remmert around 1965. We give a complete answer to this question:

THEOREM 3: *If X is a homogeneous-rational manifold, then the divisor class group* $\mathrm{Cl}(X_a)$ *of an affine kernel X_a of X is isomorphic to* $\mathbb{Z}^{\mathrm{rk}(X)-1}$. *In particular, X_a is factorial if and only if* $\mathrm{rk}(X) = 1$.

The proof of this theorem is similar to that of Theorem 1 and is also given in Section 2. It is done by investigating the canonical surjective mapping $\mathrm{Cl}(X) \to \mathrm{Cl}(X_a)$ between the divisor class groups of X and X_a.

In Section 3, we give two applications of Theorem 1. First, *the homogeneous coordinate ring S of a homogeneously minimally embedded rank 1-homogeneous manifold X is Gorenstein* (for Grassmannians, this has been shown by Hochster in [11]). This is proved by first showing that S is Cohen-Macaulay and then, of course, applying Murthy's Theorem ([17]). For the second application, let X be a rank 1-homogeneous-rational manifold, homogeneously minimally embedded in $\mathbb{P}_N$, and let R be the local ring of the vertex of the affine cone over X in affine $(N+1)$-space. Using a result of Danilov ([6]), we prove that, unless X is isomorphic to a projective space (in which case R is regular), *R is a non-regular local unique factorization domain whose completion $\hat{R}$ is again factorial.*

It should be noted that the proofs of the theorems as well as the applications, though not being very complicated, depend on an interplay of several mathematical fields: from representation theory of semisimple complex Lie algebras and Lie groups, we use Tits' embedding theorem and the Borel-Weil Theorem; from complex analysis, we use Bott's Theorem and results of Remmert-van de Ven; from algebraic geometry and commutative algebra, we use Samuel's Criterion of Factoriality, Murthy's Theorem, results of Danilov on the divisor class group of a complete local ring, etc.

It seems that most of our results carry over to varieties G/H over more general algebraically closed ground fields K, at least for char $K = 0$.

It is the author's pleasure to thank Prof. R. Remmert for bringing the above mentioned problems to his attention as well as for many helpful conversations during the preparation of this paper.

1. Proof of Theorem 2

We first discuss the Borel-Weil Theorem, which will turn out to be crucial for the proof of our Normality Criterion. Let always $X = G/H$ be a homogeneous-rational manifold, where G is a connected simply-connected semisimple complex Lie group acting transitively on X and H is a proper parabolic subgroup of G. We further denote by $\mathrm{Cl}(X)$ the divisor class group of X. We begin with the following simple

LEMMA 1: $\mathrm{Cl}(X) \cong H^1(X, \mathcal{O}^*) \cong H^2(X, \mathbb{Z}) \cong \mathbb{Z}^{\mathrm{rk}(X)}$.

PROOF: The first isomorphism follows from [10, p. 145]. Next, since $H^q(X, \mathcal{O}) = 0$ for $q \geqslant 1$ (cf. [5, Lemma 14.2]), from the exact cohomology sequence belonging to the short exact exponential sequence we obtain $H^1(X, \mathcal{O}^*) \cong H^2(X, \mathbb{Z})$. Finally, since $b_1(X) = 0$, $H^2(X, \mathbb{Z})$ is torsion-free, whence $H^2(X, \mathbb{Z}) \cong \mathbb{Z}^{\mathrm{rk}(X)}$ because of $b_2(X) = \mathrm{rk}(X)$. □

Now let $\mathcal{L}$ be a line bundle on X, $\mathcal{L} = \mathcal{O}(D)$ with D a divisor on X. Let $|D| \cong \mathbb{P}(H^0(X, \mathcal{L}))$ be the corresponding linear system (which may be empty). Since G is connected, G acts trivially on $H^2(X, \mathbb{Z})$, hence on $\mathrm{Cl}(X)$. Thus, for $D^* \in |D|$, $g \in G$, $g(D^*) \in |D|$, and hence G acts on $|D| \cong \mathbb{P}(H^0(X, \mathcal{L}))$. Since G is simply-connected, this action lifts to a linear action of G on $H^0(X, \mathcal{L})$. In particular, $H^0(X, \mathcal{L})$ is a G-module. Now we state the following special case of the Borel-Weil Theorem ([4]):

THEOREM BW: *If $\mathcal{L}$ is a very ample line bundle on X, then the G-module $H^0(X, \mathcal{L})$ is irreducible.*

PROOF OF THEOREM 2: Denote by $S = \Sigma_{n \geqslant 0} S_n$ the homogeneous coordinate ring of $f(X)$, and let $\tilde{S}$ be the integral closure of S. Then we have (cf. [10, Ch. II, Ex. 5.14]): $\tilde{S} = \Sigma_{n \geqslant 0} H^0(f(X), \mathcal{O}(n))$. Here, $\mathcal{O}(1)$ denotes the twisting sheaf of Serre (cf. [10, p. 117]), i.e. considered as a line bundle on X, $\mathcal{O}(1) = f^*\mathcal{H}$, where $\mathcal{H}$ is the hyperplane section bundle on $\mathbb{P}_N$, and $\mathcal{O}(n) = \mathcal{O}(1)^n$. Then, by Theorem BW, for all $n \geqslant 0$, $H^0(f(X), \mathcal{O}(n))$ is an *irreducible* G-module.

Now let f be homogeneously normal. Then the linear action ϕ_f: $G \to \mathrm{SL}(N+1, \mathbb{C})$ induces a natural action $\tilde{\phi}_f$ of G on S. This action preserves the grading of S, i.e. $\tilde{\phi}_f(G) S_n \subset S_n$ for all $n \geqslant 0$, and, in fact, furnishes S_n with the structure of a G-submodule of $H^0(f(X), \mathcal{O}(n))$. But $H^0(f(X), \mathcal{O}(n))$ is an irreducible G-module. Hence, for a fixed $n \geqslant 0$, we have either $S_n = 0$ which is clearly impossible, or $S_n = H^0(f(X), \mathcal{O}(n))$. Thus we obtain $S = \tilde{S}$, and $f(X)$ is projectively normal.

Finally, if f is not homogeneously normal, then evidently $S_1 \subsetneqq H^0(f(X), \mathcal{O}(1))$, and hence $f(X)$ is not projectively normal. □

EXAMPLE: As a special case of Theorem 2, for $X = \mathbb{P}_1$, we obtain the following well-known facts:

(1) The d-uple embedding of $\mathbb{P}_1$ in $\mathbb{P}_d$ (this embedding is given by the monomials in two variables of degree d) is projectively normal (cf. [10, Ch. IV, Ex. 3.4]).

(2) The twisted quartic curve in $\mathbb{P}_3$ (this is given by the (non-homogeneously normal) embedding $[z_0 : z_1] \to [z_0^4 : z_0^3 z_1 : z_0 z_1^3 : z_1^4]$ of $\mathbb{P}_1$ in $\mathbb{P}_3$) is not projectively normal (cf. [10, Ch. I, Ex. 3.18]).

2. Proofs of Theorem 1 and Theorem 3

We begin with some useful remarks concerning very ample line bundles on homogeneous-rational manifolds. In this whole section, $X = G/H$ is a homogeneous-rational manifold, where, as usual, G is a connected simply-connected semisimple complex Lie group acting transitively on X and H a proper parabolic subgroup of G. We further let $r = \mathrm{rk}(X)$.

By Lemma 1, $H^1(X, \mathcal{O}^*) \cong \mathbb{Z}^r$, so it is reasonable to speak of line bundles of type $(n_1,\dots,n_r) \in \mathbb{Z}^r$ on X. However, since a line bundle of type $(1,\dots,1)$ should be "positive", this has to be rendered precise: Let B be a Borel subgroup of G, and consider the B-action on X. Then there is an open B-orbit U in X (the open "Bruhat-cell"), and the complement $X - U$ is a divisor on X consisting of r irreducible components $D_1,\dots,D_r$ (see [15] for details in the case $X = G/B$). Now, for $(n_1,\dots,n_r) \in \mathbb{Z}^r$, define the line bundle of type $(n_1,\dots,n_r)$ to be the line bundle belonging to the divisor $n_1 D_1 + \dots + n_r D_r$. Equivalently, line bundles of type $(n_1,\dots,n_r)$ may be defined as follows: Let $R = T \cdot S$ be the reductive part of H, where $T \cong (\mathbb{C}^*)^r$ and S is semisimple. Then the groups T^* and H^* of holomorphic characters of T and H are isomorphic, $T^* \cong H^*$. Obviously, $T^* \cong \mathbb{Z}^r$, the isomorphism being given by $\mathbb{Z}^r \ni (n_1,\dots,n_r) \to \chi_{(n_1,\dots,n_r)} \in T^*$, where $\chi_{(n_1,\dots,n_r)}(z_1,\dots,z_r) := z_1^{n_1} \cdot \ldots \cdot z_r^{n_r}$. Now the line bundle of type $(n_1,\dots,n_r)$ on X is the homogeneous line bundle given by the character $\chi_{(n_1,\dots,n_r)} \in T^* \cong H^*$. Furthermore, line bundles of type $(n_1,\dots,n_r)$ on X may be described in the following way: Let $P_1,\dots,P_r$ the maximal parabolic subgroups of G which contain H, let $X_i := G/P_i$, and let $\pi_i\colon X \to X_i$ the natural fibrations. Since $\mathrm{rk}(X_i) = 1$, $H^1(X_i, \mathcal{O}^*) \cong \mathbb{Z}$. Now take positive generators $\mathcal{L}_i$ of $H^1(X_i, \mathcal{O}^*)$, $i = 1,\dots,r$. Then the line bundle $\mathcal{L}$ of type $(n_1,\dots,n_r)$ on X is given by $\mathcal{L} = \pi_1^*(\mathcal{L}_1^{n_1}) \otimes \dots \otimes \pi_r^*(\mathcal{L}_r^{n_r})$.

Now, very ample line bundles on X can be easily characterised:

REMARK: *A line bundle $\mathcal{L}$ of type $(n_1,\dots,n_r)$ on X is very ample if and only if $n_i > 0$, $i = 1,\dots,r$. In particular, a holomorphic embedding $f\colon X \to \mathbb{P}_N$ is homogeneously minimal if and only if f is given by a base $s_0,\dots,s_N$ of $H^0(X, \mathcal{L})$, where $\mathcal{L}$ is a line bundle of type* $(1,\dots,1)$ *on X.*

PROOF: The first part is contained in [4, §4], the second assertion follows from Tits' Theorem (see [22, Korollar 2.2.2]). □

We now come to the proof of Theorem 1. We employ Samuel's

CRITERION OF FACTORIALITY (cf. [10, Ch. II, Ex. 6.3]): *A projective variety V is factorial if and only if* (1) *V is projectively normal, and* (2) *the divisor class group* $\mathrm{Cl}(V)$ *of V is isomorphic to $\mathbb{Z}$ and is generated by the class of a (suitable) hyperplane section.*

Proof of Theorem 1: By Samuel's Criterion and Lemma 1, clearly $\mathrm{rk}(X)=1$ if $f(X)$ is factorial. So assume $\mathrm{rk}(X)=1$. Take a line bundle $\mathscr{L}_0$ of type (1) on X. Hence $\mathscr{L}_0$ generates $H^1(X, \mathscr{O}^*)\cong \mathbb{Z}$. By the Remark, f is given by sections $s_0,\dots,s_N \in H^0(X,\mathscr{L})$, where $\mathscr{L}$ is a line bundle of type (n) on X, $n \geqslant 1$, i.e. $\mathscr{L}=\mathscr{L}_0^n$ with $n\geqslant 1$. Let V^* be a hyperplane in $\mathbb{P}_N$ not containing $f(X)$ and V the corresponding hyperplane section. Then we have $\mathrm{Cl}(f(X))/(V)\cong H^1(X, \mathscr{O}^*)/(\mathscr{L})=(\mathscr{L}_0)/(\mathscr{L})\cong \mathbb{Z}/n\mathbb{Z}$. Now, by Samuel's Criterion, $f(X)$ is factorial if and only if $n=1$ and $f(X)$ is projectively normal. Hence the assertion follows from Theorem 2 and the Remark. □

Proof of Theorem 3: Let $f\colon X\to \mathbb{P}_N$ be a homogeneously minimal embedding of X, and let an affine kernel X_a of X be given by $X_a=f(X)-Z$, where Z is a general (i.e. smooth) hyperplane section of $f(X)$. We consider the exact sequence $\mathbb{Z}\xrightarrow{i}\mathrm{Cl}(X)\to \mathrm{Cl}(X_a)\to 0$, where the map i is given by $1\to 1\cdot Z$ (cf. [10, Ch. II, Prop. 6.5]). Since the group $\mathrm{Cl}(X)$ is torsion-free, the map i is injective. We have to determine the image $\mathrm{Im}(i)$ of i in $\mathrm{Cl}(X)$. First, $\mathrm{Cl}(X)\cong H^1(X, \mathscr{O}^*)\cong \mathbb{Z}^r$. Next, by the Remark, f is given by a base $s_0,\dots,s_N$ of $H^0(X,\mathscr{L}_0)$, where $\mathscr{L}_0$ is a line bundle of type $(1,\dots,1)$ in $H^1(X, \mathscr{O}^*)$. Hence, for the divisor class group of X_a we obtain: $\mathrm{Cl}(X_a)\cong \mathrm{Cl}(X)/\mathrm{Im}(i)\cong H^1(X, \mathscr{O}^*)/(\mathscr{L}_0)\cong \mathbb{Z}^r/\langle(1,\dots,1)\rangle\cong \mathbb{Z}^{r-1}$, whence the assertion. In particular, if $r=1$, then $\mathrm{Cl}(X_a)=0$, and hence X_a is factorial (cf. [10, Ch. II, Prop. 6.2]).

3. Applications

In this section, we give two applications of Theorem 1 and the following special case of a theorem of Bott (cf. [5, Thm. IV′]):

Theorem B: *If $\mathscr{L}$ is a very ample line bundle on a homogeneous-rational manifold X, then $H^q(X,\mathscr{L})=0$ for $q\geqslant 1$ and $H^q(X,\mathscr{L}^{-1})=0$ for $q<\dim X$.*

For the first application, recall that a noetherian ring A is called *Cohen-Macaulay* (*Gorenstein*, resp.) if, for every maximal ideal $\mathfrak{m}$ of A, the local ring $A_{\mathfrak{m}}$ is Cohen-Macaulay (Gorenstein, resp.), i.e. $\dim A_{\mathfrak{m}}=\operatorname{depth} A_{\mathfrak{m}}$ (the injective dimension of $A_{\mathfrak{m}}$ is finite, resp.). For generalities on Cohen-Macaulay and Gorenstein rings, see e.g. [14]. Now we can state:

Corollary 1: *The homogeneous coordinate ring S of a homogeneously minimally embedded rank 1-homogeneous-rational manifold X is Gorenstein.*

PROOF: According to a theorem of Murthy ([17], see also [7, Thm. 12.3]), a factorial Cohen-Macaulay factor ring of a Gorenstein ring is Gorenstein. Hence, by Theorem 1, it is sufficient to show that S is Cohen-Macaulay. In fact, we have quite generally

PROPOSITION: *If f: $X \to \mathbb{P}_N$ is a homogeneously normal embedding of a homogeneous-rational manifold X, then the homogeneous coordinate ring $S(f(X))$ of $f(X)$ is Cohen-Macaulay.*

PROOF: It is well-known that the homogeneous coordinate ring $S(V)$ of a nonsingular projectively normal projective variety V is Cohen-Macaulay provided $H^q(V, \mathcal{O}(n)) = 0$ for all $n \in \mathbb{Z}$ and $1 \leqslant q \leqslant \dim V - 1$ (this is e.g. a special case of Prop. B, p. 131, and Prop. 5.1 of [12]). Applying this theorem to our situation, it follows from Theorem 2 and Theorem B that $S(f(X))$ is Cohen-Macaulay. □

For the second application, let X be a homogeneous-rational manifold, and let f: $X \to \mathbb{P}_N$ be a homogeneously minimal embedding of X. Denote by $V(X)$ the affine cone over $f(X)$ in affine $(N+1)$-space. Let P be the vertex of $V(X)$ and $R = \mathcal{O}_{V(X),P}$ the local ring of P on $V(X)$. Thus, by Theorem 2, R is a normal domain, and, unless X is isomorphic to $\mathbb{P}_n$ for some n, R is *not* regular.

Now assume additionally $\mathrm{rk}(X) = 1$. Then, by Theorem 1, $f(X)$ is factorial, and hence R is a unique factorization domain (cf. [7, Cor. 10.3]). One may ask whether the completion $\hat{R}$ of R with respect to its maximal ideal is again factorial. In general, if A is a local noetherian Krull domain and $\hat{A}$ its completion, then, by Mori's Theorem (cf. [7, Cor. 6.12]), A is factorial if $\hat{A}$ is, but the converse is false in general (see [7, Example 19.9] for a counterexample). In our case, however, we have

COROLLARY 2: *The ring $\hat{R}$ is a unique factorization domain.*

PROOF: We use the following result of Danilov (cf. [6, Theorem in §2 and Prop. 8], see also [16, p. 532 f.]): If V is a nonsingular projectively normal projective variety, A the local ring of the vertex of the affine cone over V, and $\hat{A}$ the completion of A, then the divisor class groups of A and $\hat{A}$ are isomorphic if and only if $H^1(V, \mathcal{O}(n)) = 0$ for all $n \geqslant 1$. Applying this theorem to our situation, by Theorem B we obtain $\mathrm{Cl}(\hat{R}) = \mathrm{Cl}(R) = 0$, whence the assertion. □

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(Oblatum 22-VII-1982)

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