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INJECTIVITY PROPERTIES OF LIFTINGS ASSOCIATED TO WEIL REPRESENTATIONS

S. Rallis

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Introduction

We have presented in [R-2] an approach to studying the space of cusp forms of classical groups of type Sp or $O(Q)$. This approach uses the lifting theory of representations between dual pairs arising from oscillator representations. We have given a natural decomposition of the space of cusp forms on $L^2_{\text{cusp}}(\text{Sp}_r(\mathbb{A}))$ into orthogonal pieces X_i where if $X_i \neq \{0\}$, then X_i "lifts" to a space of cusp forms on $O(Q_i)(\mathbb{A})$, (Q_i, a quadratic form of dimension 2λ with split part $2\lambda - 2i$). We have shown in [R-2] several cases where the Howe duality conjecture is valid for the lifting from the space X_i to $L^2_{\text{cusp}}(O(Q_i)(\mathbb{A}))$; indeed this says that the lifting on the set of automorphic representations occurring in X_i is an *injective mapping* and that *multiplicities are preserved* under the lifting. It is reasonable to expect that the methods of [R-2] can be extended so that this conjecture (or maybe a minor variant of it) is valid for each X_i. However, what is not considered in [R-2] is to give a precise description of the representations occurring in X_i or the representations occurring in the image of X_i. We note that the first such instance where such a precise description is given at the global level is for the pair $\text{GL}_2(\mathbb{A})$ and $D_{\mathbb{A}}$, D a quaternion algebra over K, relative to the lifting between $L^2(D_K \backslash D_{\mathbb{A}})$ and $L^2_{\text{cusp}}(GL_2(\mathbb{A}))$ given in [*J-L*]. Indeed the lifting map is shown to be injective, and the image is characterized as the set of cuspidal representations of $GL_2(\mathbb{A})$ whose components at the *ramified* primes of D are

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square integrable representations. The next set of cases considered were the Saito-Shintani base change, the symmetric square map from GL_2 to GL_3, and the Shimura lifting from GL_2 to its two-fold cover. In each of these instances a comparison of traces formula is proved; from this the existence of a global (and then a local) lifting is deduced with specific information given on the injectivity of the map. Moreover there is very fine information characterizing the representations occurring in the image of the lifting map.

We are interested in considering the same questions for a general dual reductive pair. In principle, it is straightforward to prove the existence and injectivity of the lifting as shown by the various cases of the Howe conjecture in [R-2]. However, to give an adequate characterization of the image of a lift, it seems mandatory to use some type of trace formula. Indeed the basic question that initiated this study was to determine what isometry properties the lifting map possessed when it was given in concrete terms by a *kernel function*; the fundamental question is to have a method of relating the Hilbert inner product of two "lifted" modular forms ℓ_{φ,ψ_i} to the initial Hilbert inner product of the ψ_i. To this end we have discovered an inner product formula of the following form:

$$\langle \ell_{\varphi,\psi_i} | \ell_{\varphi',\psi'_i} \rangle_{O(Q)(\mathbb{A})} = \langle \psi_i * \mathscr{L}(\varphi, \varphi') | \psi_i'^{\Delta} \rangle_{\mathrm{Sp}_n(\mathbb{A})},$$

where $\mathscr{L}(\varphi, \varphi')$ is an L^1 function of $\mathrm{Sp}_n(\mathbb{A})$ (depending on φ and φ') and Δ is a certain involution on $L^2_{\mathrm{cusp}}(\mathrm{Sp}_n(\mathbb{A}))$. The import of such a formula is that to test the nonvanishing of ℓ_{φ,ψ_i}, it suffices to test the nonvanishing of $\langle \psi_i * \mathscr{L}(\varphi, \varphi) | \psi_i^{\Delta} \rangle$. What is remarkable is that this term can be effectively computed! Indeed we note that $\mathscr{L}(\varphi, \varphi)$ will be a matrix coefficient of a particular global Weil representation; then with mild restrictions $\langle \psi_i * \mathscr{L}(\varphi, \varphi) | \psi_i^{\Delta} \rangle$ will be an Euler product in which each local integral will be of the form

$$\int_{\mathrm{Sp}_n(K_v)} \langle \varphi_v * G_v | \varphi_v \rangle_v \langle (\psi_i)_v * G_v | \psi_i^{\Delta} \rangle_v \mathrm{d}G_v,$$

where $\langle \varphi_v * G_v | \varphi_v \rangle_v$ is the matrix coefficient of a local Weil representation and $\langle (\psi_i)_v * G_v | (\psi_i)_v \rangle_v$ is the matrix coefficient of a local component of ψ_i. In particular we note that the Euler product will, in fact, decompose into two pieces: i.e., a product $L_1 \cdot L_2$, where L_1 is the *special value* of a Langlands L function associated to the representation Π_i (which carries ψ_i) and L_2 is a *finite product* of local integrals as above (over those primes where the local component $(\Pi_i)_v$ is not a spherical representation). It is then easy to see that $L_1 \neq 0$ and that the nonvanishing of L_2 depends on whether $(\Pi_i)_v$ will "occur" in the associated local Weil representation. Thus, in principle, we have reduced the question of determining the image of a lift to determining locally which representations occur in the Weil representation. We note that we do not expect this always to be the case (see [P]); however for Sp_n large compared to $\mathrm{O}(Q)(\mathbb{A})$, it should be true, as we show in this paper for the family of cases $\mathrm{Sp}_1 \times \mathrm{O}(Q)$ where $\dim Q \geqslant 8$.

We note that as a special consequence of the above work, we are able to construct global automorphic representations σ of $\mathrm{O}(Q)(\mathbb{A})$ appearing in $L^2_{\mathrm{cusp}}(\mathrm{O}(Q)(\mathbb{A}))$ such that $\sigma_\infty =$ the component at some Archimedean prime "∞" will be a representation of $\mathrm{O}(Q)(\mathbb{R})$ with nonvanishing Lie algebra cohomology (see [B-W] and [M-R] for this general problem).

On the other hand, we know from the inner product formula shown above that a type of formal comparison of traces of Hecke operators is possible. That is, we recall from [R-1] that there exists a surjective homomorphism ω_{Q_v} of the local Hecke algebra $\mathscr{H}(\mathrm{O}(Q_v)//\tilde{K}_v)$ to $\mathscr{H}(\mathrm{Sp}_1(K_v)//\mathrm{SP}_1(\theta_v))$ which is compatible with the local Weil representation π_{Q_v}, i.e., $\pi_{Qv}(\omega_{Q_v}(f) = \pi_{Q_v}(f)$ for all $f \in \mathscr{H}(\mathrm{O}(Q_v)//\tilde{K}_v)$. Then for the following classical case, we have a comparison of trace formula. Namely, let $S_{4t}(\mathbb{Z})$ be the space of cusp forms of weight $4t$ for the group $SL_2(\mathbb{Z})$. Let Q be a unimodular quadratic form ($8|m$, $m = \dim Q$) and let $F_Q =$ the finite dimensional space of functions on the double coset space $\mathrm{O}(Q)(\mathbb{Q})\backslash \mathrm{O}(Q)(\mathbb{A})/U_Q$, where $U_Q =$ stabilizer in $\mathrm{O}(Q)(\mathbb{A})$ of the standard lattice $\mathbb{Z}^m$. Then we know that the lifting map defines a linear injection of $S_{4t}(\mathbb{Z})$ to F_Q, which is compatible with the local maps ω_{Q_v} (for all finite v in $\mathbb{Q}$). Moreover we have the formula

$$T_r\big(\rho(f_v)|_{X_Q}\big) = \mathrm{Tr}\big(\rho\big(\omega_{Q_v}(\check{f}_v)\big)\big)$$

for all $f_v \in \mathscr{H}(\mathrm{O}(Q_v)//\tilde{K}_v)$ (here ρ denotes the respective representations of local Hecke algebras on the given spaces of cusp forms), and $X_Q =$ the image of the lifting.

Also we note another possible use of the above work. Namely if we assume that ψ_t is an eigenfunction of $\mathscr{L}(\varphi, \varphi')$ (which is the case when ψ_t lies in an irreducible component of $L^2_{\mathrm{cusp}}(\mathrm{Sp}_n(\mathbb{A}))$), then the inner product gives, in principle, a way to determine the algebraic nature of the ratio

$$\frac{\langle \ell_{\varphi,\psi_t} | \ell_{\varphi',\psi'_t} \rangle}{\langle \psi_t | \psi'^{\Delta}_t \rangle}.$$

Indeed in Remark 2.4, we see that, for the pair $(\mathrm{Sp}_1, \mathrm{O}(Q))$ with certain restrictions on Q, this ratio (aside from a universal constant not depending on the ψ_t) can be expressed as a ratio of *special values* of L functions associated to the representation space Π of ψ_t. We can view such a formula as a possible generalization of the exact formulae of Waldspurger and Kohnen-Zagier ([W] and [K-Z]) relating the ratios of Petersson inner products of *half-integral* weight modular forms to *integral* weight modular forms (via the Shimura lifting).

We organize the paper in the following fashion.

In § 0 we present preliminary definitions and notation. We also recall certain results in [R-S].

In § 1 we consider the lifting ℓ from $L^2_{\mathrm{cusp}}(\mathrm{Sp}_k(\mathbb{A}))$ and $L^2_{\mathrm{cusp}}(\mathrm{Sp}_{k'}(\mathbb{A}))$ to the space of smooth automorphic forms $S(\mathrm{O}(Q)(K)\backslash\mathrm{O}(Q)(\mathbb{A}))$ (for a fixed form Q). Then it is possible to compute the Petersson inner product of $\ell_{\varphi,\psi}$ and $\ell_{\varphi',\psi'}$ if $k+k'<m/4$ ($m=\dim Q$) on $\mathrm{O}(Q)(\mathbb{A})$. In Proposition 1.1 (I), we show that the lifts $\ell_{\varphi,\psi}$ and $\ell_{\varphi',\psi'}$ are perpendicular if $k\neq k'$. Thus the main case of interest is when $k=k'$, and we prove in Proposition 1.1 (II) the inner product formula alluded to above. The key point in proving this formula is the use of Siegel's formula and the determination of the set of double cosets of the form $\mathrm{Sp}_k\times\mathrm{Sp}_{k'}\backslash \mathrm{Sp}_{k+k'}/P_{k+k'}$. We note that we have used a similar such decomposition in [R-2] as one of the main technical tools in proving the local Howe duality conjecture. The construction of $\mathscr{L}(\varphi,\varphi')$ as a matrix coefficient of an appropriate Weil representation, which is factorizable as a product of matrix coefficients of the associated local Weil representations, is given in the proof of Proposition 1.1 and Remark 1.2. Then we look at the local factors of $\mathscr{L}(\varphi,\varphi)$ which are of spherical type and determine in Remark 1.3 the corresponding local factor of the *inner product* $\langle\psi_1 * \mathscr{L}(\varphi,\varphi')|\psi_2^\Delta\rangle$. Moreover in § 4 we give the details of this computation when $k=1$. Then as an application of this factorization property of $\langle\psi_1 * \mathscr{L}(\varphi,\varphi')|\psi_2^\Delta\rangle$ we determine in Corollary 1 to Proposition 1.1 the representations occurring in $I_1(Q)$ (Q anisotropic over K with $m>6$). Here we use also certain results from [*R-S*], i.e., the possible discrete series representations of $\mathrm{SL}_2(\mathbb{R})$ that occur in the Weil representation for the pair $(\mathrm{O}(Q),\ \mathrm{Sp}_1=SL_2)$. In Theorem 1.6 we show, by dualizing the above Corollary, that the lifting map is *injective* between $R_1(Q)$ and $I_1(-Q)$ (see § 0 for precise definitions); moreover using the results of [R-2], we show that the global Howe duality conjecture is valid for this range of cases. Finally in Remark 1.7 we show how to construct global *cuspidal* automorphic representations of $\mathrm{O}(Q)(\mathbb{A})$ such that a local component at $\{\infty\}$ has nonvanishing Lie algebra cohomology of a certain prescribed level.

In § 2 we give another application of Proposition 1.1. Namely we prove the "Comparison of Trace" Theorem (Theorem 2.1) discussed above. We also indicate in Remark 2.3 the arithmetic nature of the ratio $\langle\ell_{\varphi,\psi_1}|\ell_{\varphi',\psi_2}\rangle/\langle\psi_1|\psi_2^\Delta\rangle$ in terms of special values of classical-type zeta functions and L-functions of Langlands type.

Sections 3 and 4 are devoted to proving convergence of certain integrals in § 1 and the determination of local factors discussed above.

We would like to thank John Millson, whose incisive questions initiated part of this study.

§ 0. Notation and preliminaries

(I) Let k be a local field of characteristic 0. We fix a nontrivial additive character τ on k. Let $\langle,\rangle_k$ be the usual Hilbert symbol on k. Let $\mathrm{d}x$ be a Haar measure on k which is self dual relative to τ. We let $|\ |_k$ be an absolute value of k.

If k is a nonarchimedean field, we let $\mathcal{O}_k$ = ring of integers of k, π_k = the maximal ideal in $\mathcal{O}_k$, and q = the cardinality of $\mathcal{O}_k/\pi_k$.

(II) Let K be a number field (i.e., finite degree extension of $\mathbb{Q}$, the rational numbers). Let $\mathbb{A}_K$ be the corresponding adelic group. Then embed K as a discrete subring in $\mathbb{A}_K$. Let K_v be the completion of K relative to a prime v in K. Let τ be a nontrivial character on $\mathbb{A}_K$ which equals 1 on K; then there exist compatible characters τ_v on K_v (for all primes v in K) such that $\tau(X)=\prod_v \tau_v(X_v)$. Let $\mathrm{d}X$ be the measure (Tamagawa measure) on $\mathbb{A}_K$ such that the group $\mathbb{A}_K/K$ is self dual relative to τ and $\mathbb{A}_K/K$ has mass 1. When the context is clear, we drop K in $\mathbb{A}_K$ and use just $\mathbb{A}$ for $\mathbb{A}_K$.

(III) Let Q be a nondegenerate quadratic form on K^m. Let Q_v be the corresponding local versions on K_v^m. If Q_v is a totally split form which is the direct sum of r hyperbolic planes, then we let $Q_v = H_r$. Let $\mathrm{O}(Q)$ be the orthogonal group of Q. Then we can form the corresponding adelic group $\mathrm{O}(Q)(\mathbb{A})$ and the corresponding local orthogonal groups $\mathrm{O}(Q_v)$ of Q_v at K_v. Let $\mathrm{O}(Q)(K)$ = the K rational points in $\mathrm{O}(Q)$ and embed $\mathrm{O}(Q)(K)$ into $\mathrm{O}(Q)(\mathbb{A})$ in the standard way. Choose a Tamagawa measure on the quotient $\mathrm{O}(Q)(K)\backslash \mathrm{O}(Q)(\mathbb{A})$ as given in [A].

Similarly let A be a nondegenerate alternating form on K^{2n}. Let Sp_n be the corresponding symplectic group and $\mathrm{Sp}_n(\mathbb{A})$, $\mathrm{Sp}_n(K_v)$ the associated adelic and local objects. Let $\mathrm{Sp}_n(K)$ = the K rational points in Sp_n and embed $\mathrm{Sp}_n(K)$ into $\mathrm{Sp}_n(\mathbb{A})$ again in the standard fashion and choose a Tamagawa measure on the quotient $\mathrm{Sp}_n(K)\backslash \mathrm{Sp}_n(\mathbb{A})$ as given in [A].

(IV) If $\tilde{G}_{\mathbb{A}}$ is a global group, then we denote the space of cusp forms on $\tilde{G}_K\backslash \tilde{G}_{\mathbb{A}}$ by $L^2_{\mathrm{cusp}}(\tilde{G}_{\mathbb{A}})$. We note (by our convention) that if $\tilde{G}_K\backslash \tilde{G}_{\mathbb{A}}$ is compact, then $L^2_{\mathrm{cusp}}(\tilde{G}_{\mathbb{A}}) = \{\text{all functions } f \perp \text{ constants}\}$. We know that $L^2_{\mathrm{cusp}}(\tilde{G}_{\mathbb{A}})$ is discretely decomposable as a $\tilde{G}_{\mathbb{A}}$ module, and each unitary irreducible representation occurring in $L^2_{\mathrm{cusp}}(\tilde{G}_{\mathbb{A}})$ has a finite multiplicity. We denote

$$\langle f_1 | f_2 \rangle_{\tilde{G}(\mathbb{A})} = \int_{\tilde{G}(K)\backslash \tilde{G}(\mathbb{A})} f_1(g)\overline{f_2(g)}\,\mathrm{d}g,$$

where $\mathrm{d}g$ is some Tamagawa measure on the quotient $\tilde{G}(K)\backslash \tilde{G}(\mathbb{A})$.

(V) We consider the following Weil representation. Namely, we fix a nondegenerate form Q on K^m and a nondegenerate alternating form A on K^{2n}. Then we consider the alternating form $Q \otimes A$ on $M_{m2n}(K)$. To this form we associate a Weil representation of the two-fold cover $\tilde{\mathrm{Sp}}_{m\cdot n}$ of Sp_{mn} given in [R-2]. Then we restrict the representation to a subgroup $\tilde{\mathrm{Sp}}_n \times \mathrm{O}(Q)$ = inverse image of $\mathrm{Sp}_n \times \mathrm{O}(Q)$ in $\tilde{\mathrm{Sp}}_{mn}$. If m is even, then we know that we get an honest representation of $\mathrm{Sp}_n \times \mathrm{O}(Q)$. This we call the *Weil representation* π_Q of $\mathrm{Sp}_n \times \mathrm{O}(Q)$ on the space $S[M_{mn}(k)]$. In fact, we recall formulae for generators of $\mathrm{Sp}_n \times \mathrm{O}(Q)$ in π_Q in § 1 of [R-2] where $Q = Q_v$, a local form of Q.

(VI) Using the local data in (V), we can define a global Weil representation π_Q of $\mathrm{Sp}_n(\mathbb{A}) \times \mathrm{O}(Q)(\mathbb{A})$ on the space $S[M_{mn}(\mathbb{A})]$ (for details, see [R-2]). That is, to every $\varphi \in S[M_{mn}(\mathbb{A})]$, we can construct a function $((G, g) \in \mathrm{Sp}_n(\mathbb{A}) \times \mathrm{O}(Q)(\mathbb{A}))$:

$$\theta_\varphi(G, g) = \sum_{\xi \in M_{mn}(K)} \pi_Q(G, g)(\varphi)(\xi).$$

Then θ_φ is an automorphic function on $\mathrm{Sp}_n(\mathbb{A}) \times \mathrm{O}(Q)(\mathbb{A})$. That is, $\theta_\varphi(\gamma_1 G, \gamma_2 g) = \theta_\varphi(G, g)$ for all $\gamma_1 \in \mathrm{Sp}_n(K)$, $\gamma_2 \in \mathrm{O}(Q)(K)$. Moreover, θ_φ is a slowly increasing function on $\mathrm{Sp}_n(K) \times \mathrm{O}(Q)(K) \backslash \mathrm{Sp}_n(\mathbb{A}) \times \mathrm{O}(Q)(\mathbb{A})$ in the sense of $[A]$.

(VII) We construct here representatives for the *maximal parabolic subgroups* of Sp_n.

Sp_n has parabolics given by P_{n-i}, $i = 0, \ldots, n-1$, where

$$P_{n-i} \cong \mathrm{Sp}_i \times G\ell_{n-i} \times U_i^n$$

with

$$\mathrm{Sp}_i = \left\{ \left[\begin{array}{cc|cc} A & 0 & B & 0 \\ 0 & I & 0 & 0 \\ \hline C & 0 & D & 0 \\ 0 & 0 & 0 & I \end{array}\right] \,\middle|\, \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathrm{Sp}_i \right\}$$

and

$$G\ell_{n-i} = \left\{ \left[\begin{array}{cc|cc} I & 0 & & \\ & & 0 & \\ 0 & A & & \\ \hline & & 0 & 0 \\ 0 & & & \\ & & 0 & (A')^{-1} \end{array}\right] \,\middle|\, A \in G\ell_{n-i} \right\}$$

and

U_i^n is the semidirect product $U_i^{n,1} \times U_i^{n,2}$, where

$$U_\iota^{n,1} = \left\{ \left[\begin{array}{cc|cc} I & 0 & 0 & \\ T & I & & \\ \hline 0 & & I & -T^t \\ & & 0 & I \end{array} \right] \middle| T \in M_{n-\iota,\iota}(k) \right\}$$

and

$$U_\iota^{n,2} = \left\{ \left[\begin{array}{c|cc} I & 0 & T \\ & T^t & X \\ \hline 0 & I & \end{array} \right] \middle| T \in M_{\iota,n-\iota}(k),\ X \in \mathrm{Sym}_{n-\iota}(k) \right\}.$$

(VIII) We recall here Theorem 1.2.1 of [R-2]: $L^2_{\mathrm{cusp}}(\mathrm{O}(Q(\mathbb{A}))$ is an orthogonal direct sum of

$$R_1(Q) \oplus R_2(Q) \oplus \ldots \oplus R_m(Q)$$

where $R_1(Q) = \{f \in L^2_{\mathrm{cusp}} | \langle \theta_\varphi(G, g) | f(g) \rangle \equiv 0$ for all $G \in \mathrm{Sp}_1(\mathbb{A})$ and all $\varphi \in S[M_{m1}(\mathbb{A})]\}^\perp$ in L^2_{cusp} and inductively $R_\iota(Q) = \{f \in L^2_{\mathrm{cusp}} | \langle \theta_\varphi(G, g) | f(g) \rangle \equiv 0$ for all $G \in \mathrm{Sp}_\ell(\mathbb{A})$ and all $\varphi \in S[M_{m\ell}(\mathbb{A})]$ with $\ell \leqslant i\}^\perp$ in $\{f \in L^2_{\mathrm{cusp}} | \langle \theta_\varphi(G, g) | f(g) \rangle \equiv 0$ for all $G \in \mathrm{Sp}_\ell(\mathbb{A})$ and all $\varphi \in S[M_{m\ell}(\mathbb{A})]$ with $\ell < i\}$. Similarly $L^2_{\mathrm{cusp}}(\mathrm{Sp}_n(\mathbb{A}))$ has an orthogonal decomposition

$$I_1(Q) \oplus I_2(Q \oplus H_1) \oplus \ldots \oplus I_{r-1}(Q \oplus H_r)$$

where the spaces I_i are defined in a similar manner as the R_ι above. (Here Q is a nondegenerate anisotropic form over K and is possibly zero.)

(IX) We use the parametrization of the representations of $SL_2(\mathbb{R})$ as given in [L]. Then H_0^+ and H_0^- are the mock discrete series and $D_{m.}^{\pm}$ is the family of discrete series with $m \geqslant 2$.

(X) We let $e(\ldots)$ be the exponential function e raised to the $(\ldots)$ power.

(XI) We let

$$w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \mathrm{Sp}_n(\mathbb{Q}) \text{ and } \mathrm{w}^{-1} = \mathrm{w}_1.$$

(XII) We let

$$\Omega_\iota = w \left[\begin{array}{c|c} I & T_\iota \\ \hline 0 & I \end{array} \right] \in \mathrm{Sp}_{2n}(\mathbb{Q})$$

where

$$T_i = \left[\begin{array}{cc|cc} 0 & & I_{n-i} & 0 \\ & & 0 & 0 \\ \hline I_{n-i} & 0 & 0 & \\ 0 & 0 & & \end{array}\right],$$

$I_{n-i} = (n-i) \times (n-i)$ identity matrix.

§ 1. An inner product formula

One of the main goals mentioned in [R-2] is to have an effective comparison of Hecke operators on the spaces $L^2_{\text{cusp}}(\ )$ and R_i (also L^2_{cusp} $(\ )$ and $S_i(\ )$). However prior to having such a statement, we must determine a relation between the inner product of the lift of cusp forms on $\mathrm{O}(Q)(\mathbb{A})$ to the inner product of the initial cusp forms themselves on $\mathrm{Sp}_r(\mathbb{A})$.

Indeed we fix cusp forms ψ_1 and ψ_2 on the groups $\mathrm{Sp}_k(\mathbb{A})$ and $\mathrm{Sp}_{k'}(\mathbb{A})$ respectively. Then we consider the associated lifts

$$\ell^k_{\varphi,\psi_1}(g) = \int_{\mathrm{Sp}_k(K)\backslash \mathrm{Sp}_k(\mathbb{A})} \theta_\varphi(G, g)\psi_1(G)\mathrm{d}G$$

and

$$\ell^{k'}_{\varphi',\psi_2}(g) = \int_{\mathrm{Sp}_{k'}(K)\backslash \mathrm{Sp}_{k'}(\mathbb{A})} \theta_{\varphi'}(G, g)\psi_2(G)\mathrm{d}G.$$

Then in order to compute the inner product of these two functions, we assume that *at least one is a cusp form*, and hence we have

$$\langle \ell^k_{\varphi,\psi_1} | \ell^{k'}_{\varphi',\psi_2}\rangle_{\mathrm{O}(Q)(\mathbb{A})}$$

$$= \int_{\mathrm{Sp}_k \times \mathrm{Sp}_{k'}(K)\backslash \mathrm{Sp}_k \times \mathrm{Sp}_{k'}(\mathbb{A})} \psi_1(G)\bar{\psi}_2\big((G')^{\Delta_0}\big)\mathscr{T}_Q(\varphi \otimes \bar{\varphi}')$$

$$(G, G')\mathrm{d}G\mathrm{d}G', \tag{1-1}$$

($(G')^{\Delta_0}$ defined below) where $\mathscr{T}_Q$ is the functional on $S[M_{m,k+k'}(\mathbb{A})] = S[M_{m,k}(\mathbb{A})] \otimes S[M_{m,k'}(\mathbb{A})]$ given by (provided we have convergence of such an integral)

$$\mathscr{T}_Q(\psi)(G) = \int_{\mathrm{O}(Q)(K)\backslash \mathrm{O}(Q)(\mathbb{A})} \theta_\psi(G, g)\mathrm{d}g,$$

with $\psi \in S[M_{m,k+k'}(\mathbb{A})]$, $G \in \mathrm{Sp}_{k+k'}(\mathbb{A})$. In the above formula (1-1), we consider $\mathrm{Sp}_k \times \mathrm{Sp}_{k'}$ as a subgroup of $\mathrm{Sp}_{k+k'}$ via the obvious diagonal embedding map.

PROOF OF (1-1): We first write down the integral for the inner product in (1-1), and we have

$$\langle \ell^k_{\varphi,\psi_1} | \ell^{k'}_{\varphi',\psi_2} \rangle$$

$$= \int_{O(Q)(K)\backslash O(Q)(\mathbb{A})} \left\{ \int_{\mathrm{Sp}_k \times \mathrm{Sp}_{k'}(K)\backslash \mathrm{Sp}_k \times \mathrm{Sp}_{k'}(\mathbb{A})} \psi_1(G)\overline{\psi_2(G')} \right.$$

$$\times \left\{ \sum_{(\xi|\xi') \in M_{m,k+k'}(K)} \pi_Q(G, g)(\varphi)(\xi)\overline{\pi_Q(G', g)(\varphi')(\xi')} \right\}$$

$$\left. \mathrm{d}G\mathrm{d}G' \right\} \mathrm{d}g.$$

But then we know that $\overline{\pi_Q(G', g)(\varphi')(X)} = \pi_Q((G')^{\Delta_0}, g)(\bar{\varphi}')(X)$ for all $\varphi' \in S[M_{mk'}(\mathbb{A})]$, $X \in M_{mk'}(\mathbb{A})$ and

$$(G')^{\Delta_0} = \left[\begin{array}{c|c} I_{k'} & 0 \\ \hline 0 & -I_{k'} \end{array}\right] G' \left[\begin{array}{c|c} I'_k & 0 \\ \hline 0 & -I_{k'} \end{array}\right].$$

Thus combining the terms in the above series via this latter rule and changing the order of integration we get (1-1). We note that it is possible to switch the order of integration when $\mathscr{T}_Q(\psi)$ is an absolutely convergent integral (i.e., this happens when Q is anisotropic or when $\dim(Q) > \mathrm{rank}(Q) + (k + k') + 1$ if Q is isotropic); then $\mathscr{T}_Q(\psi)$ is a slowly increasing function on its restriction to $\mathrm{Sp}_k \times \mathrm{Sp}_{k'}(\mathbb{A})$ and is integrated against the product of cusp forms $\psi_1 \cdot \bar{\psi}_2$.

On the other hand, we can infer something about the functional $\psi \to \mathscr{T}_Q(\psi)(G)$, $G \in \mathrm{Sp}_{k+k'}(\mathbb{A})$. Namely we can apply Siegel's formula to $\mathscr{T}_Q$ (when $\dim Q > 2(k + k') + 2$), and we have

$$\mathscr{T}_Q(\psi)(G) = \sum_{\gamma \in P_{k+k'}(K)\backslash \mathrm{Sp}_{k+k'}(K)} \pi_Q(\gamma G)(\psi)(\mathbf{0}).$$

But we know that $\mathrm{Sp}_k(K) \times \mathrm{Sp}_{k'}(K)$ operating on the coset space $P_r(K)\backslash \mathrm{Sp}_r(K)$ ($r = k + k'$) has a finite number of orbits. Thus if Ω_t is a set of representatives of such orbits and $(\mathrm{Sp}_k \times \mathrm{Sp}_{k'})(\Omega_t)$ is the stabilizer of $P_r(K) \cdot \Omega_t$ in $\mathrm{Sp}_k \times \mathrm{Sp}_{k'}(K)$, then we see easily that (*see* § 3 *for the* *proof of convergence in* (1-2))

$$\langle \ell^{k}_{\varphi,\psi_1} | \ell^{k'}_{\varphi',\psi_2} \rangle_{O(Q)(\mathbb{A})}$$

$$= \sum_{\Omega_i} \left\{ \int_{\mathrm{Sp}_k \times \mathrm{Sp}_{k'}(\Omega_i) \backslash \mathrm{Sp}_k \times \mathrm{Sp}_{k'}(\mathbb{A})} \psi_1(G) \overline{\psi_2\left((G')^{\Delta_0}\right)} \right.$$

$$\left. \times \pi_Q(\Omega_i(G, G'))(\varphi \otimes \bar{\varphi}')(\mathbf{0}) \mathrm{d}\sigma_i(G, G') \right\}. \tag{1-2}$$

Thus we have completed the first reduction using Siegel's formula to express $\langle \ell^{k}_{\varphi,\psi_1} | \ell^{k'}_{\varphi',\psi_2} \rangle$ as a finite sum of adelic integrals over $\mathrm{Sp}_k \times \mathrm{Sp}_{k'}(\mathbb{A})$.

Then we know that if $k \geqslant k'$, then $\mathrm{Sp}_k \times \mathrm{Sp}_{k'}(\Omega_i)$ is isomorphic to the group ($i = 0, \ldots, k'$) (see (XII of §0)

$$\Delta_i \times (S_i \times R_i)$$

where

$$\Delta_i = \left\{ (g, g^{\Delta_1}) | g \in \mathrm{Sp}_i(K) \right\}$$

and

$$S_i = GL_{k-i} \times U_i^k$$

$$R_i = GL_{k'-i} \times U_i^{k'}$$

(for this, see similar computations given in [R-2]). We recall that $G^{\Delta_1} = ((G')^{-1})^{\Delta_0}$. We let $G^{\Delta} = (G')^{-1}$.

Thus the second reduction step is to evaluate the adelic integrals in (1-2). We have that

$$\int_{\mathrm{Sp}_k \times \mathrm{Sp}_{k'}(\Omega_i) \backslash \mathrm{Sp}_k \times \mathrm{Sp}_{k'}(\mathbb{A})} \psi_1(G) \overline{\psi_2\left((G')^{\Delta_0}\right)} \pi_Q(\Omega_i(G, G'))$$

$$= (\varphi \otimes \bar{\varphi}')(\mathbf{0}) \mathrm{d}G \mathrm{d}G'$$

$$\int_{\Delta_i \times \tilde{S}_i(\mathbb{A}) \times \tilde{R}_i(\mathbb{A}) \backslash \mathrm{Sp}_k \times \mathrm{Sp}_{k'}(\mathbb{A})} \pi_Q(\Omega_i(G, G'))(\varphi \otimes \bar{\varphi}')(\mathbf{0})$$

$$\left\{ \int_{U_i^k(K) \backslash U_i^k(\mathbb{A}) \times U_i^{k'}(K) \backslash U_i^{k'}(\mathbb{A})} \psi_1(uG) \overline{\psi_2^{\Delta_0}(u'G')} \mathrm{d}u \mathrm{d}u' \right\} \mathrm{d}\sigma(G, G'),$$

where $\tilde{S}_i(\mathbb{A}) = GL_{k-i}(K) \times U_i^k(\mathbb{A})$ and $\tilde{R}_i(\mathbb{A}) = GL_{k'-i}(K) \times U_i^{k'}(\mathbb{A})$ and $\mathrm{d}\sigma$ is a suitably normalized measure on the quotient $\Delta_i \times \tilde{S}_i \times \tilde{R}_i \backslash$

$\mathrm{Sp}_k \times \mathrm{Sp}_{k'}(\mathbb{A})$. Here $U_i^k \times U_i^{k'}(\mathbb{A})$ fixes Ω_i, i.e. $\pi_Q(\Omega_i \cdot (x, y))(\psi)(\mathbf{0}) = \pi_Q(\Omega_i)(\psi)(\mathbf{0})$ for all $\psi \in S[M_{m,k+k'}(\mathbb{A})]$ (i.e. using factorizability of π_Q into local π_{Q_v} and the validity of the statement for local Weil representations π_{Q_v}).

Thus we have that if $k > k'$ or $i > 0$ (if $k = k'$), then (since ψ_1 and ψ_2 are cusp forms) $\langle \ell^k_{\varphi,\psi_1} | \ell^{k'}_{\varphi',\psi_2} \rangle \equiv 0$. Thus if $i = 0$ and $k = k'$, the only adelic integral that we must consider is

$$\int_{\Delta_0\backslash \mathrm{Sp}_k \times \mathrm{Sp}_k(\mathbb{A})} \psi_1(G)\overline{\psi_2^{\Delta_0}(G')}\pi_Q(\Omega_0(G, G'))(\varphi \otimes \bar{\varphi}')(\mathbf{0})\mathrm{d}\sigma(G, G'). \tag{1-3}$$

But it is possible to telescope the integration above so that (1-3) equals

$$\int_{\Delta_0(\mathbb{A})\backslash \mathrm{Sp}_k \times \mathrm{Sp}_k(\mathbb{A})} \pi_Q(\Omega_0(G, G'))(\varphi \otimes \bar{\varphi}')(\mathbf{0}) \left(\int_{\Delta_0(K)\backslash\Delta_0(\mathbb{A})} \psi_1(hG)\overline{\psi_2\left(h^{\Delta}(G')^{\Delta_0}\right)}\mathrm{d}h\right)\mathrm{d}\sigma(G, G').$$

(Here again we use the factorizability properties of the Weil representation to get $\pi_Q(\Omega_0(g, g^{\Delta_1}))(\psi)(\mathbf{0}) = \pi_Q(\Omega_0)(\psi)(\mathbf{0})$ for all $\psi \in S[M_{mr}(\mathbb{A})]$.)

But then we note that the map $\mathrm{Sp}_r(\mathbb{A}) \to \Delta_0(\mathbb{A})\backslash \mathrm{Sp}_r(\mathbb{A}) \times \mathrm{Sp}_r(\mathbb{A})$ given by $g \rightsquigarrow (g, 1)\Delta_0(\mathbb{A})$ is a homomorphism of topological spaces carrying the Haar measure on $\mathrm{Sp}_r(\mathbb{A})$ onto the measure $\mathrm{d}\sigma$ (given above) on $\Delta_0(\mathbb{A})\backslash \mathrm{Sp}_r(\mathbb{A}) \times \mathrm{Sp}_r(\mathbb{A})$. Thus we can rewrite (1-3) as

$$\int_{\mathrm{Sp}_k(\mathbb{A})} \pi_Q(\Omega_0(G, 1))(\varphi \otimes \bar{\varphi}')(\mathbf{0})\langle \psi_1 * G | \psi_2^{\Delta} \rangle_{\mathrm{Sp}_k(\mathbb{A})}\mathrm{d}\omega(G),$$

where

$$\langle \psi_1 * G | \psi_2^{\Delta} \rangle_{\mathrm{Sp}_k(\mathbb{A})} = \int_{\mathrm{Sp}_k(K)\backslash \mathrm{Sp}_k(\mathbb{A})} \psi_1(hG)\overline{\psi_2^{\Delta}(h)}\mathrm{d}h.$$

Thus we define for any pair (φ, φ') in $S[M_{mk}(\mathbb{A})] \times S[M_{mk}(\mathbb{A})]$ a function on $\mathrm{Sp}_k(\mathbb{A})$,

$$\mathscr{L}(\varphi, \varphi')(G) = \pi_Q(\Omega_0(G, 1))(\varphi \otimes \bar{\varphi}')(\mathbf{0}).$$

Then we have the following Proposition determining $\langle \ell^k_{\varphi,\psi_1} | \ell^{k'}_{\varphi',\psi_2} \rangle$.

Proposition 1.1: *Assume that $m > 2(k + k') + 2$. Assume ℓ^k_{φ,ψ_1} or $\ell^{k'}_{\varphi',\psi_2}$ is a cusp form. Then*

(I) *Let $k \neq k'$. Then $\langle \ell^k_{\varphi,\psi_1} | \ell^{k'}_{\varphi',\psi_2} \rangle \equiv 0$.*

(II) *If* $k = k'$, *then*

$$\langle \ell^k_{\varphi,\psi_1} | \ell^{k'}_{\varphi',\psi_2} \rangle = \langle \psi_1 * \mathscr{L}(\varphi, \varphi') | \psi_2^{\Delta} \rangle$$

where $\mathscr{L}(\varphi, \varphi')$ *is an* L^1 *function on* $\mathrm{Sp}_k(\mathbb{A})$.

PROOF: We recall the construction of the Tamagawa measure on $\mathrm{Sp}_n(\mathbb{A})$. Locally we fix for each prime v in K a Haar measure $\mathrm{d}\sigma_v$ on $\mathrm{Sp}_n(K_v)$ such that if v is finite, then

$$\int_{\mathrm{Sp}_n(\mathcal{O}_{K_v})} \mathrm{d}\sigma_v = q^{-\dim(\mathrm{Sp}_n)} \# \left[\mathrm{Sp}_n\left(\mathcal{O}_{K_v} / \pi_v \mathcal{O}_{K_v} \right) \right].$$

Then the Tamagawa measure on $\mathrm{Sp}_n(\mathbb{A})$ is the *unique measure* $\mathrm{d}\sigma$ on $\mathrm{Sp}_n(\mathbb{A})$ such that $\mathrm{d}\sigma$ induces on each product

$$\mathrm{Sp}_n(\mathbb{A}_s) = \prod_{v \in S} \mathrm{Sp}_n(\mathcal{O}_{K_v}) \times \prod_{v \notin S} \mathrm{Sp}_n(K_v)$$

(S = all primes in K except a finite number), the *product measure* $\prod_v \mathrm{d}\sigma_v$. Thus to test whether $\mathscr{L}(\varphi, \varphi')$ is an L^1 function on $\mathrm{Sp}_n(\mathbb{A})$, it suffices to consider $\varphi = \prod \varphi_v$, $\varphi' = \prod \varphi'_v$ and to show that for the filtered set

$$\psi_S(G) = \pi_Q(\Omega(G, 1))(\varphi \otimes \bar{\varphi}')(\mathbf{0}),$$

where $G \in \mathrm{Sp}_n(\mathbb{A}_S)$,

$$\lim_S \int |\psi_S(G)| \mathrm{d}\sigma < \infty.$$

On the other hand, we know that

$$\pi_Q(\Omega_0(G, 1))(\varphi \otimes \bar{\varphi}')(\mathbf{0})$$

$$= \int_{M_{mk}(\mathbb{A}) \times M_{mk}(\mathbb{A})} \tau\left[\mathrm{Tr}\left[\mu^t Q \sigma \right] \right] \pi_Q(G)(\varphi)(\mu) \bar{\varphi}'(\sigma) \mathrm{d}\mu \mathrm{d}\sigma,$$

while the latter term is (up to a nonzero scalar) of the form

$$\int_{M_{mk}(\mathbb{A})} \pi_Q(G)(\varphi)(\mu) \pi_Q(w)(\bar{\varphi}')(\mu) \mathrm{d}\mu.$$

But then we can express the above integral as an absolutely convergent product

$$\prod_v \int_{M_{mk}(K_v)} \pi_{Q_v}(G_v)(\varphi_v)(\mu_v) \pi_{Q_v}(w)(\bar{\varphi}'_v)(\mu_v) \mathrm{d}\mu_v.$$

But on the other hand, we know that for *almost* all primes v in K, Q_v is isotropic (here $m > 4$) and the Weil representation π_{Q_v} of $\mathrm{Sp}_k(K_v)$ on $S[M_{mk}(K_v)]$ has the characteristic function χ_v of the lattice $M_{mk}(\mathcal{O}_v)$ in $M_{mk}(K_v)$ as an $\mathrm{Sp}_k(\mathcal{O}_v)$ *fixed vector*. Thus for *almost* all primes v in K (let R be this set),

$$c_v = \int_{\mathrm{Sp}_k(\mathcal{O}_v)} \left(\int_{M_{mk}(K_v)} \pi_{Q_v}(k_v)(\chi_v)(\mu_v)\bar{\chi}'_v(\mu_v)\mathrm{d}\mu_v \right) \mathrm{d}\sigma_v$$

$$= q^{-\dim(\mathrm{Sp}_k)} \# \left[\mathrm{Sp}_k(\mathcal{O}_v)/\mathrm{Sp}_k(\pi\mathcal{O}_v)\right] vol(M_{mk}(\mathcal{O}_v)).$$

Hence we have

$$\int_{\mathrm{Sp}_n(\mathbf{A}_S)} |\psi_S(G)|\mathrm{d}\sigma \leqslant \left(\prod_{v \in (S \cap R)} c_v \right) \prod_{v \notin S \cap R} \left\{ \int_{\mathrm{Sp}_k(K_v)} \right.$$

$$\left. \int_{M_{mk}(K_v)} |\pi_{Q_v}(g_v)(\varphi_v)(\mu_v)|\, |\pi_{Q_v}(w)\bar{\varphi}'_v(\mu_v)|\mathrm{d}\mu_v \mathrm{d}\sigma_v \right\}. \qquad (*)$$

But then following the general arguments of $[H\text{-}M]$ we have that

$$\int_{M_{mk}(K_v)} \pi_Q(g_v)(\psi_v)(\mu_v)\tilde{\psi}_v(\mu_v)\mathrm{d}\mu_v$$

for

$$g_v = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & D^{-1} \end{array}\right]$$

with $D = \mathrm{diag}(a_1, \ldots, a_k)$ is majorized by

$$|a_1|^{-m/2} \ldots |a_k|^{-m/2}.$$

Thus since $m > 4k + 2$, we have that in $(*)$

$$\prod_{v \notin S \cap R} \int_{\mathrm{Sp}_k(K_v)} \left(\int |\pi_{Q_v}(g_v)(\varphi_v)(\mu_v)|\, |\pi_{Q_v}(w)\bar{\varphi}'_v(\mu_v)|\mathrm{d}\mu_v \right) \mathrm{d}\sigma_v$$

is majorized independent of the set S. (Here we use the polar decomposition of $\mathrm{Sp}_k(K_v)$ relative to $\mathrm{Sp}_k(\mathcal{O}_v)$ and the diagonals.)

Thus we have by monotonic convergence that

$$\lim_S \int_{\mathrm{Sp}_k(\mathbf{A}_s)} |\psi_S(G)|\mathrm{d}\sigma_S(G)$$

converges to

$$\int_{\mathrm{Sp}_k(\mathbb{A})} |\psi(G)| \mathrm{d}\sigma(G).$$

Thus ψ is an L^1 function on $\mathrm{Sp}_k(\mathbb{A})$.

Q.E.D.

REMARK 1.2: In the course of the proof of Proposition 1.1, we have shown that $\mathscr{L}(\varphi, \varphi')$ is, up to a nonzero scalar multiple (independent of φ and φ'),

$$\langle \pi_Q(G)\varphi | \pi_Q(w_1)\varphi' \rangle_{L^2(M_{mk}(\mathbb{A}))},$$

($\langle , \rangle$ Hilbert product on $L^2(\dots)$ and see (XI) of §0) which is a matrix coefficient of the Weil representation of $\mathrm{Sp}_k(\mathbb{A}) \times \mathrm{O}(Q)(\mathbb{A})$ on $S[M_{mk}(\mathbb{A})]$. Moreover, assuming that $\varphi = \otimes \varphi_v$ and $\varphi' = \otimes \varphi'_v$, we show that

$$\langle \pi_Q(G)\varphi | \pi_Q(w_1)\varphi' \rangle = \prod_{v \in K} \langle \pi_{Q_v}(G_v)\varphi_v | \pi_{Q_v}(w_1)\varphi'_v \rangle.$$

Then it is possible to draw certain interesting consequences from Proposition 1.1 and Remark 1.2.

Indeed we see directly that the subspace of $L^2_{\mathrm{cusp}}(\mathrm{Sp}_n(\mathbb{A}))$ where each irreducible component maps to a nonzero subspace in $L^2_{\mathrm{cusp}}(\mathrm{O}(Q)(\mathbb{A}))$ via the lifting map can be characterized as the closure of $\{\mathscr{L}(\varphi, \varphi')(L^2_{\mathrm{cusp}}(\mathrm{Sp}_n(\mathbb{A})))\}$ (φ, φ' vary in $S[M_{mn}(\mathbb{A})]$) in $L^2_{\mathrm{cusp}}(\mathrm{Sp}_n(\mathbb{A}))$. Thus it becomes a matter of determining the range of the operators $\mathscr{L}(\varphi, \varphi')$ acting on $L^2_{\mathrm{cusp}}(\mathrm{Sp}_n(\mathbb{A}))$. But using Remark 1.2 and the proof of Proposition 1.1, we see that if ψ_1 and ψ_2^Δ lie in the same irreducible component $\prod$, then

$$\langle \psi_1 * \mathscr{L}(\varphi, \varphi') | \psi_2^\Delta \rangle = \prod_{v \in K} \int_{\mathrm{Sp}_n(K_v)} \langle \pi_{Q_v}(G_v)\varphi_v | \pi_{Q_v}(w_1)\varphi'_v \rangle$$

$$\times \langle (\psi_1)_v * G_v | (\psi_2)_v^\Delta \rangle_{\Pi_v} \mathrm{d}\sigma_v(G_v)$$

if $\varphi = \otimes_{v \in K}\varphi_v$ and $\varphi' = \otimes_v \varphi'_v$ are factorizable functions. Thus the essential *local question* is to determine which local unitary representations of $\mathrm{Sp}_n(K_v)$ have the property that the space $A(\delta)$ of matrix coefficients of δ can be nonsingularly paired with the space $A(\pi_{Q_v})$, the space of matrix coefficients of the local Weil representation π_{Q_v} of $\mathrm{Sp}_n(K_v)$ on $S[M_{mn}(K_v)]$, via the bilinear form

$$A(\delta) \otimes A(\pi_{Q_v}) \rightsquigarrow C$$

given by

$$f_1 \otimes f_2 \mapsto \int_{\mathrm{Sp}_n(K_v)} f_1(x) f_2(x) \mathrm{d}\sigma_v(x).$$

However we know that if $A(\pi_{Q_v}) \subseteq L^1(\mathrm{Sp}_n(K_v), \mathrm{d}\omega_v)$, then a necessary and sufficient condition for the above form to be zero (or, simply put, $A(\pi_{Q_v}) \subseteq \text{Annihilator}(\delta) = \{f \in L^1 | \pi_\delta(f) \equiv 0\}$) is that $\mathrm{Hom}_{\mathrm{Sp}_n(K_v)}(\pi_{Q_v}, \check{\delta}) = \{0\}$ (where $\check{\delta}$ = the contragredient representation of δ). We shall use this criterion below to determine the nonvanishing properties of the inner product

$$\langle \psi_1 * \mathscr{L}(\varphi, \varphi') | \psi_2^{\Delta} \rangle.$$

REMARK 1.3: We can compute

$$\int_{\mathrm{Sp}_k(K_v)} \langle \pi_{Q_v}(G_v)\varphi_v | \pi_{Q_v}(w_1)\varphi_v' \rangle \langle (\psi_1)_v * G_v | (\psi_1)_v \rangle_{\Pi_v} \mathrm{d}\sigma_v(G_v)$$

explicitly in the case when ψ_1 is an $\mathrm{Sp}_k(\mathcal{O}_v)$ invariant vector (spherical-type local component) of an irreducible representation Π_v (v finite) and $\varphi_v = \varphi_v'$ = characteristic function of the lattice $M_{mk}(\mathcal{O}_v)$ in $M_{mk}(K_v)$. Indeed, if

$$G_v = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & D^{-1} \end{array}\right]$$

where D is the diagonal matrix

$$\begin{bmatrix} \pi^{m_1} & & 0 \\ & \ddots & \\ 0 & & \pi^{m_k} \end{bmatrix},$$

then the above integral equals the infinite sum

$$\sum_{D \in \Sigma} \omega_{\Pi_v}(D) r(D) \operatorname{vol}(D) \langle \det D | \Delta_Q \rangle$$

(Δ_Q = discriminant of Q), where ω_{Π_v} is the spherical function associated to the representation Π_v (i.e. $\omega_{\Pi_v}(g) = \langle (\psi_1)_v * g_v | (\psi_1)_v \rangle_{\Pi_v}$, $r(D) = q^{-(m_1 + \dots m_k)\cdot m/2}$, and $\operatorname{vol}(D)$ is the volume of the double coset $\mathrm{Sp}_k(\mathcal{O}_v) D\, \mathrm{Sp}_k(\mathcal{O}_v)$ *relative to* $\mathrm{d}\sigma_v$ and Σ runs through the set of distinct $\mathrm{Sp}_k(\mathcal{O}_v)$ double cosets in $\mathrm{Sp}_k(K_v)$. Then it is possible to compute the infinite sum as a rational function (see § 4), and we get for $k = 1$ that the sum equals

$$\frac{\left(1 + \langle \Delta_Q | \pi \rangle q^{-((m/2)-1)}\right)\left(1 - \langle \Delta_Q | \pi \rangle q^{-m/2}\right)}{\left(1 - \langle \Delta_Q | \pi \rangle q^{-\sigma-((m/2)-1)}\right)\left(1 - \langle \Delta_Q | \pi \rangle q^{\sigma-((m/2)-1)}\right)}$$

where σ is the complex number which determines the representation Π_v, i.e. Π_v is equivalent to the unitary representation contained in the induced representation of $\mathrm{Sp}_1(K_v)$ induced from the parabolic subgroup

$$\begin{bmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{bmatrix} \rightsquigarrow |\alpha|^{\sigma+1}.$$

In particular, noting that Π_v must be unitary, we have that the Euler product

$$\prod_v \frac{\left(1 + \langle \Delta_Q | \pi \rangle q^{-((m/2)-1)}\right)\left(1 - \langle \Delta_Q | \pi \rangle q^{-m/2}\right)}{\left(1 - \langle \Delta_Q | \pi \rangle q^{-\sigma_v - ((m/2)-1)}\right)\left(1 - \langle \Delta_Q | \pi \rangle q^{\sigma_v - ((m/2)-1)}\right)}$$

(where the product runs over all finite primes v in K where $\mathrm{Sp}_1(K_v)$ has a nonzero fixed vector in Π_v) is a *nonzero* function (i.e. σ_v satisfies $|\mathrm{Re}(\sigma_v)| < 1$).

Thus with the above computation, we can obtain the following consequence of Proposition 1.1.

For the remainder of this section, we assume that K is a totally real field over $\mathbb{Q}$.

COROLLARY 1 TO PROPOSITION 1.1:

Let Q be anisotropic form over K with $m > 6$. *Then*

$$I_1(Q) = \bigoplus_{\Pi \in X_Q} \Pi$$

where $X_Q = \{$all cuspidal $\mathrm{Sp}_1(\mathbb{A})$ representations Π satisfying (i) if v is a prime where Q_v is anisotropic, then the local factor Π_v of Π occurs discretely in the Weil representation π_{-Q_v}; (ii) if v is a *real prime* where Q_v is isotropic of signature $(a_v, 1)$ ($(1, a_v)$ resp.) and if Π_v admits a highest weight vector, then Π_v is either (-) mock discrete series ((+) mock discrete series, resp.) or appears discretely in the Weil representation π_{-Q_v}(π_{Q_v}, resp.).

REMARK 1.4: We note that since $m > 6$, then all the primes where Q_v is *anisotropic* are *Archimedean real primes*. Thus we can characterize the set X_Q in a simpler fashion as follows: $X_Q = \{$all cuspidal $\mathrm{Sp}_1(\mathbb{A})$ representations Π satisfying

(i) if v is a prime where Q_v is anisotropic, then Π_v is a discrete series of the form D_r^- with $r \geqslant m/2$ (D_r^+, resp., with $r \geqslant m/2$) if Q_v is positive definite (negative definite).

(ii) if v is a real prime where Q_v has signature $(a_v, 1)$ or $(1, a_v)$ and if Π_v admits a highest weight vector, then Π_v is a discrete series of the form D_i^- with $i \geqslant 2$ or a mock discrete series H_0^- (D_i^+ resp. with $i \geqslant 2$ or a mock discrete series H_0^+)$\}$.

PROOF: We must show two statements. First, if Π occurs in X_Q, then for some $\psi \in \Pi$, $\ell_{\varphi,\psi} \neq 0$ (for a suitable choice of φ in $S[M_{m1}(\mathbb{A})]$). Secondly, if Π does not occur in X_Q, we must show $\ell_{\varphi,\psi} \equiv 0$ for all ψ in Π and φ in $S[M_{m1}(\mathbb{A})]$.

From Remark 1.3 (noting that φ and φ' are factorizable), we have for any *cuspidal* representation Π occurring in $L^2_{\text{cusp}}(\text{Sp}_1(\mathbb{A}))$ that (ψ_1, ψ_2 are elements of Π)

$$\begin{aligned}&\langle \psi_1 * \mathscr{L}(\varphi, \varphi')|\psi_2^{\Delta}\rangle\\ &= \prod_{v\in R'} \frac{\left(1+\langle \Delta_Q|\pi\rangle q_v^{-((m/2)-1)}\right)\left(1-\langle \Delta_Q|\pi\rangle q_v^{-m/2}\right)}{\left(1-\langle \Delta_Q|\pi\rangle q_v^{-\sigma_v-(m/2-1)}\right)\left(1-\langle \Delta_Q|\pi\rangle q^{\sigma_v-(m/2-1)}\right)}\\ &\times\left\{\prod_{v\in R''}\int_{\text{Sp}_n(K_v)}\langle \pi_{Q_v}(G_v)\varphi_v|\pi_{Q_v}(w_1)\varphi_v'\rangle\right.\\ &\qquad\left.\langle(\psi_1)_v * G_v|(\psi_2)_v^{\Delta}\rangle \mathrm{d}\sigma_v(G_v)\right\}\end{aligned}$$

where R' = set of all primes in K which are finite and where $(\psi_1)_v$ and $(\psi_2^{\Delta})_v$ are $\text{Sp}_1(\mathcal{O}_v)$ invariant vectors in Π_v and $\varphi_v = \varphi_v'$ = characteristic function of the lattice $M_{m1}(\mathcal{O}_v)$. We note that R'', the remaining set of primes in K_v, is a finite set. In any case, we note that the nonvanishing of $\langle \psi_1 * \mathscr{L}(\varphi, \varphi')|\psi_2^{\Delta}\rangle$ depends only on the set R'' and the corresponding local integrals appearing in its product.

Then we let $v \in R''$ and suppose v is finite. By using the tensor product properties of the Weil representation, it is possible to write $\langle \pi_{Q_v}(g_v)\varphi_v|\pi_{Q_v}(w_1)\varphi_v'\rangle$ as a product

$$\langle \pi_{(Q_1)_v}(g_v)\varphi_v^1|\pi_{(Q_1)_v}(w_1)\varphi_v^{1\prime}\rangle\langle \pi_{(Q_2)_v}(g_v)\varphi_v^2|\pi_{(Q_2)_v}(w_1)\varphi_v^{2\prime}\rangle,$$

where Q_v splits as an orthogonal direct sum $(Q_1)_v \oplus (Q_2)_v$ with $(Q_1)_v$ a split form and $(Q_2)_v$ an anisotropic form (of dimension 0, 2, or 4) and $\varphi_v = \varphi_v^1 \otimes \varphi_v^2$, $\varphi_v' = \varphi_v^{1\prime} \otimes \varphi_v^{2\prime}$. But this means that the local integral is given by integrating $\langle \pi_{(Q_1)_v}(g_v)\varphi_v^1|\pi_{(Q_1)_v}(w_1)\varphi_v^{1\prime}\rangle$ against the product $\langle \pi_{(Q_2)_v}(g_1)\varphi_v^2|\pi_{(Q_2)_v}(w_1)\varphi_v^{2\prime}\rangle\langle(\psi_1)_\text{v} * g_\text{v}|(\psi_2^{\Delta})_\text{v}\rangle$. Choosing φ_v^2 and $\varphi_\text{v}^{2\prime}$ appropriately, it is simply a matter (to show the non-vanishing of this local integral) of showing that $\langle \pi_{(Q_1)_v}(g_v)\varphi_v^1|\pi_{(Q_1)_v}(w_1)\varphi_v^{1\prime}\rangle$ is a sufficiently generic function, i.e., arbitrary of compact support. However we recall from [R-2] that the family of functions

$$g_v \rightsquigarrow \pi_{(Q_1)_v}(\Omega_0 g_v)(\rho_v \otimes \bar{\rho}_v')(\mathbf{0})$$

($g_v \in \mathrm{Sp}_2(K_v)$ and ρ_v, ρ'_v arbitrary functions in $S[M_{m1}(K_v)]$), span

$$\mathrm{Ind}_{P_2}^{\mathrm{Sp}_2}\left(\left[\begin{array}{c|c} A & X \\ \hline 0 & (A^t)^{-1} \end{array}\right] \to \det|A|^{m'_v/2}\right).$$

(Here m'_v = dimension of the space on which $(Q_1)_v$ lives). Then by suitable choice of ρ_v and ρ'_v and by restriction to $\mathrm{Sp}_1 \times \{\mathbb{1}\}$, we obtain a function $g_v \rightsquigarrow \pi_{(Q_1)_v}(\Omega_0(g_v, 1))(\rho_v \otimes \bar{\rho}'_v)(\mathbf{0})$ whose integral against the product function above is nonzero!

Thus, so far, we have shown that if Π is any cuspidal irreducible representation, then Π will appear in $S_1(Q)$ if and only if $\mathrm{Hom}_{\mathrm{Sp}_1(K_v)}(\pi_{Q_v}, \check{\Pi}_v) \neq \{0\}$ for *all* Archimedean primes in K_v. Thus to finish the proof, we need only to determine the representations σ_v of $\mathrm{Sp}_1(K_v)$ which satisfy $\mathrm{Hom}_{\mathrm{Sp}_1(K_v)}(\pi_{Q_v}, \sigma_v) \neq 0$. Indeed, if Q_v is anisotropic, then we know exactly which σ_v appear in π_{Q_v} (see [*R-S*]). If Q_v is isotropic and v is real, then we known from the computation given in § 4 (see also [*R-S*, § 5 and § 6]) that $\mathrm{Hom}_{\mathrm{Sp}_1(\mathbb{R})}(\pi_{Q_v}, H_s) \neq 0$ for any principal series representation H_s with $|\mathrm{Re}(s)| < 1$ (see comments below). Thus it suffices to check all representations of discrete series type (both usual discrete series and mock discrete series) for $\mathrm{Sp}_1(\mathbb{R})$. Here we use the results of [*R-S*]; indeed if $b > 1$ (where signature$(Q) = (a, b)$), then all discrete series representations of $\mathrm{Sp}_1(\mathbb{R})$ occur in π_{Q_v}. If $b = 1$, then only discrete series of the type D_l^+ occur in π_{Q_v}. We note here that in § 4, with $b = 1$, we will show that $\mathrm{Hom}_{\mathrm{Sp}_1(\mathbb{R})}(\pi_{Q_v}, H_0^+) \neq 0$ (where $H_0^+ = (+)$ mock discrete series) and from [*R-S*, §6], we have $\mathrm{Hom}_{\mathrm{Sp}_1(\mathbb{R})}(\pi_{Q_v}, H_0^-) = \{0\}$ (where $H_0^- = (-)$ mock discrete series).

Q.E.D.

Remark 1.5: We note that the main obstacle to extending the above Corollary to the general Sp_n case consists of two technical points: (1) to establish a formula analogous to the one for $Z(\omega_s, \lambda)$ (defined and derived in § 4 for Sp_1) in order to prove the *nonvanishing* of the main part of the Euler product for $\langle \psi_1 * \mathscr{L}(\varphi, \varphi') | \psi_2^{\Delta} \rangle$ given above and (2) to characterize at the real Archimedean primes which unitary representations of $\mathrm{Sp}_n(\mathbb{R})$ occur in the Weil representation π_Q on $S[M_{mn}(\mathbb{R})]$.

By dualizing the lifting problem above, we can now give a complete characterization of the lift from $\mathrm{O}(Q)(\mathbb{A})$ to $\mathrm{Sp}_1(\mathbb{A})$.

Theorem 1.6: *Let Q be an anisotropic form with $m > 6$. Then the lifting map from $R_1(Q)$ into $L^2_{\mathrm{cusp}}(\mathrm{Sp}_1(\mathbb{A}))$ has as an image the space $I_1(-Q)$. Moreover if Q is defined over $\mathbb{Q}$, then the lifting map from $R_1(Q)$ to $I_1(-Q)$ satisfies the global Howe duality conjecture and is multiplicity preserving in the sense of* [R-2].

PROOF: The proof of the second statement follows from the first statement coupled with the results of [R-2, § 2].

On the other hand, we recall if ψ is a cusp form on $\mathrm{Sp}_1(\mathbb{A})$ and f is a cusp form on $\mathrm{O}(Q)(\mathbb{A})$ with

$$\ell_{\varphi,\psi}(g)=\int_{\mathrm{Sp}_1(K)\backslash \mathrm{Sp}_1(\mathbb{A})}\psi(G)\theta_\varphi(G,g)\mathrm{d}G$$

and

$$\mathscr{L}_{\varphi,f}(G)=\int_{\mathrm{O}(Q)(K)\backslash \mathrm{O}(Q)(\mathbb{A})}f(g)\theta_\varphi(G,g)\mathrm{d}g,$$

then for fixed φ in $S[M_{m1}(\mathbb{A})]$, the linear maps $\psi\to\ell_{\varphi,\psi}$ and $f\to\mathscr{L}_{\varphi,f}$ are "twisted" adjoint to each other, that is

$$\int_{\mathrm{O}(Q)(K)\backslash \mathrm{O}(Q)(\mathbb{A})}\ell_{\varphi,\psi}(g)\overline{f(g)}\mathrm{d}g=\int_{\mathrm{Sp}_1(K)\backslash \mathrm{Sp}_1(\mathbb{A})}\psi(G)\overline{\mathscr{L}_{\varphi,f}(G^{\Delta_0})}\mathrm{d}G_0.$$

Then we recall that $\pi_{-Q}(G)=\pi_Q(G^{\Delta_0})$ for all $G\in\mathrm{Sp}_n(\mathbb{A})$. In particular this implies that $\beta\in I_1(-Q)^{\perp}$ if and only if $\beta^{\Delta_0}(=_{\mathrm{defn}}\beta(G^{\Delta_0})$ for $G\in\mathrm{Sp}_n(\mathbb{A}))\in I_1(Q)^{\perp}$. Hence it follows by a standard argument that if $f\in R_1(Q)$, then $L_{\varphi,f}\in I_1(-Q)$. (Here we use the inner product formula given above.) Similarly we have that if $\beta\in I_1(-Q)$, then $\ell_{\varphi,\beta}\in R_1(Q)$; this implies that $I_1(-Q)$ equals the image of $R_1(Q)$ under lifting.

Q.E.D.

REMARK 1.7: We consider an automorphic cuspidal representation Π in $L^2_{\mathrm{cusp}}(\mathrm{Sp}_1(\mathbb{A}))$, where $\Pi_\infty=\otimes D^-_{m/2}$ (that is, at *every real prime* we specify the representation $D^-_{m/2}$). Such representations certainly exist in $L^2_{\mathrm{cusp}}(\mathrm{Sp}_1(\mathbb{A}))$. Then we consider a cuspidal representation σ in $L^2_{\mathrm{cusp}}(\mathrm{O}(Q)(\mathbb{A}))$ which maps onto Π via the lifting map. We know from [R-2, § 2] that the local components σ_v of σ must satisfy

$$\mathrm{Hom}_{\mathrm{Sp}_1(K_v)\times\mathrm{O}(Q_v)}\left(S\left[M_{m1}(K_v)\right],\check{\Pi}_v\otimes\sigma_v\right)\neq 0$$

(all components of σ are self-contragredient). Thus it follows that if Q_v is anisotropic, then $\sigma_v=$ trivial representation of $\mathrm{O}(Q_v)$ and if Q_v is isotropic, then σ_v is the unique representation of $\mathrm{O}(Q_v)$ which has the property that

$$H^{b_v}\left(LA(\mathrm{O}(Q_v)),M_v,\sigma_v\right)\neq 0$$

where $H^*=$ relative Lie algebra cohomology and $M_v=$ maximal compact subgroup of $O(Q)$. Thus we have constructed an automorphic cuspidal representation σ of $O(Q)(\mathbb{A})$ such that for a real prime v in K where Q_v is *isotropic,* σ_v has a nonvanishing cohomology at level b_v.

REMARK 1.8: The import of Corollary 1 to Proposition 1.1 and Theorem 1.6 is a complete characterization of the image of the lifting β_1 from $O(Q)(\mathbb{A})$ to $Sp_1(\mathbb{A})$; moreover, in the case that Q is anisotropic over $\mathbb{Q}$, we have that the lifting satisfies the global Howe duality conjecture and is multiplicity preserving. Moreover, we see that if Q_1 and Q_2 are any two anisotropic forms of equal dimension greater than 6 over K and have the same signature at every real Archimedean prime of K, then $I_1(Q_1) = I_1(Q_2)$; that is, $I_1(Q)$ *depends only on the signature of Q at the Archimedean real primes of K.*

§ 2. A "comparison of trace" formula

It is possible to draw a conclusion about classical cusp forms from the above work. Namely, we let Q be a unimodular quadratic form over $\mathbb{Z}$ on $\mathbb{R}^{8\cdot t}$. In particular, this means that if we consider the *genus* of Q, we get a finite class of lattices L_t on $\mathbb{Z}^{8\cdot t}$ such that the finite set of double cosets

$$O(Q)(\mathbb{Q})\backslash O(Q)(\mathbb{A})/U_{\mathbb{Z}^{8\cdot t}},$$

$$U_{\mathbb{Z}^{8\cdot t}} = \text{the stabilizer of } \mathbb{Z}^{8\cdot t} \text{ in } O(Q)(\mathbb{A}),$$

is in one-one correspondence with the lattices $L_t = g_t(\mathbb{Z}^{8\cdot t})$, g_t a coset representative above. Then we know that the associated family of θ-series

$$\left\{\theta_{L_t}(\tau) = \sum_{\xi\in L_t} e\left[\pi\sqrt{-1}\,\frac{\tau}{n(L_t)}Q(\xi_t)\right] | n(L_t),\right.$$

$$\left.\text{the } \textit{norm} \text{ of } L_t \text{ given in } [E, \S\, 8]\right\}$$

belongs to the space of holomorphic modular forms $M_{4t}(SL_2(\mathbb{Z})) =$

$$\left\{f\colon H\to\mathbb{C},\ \text{holomorphic}\ | f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{4t}f(z)\right.$$

with

$$\left.\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})\right\}.$$

Let X_Q = the subspace of $M_{4t}(SL_2(\mathbb{Z}))$ spanned by the θ_{L_t} above.

We know that $SL_2(\mathbb{A}) = Sp_1(\mathbb{A})$ satisfies the strong approximation Theorem over $\mathbb{Q}$, the rationals. This implies that we can identify the space $S_{4t}(SL_2(\mathbb{Z}))$, the space of cusp forms in M_{4t}, with

$$\oplus \left\{ v \in \Pi \mid \Pi = \Pi_{\mathrm{fin}} \otimes \Pi_\infty \text{ where } \Pi_\infty = D_k^+ \text{ and} \right.$$

$$\left. \text{the component of } v \text{ in } \Pi_{\mathrm{fin}} \text{ is fixed under } \mathbb{K} = \prod_{v<\infty} SL_2(\mathcal{O}_k) \right\}$$

via the map $\phi(g_\infty) j(g_\infty, \sqrt{-1})^k = f_\phi(g_\infty(\sqrt{-1}))$, where $f_\phi \in S_{4t}(SL_2(\mathbb{Z}))$ and $\phi \in L^2_{\mathrm{cusp}}(\mathrm{Sp}_1(\mathbb{A}))$. (Here j is the usual canonical automorphy factor, and $g \in \mathrm{Sp}_1(\mathbb{A})$ has the decomposition $g_{\mathbb{Q}} \cdot g_\infty \cdot g_U$, $g_Q \in \mathrm{Sp}_1(\mathbb{Q})$, $g_\infty \in \mathrm{Sp}_1(\mathbb{R})$, and $g_U \in \mathbb{K}$.

Then we know that $U_{\mathbb{Z}^{8\cdot t}} = \mathrm{O}(Q)(\mathbb{R})\mathbb{L}$ where $\mathbb{L} = \prod_{v<\infty} L_v$ with $L_v =$ the stabilizer in $\mathrm{O}(Q_v)$ of the lattice $\mathcal{O}_v \otimes \mathbb{Z}^{8\cdot t}$. Then we have that $F(\mathrm{O}(Q)(\mathbb{Q})\backslash \mathrm{O}(Q)(\mathbb{A})/U_{\mathbb{Z}^{8\cdot t}}) = \{ f : \mathrm{O}(Q)(\mathbb{A}) \to \mathbb{C} \mid \text{f is invariant on the left and right by } \mathrm{O}(Q)(\mathbb{Q}) \text{ and } U_{\mathbb{Z}^{8\cdot t}}, \text{ resp.}\}$ is a finite dimensional space and, in fact, using the usual Tamagawa measure on $\mathrm{O}(Q)(\mathbb{Q})\backslash \mathrm{O}(Q)(\mathbb{A})$, we have $F \subset L^2(\mathrm{O}(Q)(\mathbb{Q})\backslash \mathrm{O}(Q)(\mathbb{A}))$, and if $\mathbb{F} = \{ f \in F \mid \langle f \mid \mathbb{1} \rangle = 0\}$, then $\mathbb{F} \subset L^2_{\mathrm{cusp}}(\mathrm{O}(Q)(\mathbb{A}))$. Hence $\mathbb{F}$ can be identified to

$$L^2_{\mathrm{cusp}}(\mathrm{O}(Q)(\mathbb{A}))^{U_{\mathbb{Z}^{8\cdot t}}}.$$

In particular, if σ appears in L^2_{cusp}, then σ has a nonzero $U_{\mathbb{Z}^{8\cdot t}}$ invariant if and only if $\sigma_\infty = \mathbb{1}_{\mathrm{O}(Q)_{\mathbb{R}}}$ and $\sigma_{\mathrm{fin}} = \otimes_{v<\infty}\sigma_v$ with $\sigma_v^{L_v} \neq 0$ for all v. Moreover

$$\sigma^{U_{\mathbb{Z}^{8\cdot t}}}$$

is, at most, a one dimensional space. Hence $\mathbb{F}$ is in bijective correspondence with the set of cuspidal representations σ in $L^2_{\mathrm{cusp}}(\mathrm{O}(Q)(\mathbb{A}))$ which satisfy

$$\sigma^{U_{\mathbb{Z}^{8\cdot t}}} \neq 0.$$

Then we choose the function $\varphi \in S[M_{m1}(\mathbb{A})]$ such that $\varphi_\infty(X) = \mathrm{e}^{-\pi[X]}$ and $\varphi_p(X) =$ characteristic function of $M_{m1}(\mathcal{O}_p)$. Then using such a φ and choosing the characteristic function ρ_t of the double coset $\mathrm{O}(Q)(\mathbb{Q})g_t U_{\mathbb{Z}^{8\cdot t}}$, we have that $f_{\mathscr{L}_{\rho_t,\varphi}}(\tau)$ is a nonzero multiple of θ_{L_t} [R-3]. On the other hand, if we consider $\mathbb{F} \cap R_1(Q)$, then we know that the lifting map β_1 is injective on this space. In particular, we let $\{\psi_t\}$ be the basis of functions in $L^2_{\mathrm{cusp}}(\mathrm{Sp}_1(\mathbb{A}))$ which lie in a unique Π of $\mathrm{Sp}_1(\mathbb{A})$ (which correspond to $S_{4t}(SL_2(\mathbb{Z}))$ above and which are eigenfunctions of the Δ operation on $Sp_1(\mathbb{A})$ (denote by R_Δ, the operation $f(X) \rightsquigarrow f(X^\Delta)$). A simple computation shows that R_Δ acts trivially on $S_{4t}(\mathbb{Z})$. By the above arguments, it is easy to see that $\ell_{\varphi,\psi_t} \neq 0$. Hence ℓ_{φ,ψ_t} will lie in a unique irreducible component of $L^2_{\mathrm{cusp}}(\mathrm{O}(Q)(\mathbb{A}))$. On the other hand, using Proposition 1.1

$$\langle \ell_{\varphi,\psi_t} \mid \ell_{\varphi,\psi_j} \rangle = \langle \psi_t * \mathscr{L}(\varphi,\varphi) \mid \psi_j \rangle.$$

Then if $i \neq j$, we have $\ell_{\varphi,\psi_i} \perp \ell_{\varphi,\psi_j}$. Thus, in any case, we have that each ℓ_{φ,ψ_i} lies in a distinct irreducible of $R_1(Q)$ and that ℓ_{φ,ψ_i} span $\mathbb{F}$. Thus we have deduced the well known fact that $X_Q = M_{4t}(SL_2(\mathbb{Z}))$ (see [E]).

But using Proposition 1.1 and the results of [R-1], it is now possible to make a comparison of traces Theorem about Hecke operators on S_{4t} and $\mathbb{F} \cap R_1(Q)$. We recall the relevant results of [R-1]. If v is a finite prime, we let $\mathscr{H}_1 = \mathscr{H}(\mathrm{O}(Q_v)//L_v)$ = the algebra of L_v biinvariant, compactly-supported functions on $\mathrm{O}(Q_v)$ and $\mathscr{H}_2 = \mathscr{H}(\mathrm{Sp}_1(K_v)//\mathrm{Sp}_1(\mathscr{O}_v))$ = the algebra of $\mathrm{Sp}_1(\mathscr{O}_v)$ biinvariant, compactly-supported functions on $\mathrm{Sp}_1(K_v)$. Then there exists a surjective homomorphism $\omega_{Q_v}: \mathscr{H}_1 \to \mathscr{H}_2$ such that the ideal

$$I_v(Q_v) = \left\{ \psi \in \mathscr{H}_1 \otimes \mathscr{H}_2 | \pi_{Q_v}(\psi) = 0 \right\}$$

is generated by $\{ f \otimes \Pi - \Pi I \otimes \omega(f) | f \in \mathscr{H}_1 \}$. In simple terms this means that for each $f \in \mathscr{H}_1$, $\pi_{Q_v}(f)(\varphi) = \pi_{Q_v}(\omega_{Q_v}(f))(\varphi)$ for all $\varphi \in S[M_{m1}(\mathbb{Q}_v)]$. Then we have the following Theorem.

THEOREM 2.1: *Let $f \in \mathscr{H}_1$. Let ρ_* denote the corresponding representation of* $\mathrm{O}(Q)$ *or* Sp *on the corresponding space of cusp forms. Then*

$$\mathrm{Trace}\Big(\rho_{\mathrm{O}(Q)}(f)|_{\mathbb{F} \cap R_1(Q)} \Big) = \mathrm{Trace}\Big(\rho_{\mathrm{Sp}_1}\big(\omega_{Q_v}(f)^{\vee} \big)|_{S_{4t}} \Big)$$

where $^{\vee}$ denotes the operation $\varphi^{\vee}(x) = \varphi(x^{-1})$ on $\mathscr{H}_2$.

REMARK 2.2: Thus we have essentially a simple example of an abstract comparison of traces formula between the groups Sp_1 and $\mathrm{O}(Q)$ for Hecke operators acting on the appropriate spaces of cusp forms. (See [La] for the general philosophy of comparison of traces relative to the base change operation.)

PROOF OF THEOREM 2.1: Here we embed the Hecke operators $f \in \mathscr{H}_1$ into $C_c^{\infty}(\mathrm{O}(Q)(\mathbb{A}))$ in the standard fashion. Namely if we let $f = f_v$, the characteristic function of a double coset $L_v \xi_v L_v$, then we let the corresponding element in $C_c^{\infty}(\mathrm{O}(Q)(\mathbb{A}))$ be

$$\frac{1}{\mathrm{vol}(L_v \xi_v L_v)} f_v \otimes \prod_{w \neq v} \chi_w,$$

where χ_w = characteristic function of the maximal compact subgroup L_w in $\mathrm{O}(Q_w)$.

Similarly we embed $\omega_{Q_v}(f) \in \mathscr{H}_2$ into $C_c^{\infty}(\mathrm{Sp}_1(\mathbb{A}))$ in a similar fashion, but with one slight difference. That is, let g_v be the characteristic function of a double coset $\mathrm{Sp}_1(\mathscr{O}_v)\mu_v\mathrm{Sp}_1(\mathscr{O}_v)$. Then we let the corresponding element in $C_c^\infty(\mathrm{Sp}_1(\mathbb{A}))$ be

$$\frac{1}{\mathrm{vol}(\mathrm{Sp}_1(\mathcal{O}_v)\mu_v\mathrm{Sp}_1(\mathcal{O}_v))} g_v \otimes \prod_{w \neq v,\infty} \chi_w^* \otimes f_\infty^*$$

where χ_w^* = characteristic function of the maximal compact subgroup $\mathrm{Sp}_1(\mathcal{O}_v)$ and f_∞^* is a C^∞ function of compact support such that $\pi_{D_j^-}(f_\infty^*) = 0$ (for all j), $\pi_{D_j^+}(f_\infty^*) = 0$ (for $j \neq 2t$), and $\pi_{D_{2t}^+}(f_\infty^*)$ = identity operator on the lowest weight space for SO(2) and trivial on the remaining weight spaces (such a function f_∞^* is well known to exist by a result of Duflo and Langlands).

From the comments above, we have that

$$\mathrm{Tr}\Big(\rho_{\mathrm{O}(Q)}(f)|_{\mathbb{F} \cap R_1(Q)}\Big) = \sum_i \frac{\langle \ell_{\varphi,\psi_i} * f | \ell_{\varphi,\psi_i}\rangle}{\langle \ell_{\varphi,\psi_i} | \ell_{\varphi,\psi_i}\rangle}.$$

But we have that $\ell_{\varphi,\psi_i} * f = \ell_{\pi_Q(f)(\varphi),\psi_i} = \ell_{\pi_Q(\omega_{Q_v}(f))\varphi,\psi_i} = \ell_{\varphi,\psi_i * \omega_{Q_v}(f)^\vee}$. On the other hand,

$$\langle \psi * \mathscr{L}(\varphi, \varphi) | \psi \rangle = \langle \ell_{\varphi,\psi} | \ell_{\varphi,\psi} \rangle$$

for all ψ. With these facts we have

$$\frac{\langle \ell_{\varphi,\psi_i} * f | \ell_{\varphi,\psi_i}\rangle}{\langle \ell_{\varphi,\psi_i} | \ell_{\varphi,\psi_i}\rangle} = \frac{\langle \psi_i * \omega_{Q_v}(f)^\vee * \mathscr{L}(\varphi,\varphi) | \psi_i \rangle}{\langle \psi_i * \mathscr{L}(\varphi,\varphi) | \psi_i\rangle}$$

$$= \frac{\langle \psi_i * \omega_{Q_v}(f)^\vee | \psi_i \rangle}{\langle \psi_i | \psi_i \rangle}.$$

(We note here that ψ_i is an eigenfunction of $\mathscr{L}(\varphi, \varphi)$.)

Q.E.D.

REMARK 2.3: We note that the operator $\mathscr{L}(\varphi, \varphi)$ operating on the eigenvector ψ_i will have an eigenvalue (aside from a positive constant independent of ψ_i) of the form

$$\frac{2}{|m|} \frac{L((m/2)-1, \Pi_i, r_1)}{\zeta((m/2)-1)\zeta(m/2)} \Big(\prod_{q<\infty} \big(1 + q^{-(m/2-1)}\big) \Big)$$

where $2/|m| = 1/(d(m))$, $d(m)$ = the formal degree of the discrete series representation $D_{m/2}^-$ and $L(s, \Pi_i, r_1)$ is the Langlands L function given by

$$\prod_{p<\infty} L_p(s, (\Pi_i)_p, r_1)$$

(see § 4 for the definition of $L_p(\ldots)$). Here Π_t is the irreducible cuspidal representation of $\mathrm{Sp}_1(\mathbb{A})$ which contains the vector ψ_t.

§ 3. Convergence of integrals in (1–2)

We show here the absolute convergence of the integrals appearing in (1–2). Indeed, we first note that $\mathrm{Sp}_r(\mathbb{A}) = \underset{\sim}{P}(\mathbb{A}) \cdot K$ where $\underset{\sim}{P}$ is any parabolic of the following form: $\underset{\sim}{P}$ has a Levi factor of the type $\mathrm{Sp}_j \times G\ell_{r-j}$. Then the integrals in (1–2) are of 2 types:

$$\int_{\Delta_0(K)\backslash \mathrm{Sp}_k \times \mathrm{Sp}_k(\mathbb{A})} \psi_1(G)\overline{\psi_2(G')}\pi_Q(\Omega_0(G, G'))(\varphi \otimes \bar{\varphi}')(\mathbf{0})\mathrm{d}G\mathrm{d}G'. \tag{1}$$

We note that this integral can be expressed as a product

$$\int_{\mathrm{Sp}_k(\mathbb{A})}\left(\int_{\Delta_0(K)\backslash \Delta_0(\mathbb{A})} \psi_1(Gh)\overline{\psi_2(G)}\mathrm{d}G\right)\pi_Q(\Omega_0(h, 1))(\varphi \otimes \bar{\varphi}')(\mathbf{0})\mathrm{d}h.$$

However, by taking absolute values it suffices to show that

$$\int_{\mathrm{Sp}_k(\mathbb{A})}\left(\int_{\Delta_0(K)\backslash \Delta_0(\mathbb{A})} |\psi_1(Gh)\overline{\psi_2(G)}|\mathrm{d}G\right)|\pi_Q(\Omega_0(h, 1))(\varphi \otimes \bar{\varphi}')(\mathbf{0})|\mathrm{d}h$$

$$\leqslant \|\psi_1\|^2\|\psi_2\|^2$$

$$\int_{\mathrm{Sp}_k(\mathbb{A})} |\pi_Q(\Omega_0(h, 1))(\varphi \otimes \bar{\varphi}')(\mathbf{0})|\mathrm{d}h < \infty.$$

This is equivalent to showing $h \rightsquigarrow \pi_Q(\Omega_0(h, 1))(\varphi \otimes \bar{\varphi}')(\mathbf{0})$ is a L^1 function on $\mathrm{Sp}_k(\mathbb{A})$ (which we show in Proposition 1.1).

$$\int_{\mathrm{Sp}_k \times \mathrm{Sp}_{k'}(\Omega_t)\backslash \mathrm{Sp}_k \times \mathrm{Sp}_{k'}(\mathbb{A})} \cdots$$

$$= \int_{(\mathrm{Sp}_k \times \mathrm{Sp}_{k'}(\Omega_t)\backslash \underset{\sim}{P} \times \underset{\sim}{P}(\mathbb{A})) \times K \times K'} \psi_1(\alpha \cdot k)\overline{\psi_2(\alpha' \cdot k')}$$

$$\pi_Q(\Omega_i(\alpha k, \alpha' k'))(\varphi \otimes \bar{\varphi}')(\mathbf{0})\mathrm{d}\alpha\mathrm{d}\alpha'\mathrm{d}k\mathrm{d}k'. \tag{2}$$

Thus assuming ψ_1 and ψ_2 and $\varphi \otimes \bar{\varphi}'$ are K finite functions, it suffices to prove that

$$\int_{\mathrm{Sp}_k \times \mathrm{Sp}_{k'}(\Omega_t)\backslash \underset{\sim}{P} \times \underset{\sim}{P}'(\mathbb{A})} |\psi_1(\alpha)\psi_2(\alpha')$$

$$\pi_Q(\Omega_t(\alpha, \alpha'))(\varphi \otimes \bar{\varphi}')(\mathbf{0})|\mathrm{d}\alpha\mathrm{d}\alpha' < \infty.$$

Here, $\underset{\sim}{P}$ ($\underset{\sim}{P}'$ respectively) is the parabolic subgroup having $\mathrm{Sp}_t \times G\ell_{k-t}$ ($\mathrm{Sp}_t \times G\ell_{k'-t}$ resp.) as its Levi factor. But then we have that the integral above can be decomposed as follows:

$$\int_{\mathrm{Sp}_r(\mathbb{A})} |\pi_Q(\Omega_r(h, 1))(\varphi \otimes \bar{\varphi}')(\mathbf{0})|$$

$$\left\{\int_{(U_1 \times U_2)(K)\backslash U_1 \times U_2(\mathbb{A})} \left\{\int_{G_1 \times G_2(K)\backslash G_1 \times G_2(\mathbb{A})} \left\{\int_{\mathrm{Sp}_r(K)\backslash \mathrm{Sp}_r(\mathbb{A})}\right.\right.\right.$$

$$\left.\left.\left.|\psi_1(n(x)g_1g_2h)|\,|\psi_2(n(y)g_1'g_2)|\mathrm{d}g_2\right\}\mathrm{d}g_1\mathrm{d}g_1'\right\}\mathrm{d}x\mathrm{d}y\right\}\mathrm{d}h$$

where $G_1 = G\ell_{k-r}$, $G_2 = G\ell_{k'-r}$, U_1 and U_2 are unipotent radicals in $\underset{\sim}{P}$ and $\underset{\sim}{P}'$. Then by using the same idea as in (1) above, we have that the inner integral is majorized by

$$\int_{\mathrm{Sp}_r(K)\backslash \mathrm{Sp}_r(\mathbb{A})} |\psi_1(n(x)g_1g_2)|^2\mathrm{d}g_2 \cdot \int_{\mathrm{Sp}_r(K)\backslash \mathrm{Sp}_r(\mathbb{A})} |\psi_2(n(y)g_1'g_2)|^2\mathrm{d}g_2.$$

Hence, it suffices to show that

$$\int_{U_1(K)\backslash U_1(\mathbb{A})} \int_{G_1(K)\backslash G_1(\mathbb{A})} |\det g_1|^{m/2}$$

$$\left\{\int_{\mathrm{Sp}_r(K)\backslash \mathrm{Sp}_r(\mathbb{A})} |\psi_1(n(x)g_1g_2)|^2\mathrm{d}g_2\right\}\mathrm{d}g_1\mathrm{d}x < \infty$$

and that the function $h \to \pi_Q(\Omega_r(h, 1))(\varphi \otimes \bar{\varphi}')(\mathbf{0})$ is a L^1 function on $\mathrm{Sp}_r(\mathbb{A})$.

For the first type of integral, we consider integrating $|\psi_1(n(x)g_1g_2)|^2$ over the set $\Sigma_1 \times D_{t_1} \times S_{t_2}$ in $(U_1 \times G_1 \times \mathrm{Sp}_r)(\mathbb{A})$ where Σ_1 is any compact subset of U_1, D_{t_1} a fundamental domain in $G_1(K)\backslash G_1(\mathbb{A})$ of the form $\Lambda \cdot \{\mathrm{diag}(x_{r+1},\dots,x_k) |\, x_i \geq t_1 x_{i+1}\}$ (Λ a compact set) and S_{t_2} a fundamental domain in $\mathrm{Sp}_r(K)\backslash \mathrm{Sp}_r(\mathbb{A})$ of the form $\Lambda' \cdot \{\mathrm{diag}(x_1,\dots,x_r) |\, x_i \geq t_2 x_{i+1}, x_r \geq t_2\}$ (Λ' a compact set). We note that for the construction of fundamental domains it suffices to let $t_1 = t_2 = t$. Then it suffices to show that

$$\int |\psi_1(D(x_1,\dots,x_r)D(x_{r+1},\dots,x_k))|^2$$

$$\left(\prod_{i=1}^{i=r} x_i^{2i+1-r+m/2}\right)\left(\prod_{i=r+1}^{i=k} x_i^{-2k+2i-2+m/2}\right)\prod_{i=1}^{i=n}\frac{\mathrm{d}x_i}{x_i}<\infty.$$

where the range of integration is

$$\{(x_1,\dots,x_k)\in(\mathbb{R}_+^\times)^k \,|\, x_1$$
$$\geqslant tx_2,\dots,x_{r-1}\geqslant tx_r,\, x_{r+1}\geqslant tx_{r+2},\dots,x_{k-1}\geqslant tx_k,\, x_k\geqslant t\}$$

We then show that such an integral can be decomposed in integration over certain sets. We note that the fundamental domain in the integration above can be decomposed into a union of the following sets:

$$\bigcup_w w\{(x_1,\dots,x_k)\,|\,x_1\geqslant tx_2,\dots,x_{k-1}\geqslant tx_k,\, x_k\geqslant t\}$$

where w ranges over the cosets in Weyl group (Sp_k) mod(Weyl group($G\ell_r$) $\times$ Weyl group(Sp_{k-r})). Then we use the rapid decrease properties of ψ_1 (a cusp form) on the various translates of the Siegel domain for Sp_k to deduce the finiteness of the above integrals.

On the other hand, we have by a similar computation to that given in Proposition 1.1 that

$$\pi_Q(\Omega_r(h,1))(\varphi\otimes\bar{\varphi}')(\mathbf{0})$$
$$=\int_{M_{mr}(\mathbb{A})\times M_{mr}(\mathbb{A})}\tau(\mathrm{Tr}(\delta_1^t Q\delta_2))\,\pi_Q(h)(\varphi)(\delta_1)\bar{\varphi}'(\delta_2)\,d(\delta_1)\,d(\delta_2).$$

Then using the same argument as in Proposition 1.1, we see that the above function is L^1 on $\mathrm{Sp}_r(\mathbb{A})$.

§ 4. Computation of local factors

(1) We first assume that v is *finite* in K. Then we compute the series:

$$Z(\omega_s,\lambda)=\sum_{n=0}^{n=+\infty}\omega_s\begin{pmatrix}\pi^n & 0\\ 0 & \pi^{-n}\end{pmatrix}\mathrm{vol}\left[K\cdot\begin{pmatrix}\pi^n & 0\\ 0 & \pi^{-n}\end{pmatrix}\cdot K\right]q^{-n\lambda}$$

where the Haar measure on $\mathrm{Sp}_1(K_v)$ is so normalized that $\mathrm{vol}(K)=1$.

Then we know that

$$\operatorname{vol}\left[K\begin{pmatrix}\pi^n & 0\\ 0 & \pi^{-n}\end{pmatrix}K\right]=((1+q)/q)q^{2n}$$

for $n>1$ and that

$$\omega_s\begin{pmatrix}\pi^n & 0\\ 0 & \pi^{-n}\end{pmatrix}=\left(\frac{q}{q+1}\right)\left(\frac{q^{-n}}{1-q^{-s}}\right)$$
$$\times\left[q^{-sn}(q^{-1}-q^{-s})+q^{sn}(1-q^{1-s})\right].$$

Thus $Z(\omega_s,\lambda)$ becomes the series

$$1+\left(\frac{1}{1-q^{-s}}\right)$$
$$\times\sum_{n=1}^{n=+\infty}\left\{q^{(-s-\lambda+1)n}(q^{-1}-q^{-s})+q^{(s-\lambda+1)n}(1-q^{-1-s}\right\}.$$

Then using geometric series expansion we have that $Z(\omega_s,\lambda)$ equals

$$1+\left(\frac{1}{1-q^{-s}}\right)\left\{\frac{q^{(-s-\lambda+1)}}{1-q^{(-s-\lambda+1)}}(q^{-1}-q^{-s})\right.$$
$$\left.+\left(\frac{q^{(s-\lambda+1)}}{1-q^{s-\lambda+1}}\right)(1-q^{-1-s})\right\}.$$

But then rationalizing with common denominators, we have that $Z(\omega_s,\lambda)$ equals

$$\frac{(1+q^{-(\lambda-1)})(1-q^{-\lambda})}{(1-q^{-s-(\lambda-1)})(1-q^{s-(\lambda-1)})}.$$

REMARK 4.1: We note that $Z(\omega_s,\lambda)$ is related to the local Langlands L-function for the representation π_s. We know that the L-group of Sp_1 is $\mathrm{SO}(3,C)$ and that the L-factor associated to π_s and the standard 3-dimensional representation r_1 of $\mathrm{SO}(3,C)$ is given by (see [B])

$$L_p(w,\pi_s,r_1)=\frac{1}{(1-q^sq^{-w})(1-q^{-s}q^{-w})(1-q^{-w})}$$

Then we have that

$$Z(\omega_s,\lambda)=L_p(\lambda-1,\pi_s,r_1)\frac{1}{\zeta_p(\lambda-1)\zeta_p(\lambda)}\,\frac{1}{\zeta_p\left(\lambda-1+\frac{\pi\sqrt{-1}}{\log q}\right)}$$

where ζ_p is the local zeta factor function $\zeta_p(t) = 1/(1-q^{-t})$. Thus $Z(\omega_s, \lambda)$ is the special value (at $w = \lambda - 1$) of the Langlands L-function and local GL_1 zeta factors.

(2) Let v be a real infinite prime. We then consider a matrix coefficient of the Weil representation π_{Q_v} of the form

$$f(g) = \langle \pi_{Q_v}(g)\varphi_1 | \varphi_2 \rangle$$

where φ_1 and φ_2 belong to $S[M_{m1}(\mathbb{R})]$ and the SO(2) type of φ_1 and φ_2 is of type m, i.e.,

$$\pi_{Q_v}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}(\varphi_i) = e^{\sqrt{-1}\,m\theta}\varphi_i.$$

Then we note that $f(k(\theta)gk(\theta')) = e^{\sqrt{-1}\,m(\theta-\theta')}f(g)$ for all $g \in SL_2(\mathbb{R})$. Then we have that

$$\operatorname{trace}_{H_s}(\pi_s(f)) = \int_0^{+\infty}\left(\int_{-\infty}^{+\infty} f\left(\begin{bmatrix} r & 0 \\ 0 & r^{-1}\end{bmatrix}\begin{bmatrix} 1 & x \\ 0 & 1\end{bmatrix}\right)\mathrm{d}x\right)|r|^s \mathrm{d}r$$

(recall that f is a L^1-function on $\mathrm{Sp}_1(\mathbb{R})$) for $|\mathrm{Re}(s)| < 1$. We choose an orthogonal splitting of $Q = Q_+ \oplus Q_-$ so that $Q_+(Q_-$ resp.) is positive (negative) definite ($a = \dim Q_+$, $b = \dim Q_-$). Then, we choose $\varphi_1(X) = P_+(X)P_-(X)\,e^{-\pi[X]}$ where P_+ is a harmonic polynomial in the + part of Q of degree r_1 and P_- is a harmonic polynomial in the − part of Q of degree r_2 (if $b = 1$ then P_- is the r_2-th Hermite polynomial). Then we have that

$$f\left(\begin{bmatrix} r & 0 \\ 0 & r^{-1}\end{bmatrix}\begin{bmatrix} 1 & x \\ 0 & 1\end{bmatrix}\right)$$

$$= r^{(r_1+r_2)+m/2}\int_{\mathbb{R}^m} e^{2\pi\sqrt{-1}\,xr^2Q(X)}\,e^{-\pi[r^2+1][X]}|P_+(X)|^2|P_-(X)|^2\mathrm{d}X.$$

Then we separate $\mathbb{R}^m = \mathbb{R}^a \oplus \mathbb{R}^b$ and use polar coordinates on $\mathbb{R}^a$ and $\mathbb{R}^b$ to get that f above equals

$$\begin{aligned} & \|P_+\|^2_{L^2(S^{a-1})}\|P_-\|^2_{L^2(S^{b-1})}\Gamma\left(r_1 + \frac{a}{2} + 1\right)\Gamma\left(r_2 + \frac{b}{2} + 1\right) \\ (\text{if } b > 1) \quad & \times r^{r_1+r_2+m/2}\left[\pi(r^2+1) - 2\pi\sqrt{-1}\,xr^2\right]^{-(r_1+a/2)} \\ & \times\left[\pi(r^2+1) + 2\pi\sqrt{-1}\,xr^2\right]^{-(r_2+b/2)}, \\ \text{or} \quad & \|P_+\|^2_{L^2(S^{a-1})}\Gamma\left(r_1 + \frac{a}{2} + 1\right)r^{r_1+m/2} \end{aligned}$$

$$(\text{if } b=1) \quad \times\left[\pi(r^2+1)-2\pi\sqrt{-1}\,xr^2\right]^{-(r_1+a/2)}$$
$$\left\{\sum_{\ell=0}^{\ell=r_2} 2^{\ell}\ell!\binom{r_2}{\ell}^2\frac{(2r_2-2\ell)!}{(r_2-\ell)!}\right.$$
$$\left.\times\sum_{i=0}^{i=r_2-\ell}(-1)^i\left[\pi(r^2+1)+2\pi\sqrt{-1}\,xr^2\right]^{i-1/2-(r_2+\ell)}\right\}.$$

Then in both cases we integrate the resulting functions in the x variable over $(-\infty, +\infty)$ and get that modulo the *nonzero constants* not involving r

$$r^{(r_1+r_2+m/2-2)}(r^2+1)^{1-(r_1+r_2)-m/2}$$
$$(\text{if } b>1) \quad \times\int_{-\pi/2}^{+\pi/2} e^{\sqrt{-1}\,\theta[(r_1-r_2)+((a-b)/2)]}$$
$$\times(\cos\theta)^{r_1+r_2+(a+b)/2-2}\,d\theta.$$

Then we observe that this latter integral will *not vanish* provided suitable choice of r_1 and r_2 is made. Indeed, if we choose r_1 and r_2 so that $r_1-r_2+(a-b)/2=0$ or 1 then clearly the above integral is *nonvanishing*. We note here if $r_1-r_2+(a-b)/2=0$ ($=1$ resp.) then SO(2) type of f is 0 ($=1$ resp.). Then when we compute the above expression against $r^s dr$ we get an integral of the form

$$\int_0^{+\infty}(1+r^2)^{1-(r_1+r_2)-m/2}r^{r_1+r_2+m/2+s-2}\,dr.$$

This integral by a simple substitution $r^2=v$ becomes a beta integral of the form $B(\frac{1}{2}(r_1+r_2)+(m/4)+(s/2)-\frac{1}{2},\ \frac{1}{2}(r_1+r_2)+(m/4)-(s/2)-\frac{1}{2})$. But now if we assume that $\frac{1}{2}(r_1+r_2)+m/4$ is an *integer* (equivalent to $b\left\{{\text{even} \atop \text{odd}}\right\}$ iff $r_1-r_2+(a-b)/2=\left\{{0 \atop 1}\right\}$ resp.) then $B(\frac{1}{2}(r_1+r_2)+(m/4)+(s/2)-\frac{1}{2},\ \frac{1}{2}(r_1+r_2)+(m/4)-(s/2)-\frac{1}{2})$ is equal to

$$\pi\cdot\sec\left(\pi\frac{s}{2}\right)\left\{\prod_{v=1}^{v=\frac{1}{2}(r_1+r_2)+\frac{m}{4}-1}\left[\left(v-\tfrac{1}{2}\right)^2-\left(\frac{s}{2}\right)^2\right]\right\}$$
$$\times\frac{1}{\Gamma\left(r_1+r_2+\frac{m}{2}-1\right)}. \tag{4-1}$$

Similarly if $\frac{1}{2}(r_1+r_2)+m/4$ is half integral (equivalent to $b\left\{{\text{odd} \atop \text{even}}\right\}$ iff $r_1-r_2+(a-b)/2=\left\{{0 \atop 1}\right\}$ resp.) we have that $B(\frac{1}{2}(r_1+r_2)+(m/4)+$

$(s/2)-\frac{1}{2},\ \frac{1}{2}(r_1+r_2)+(m/4)-(s/2)-\frac{1}{2})$ is equal to

$$\pi\left(\frac{s}{2}\right)\operatorname{cosec}\left(\pi\left(\frac{s}{2}\right)\right)\left\{\prod_{v=1}^{v=\frac{1}{2}(r_1+r_2)+\frac{m}{4}-3/2}\left(v^2-\left(\frac{s}{2}\right)^2\right)\right\}$$

$$\times\frac{1}{\Gamma\left(r_1+r_2+\frac{m}{2}-1\right)}. \tag{4-2}$$

Now we consider the case $b=1$ separately. For this we may assume that $r_2=0$. Then the above computations yield after integrating in the x variable over $(-\infty,\ +\infty)$

$$(\mathit{if}\ b=1)\qquad \begin{aligned} &r^{r_1+m/2-2}\left(r^2+1\right)^{1-(r_1+m/2)}\\ &\times\int_{-\pi/2}^{+\pi/2}\mathrm{e}^{\sqrt{-1}\,\theta[r_1+(a-1)/2]}(\cos\theta)^{r_1+m/2-2}\mathrm{d}\theta.\end{aligned}$$

Then following the same reasoning as in the case above $(b>1)$ we deduce that after integrating against $r^s\mathrm{d}r$ we get expressions for the form (4-1) with $r_1+(a-1)/2$ *odd* and of the form (4-2) with $r_1+(a-1)/2$ *even* (in these expressions we note that $r_2=0$ and $m=a+1$). We note here that we have modified the SO(2) type to be more general (i.e., either *odd* or *even* integral). The key observation as in the case above is that an integral of the form

$$\int_{-\pi/2}^{+\pi/2}[\cos(\theta)]^{\mu-1}\,\mathrm{e}^{\sqrt{-1}\,\mu\theta}\mathrm{d}\theta$$

(for μ a *nonnegative* integer) is *nonvanishing*. Indeed, we can write this integral as

$$\sum_{j=0}^{j=\mu-1}\binom{\mu-1}{j}\left(\int_{-\pi/2}^{+\pi/2}\mathrm{e}^{\sqrt{-1}\,\theta(2j+1)}\mathrm{d}\theta\right).$$

Then the inner integral is up to a multiple of $\sqrt{-1}$

$$\frac{1}{2j+1}(-1)^j.$$

Then we know that

$$\sum_{j=0}^{j=\mu-1}\binom{\mu-1}{j}\frac{(-1)^j}{2j+1}=\frac{1}{2}\,\frac{\Gamma(1/2)\Gamma(\mu)}{\Gamma\left(\mu+\frac{1}{2}\right)}=\frac{(\mu-1)!2^{2\mu-1}\mu!}{(2\mu)!}.$$

References

[*A*] ARTHUR, J.: Eisenstein series and the trace formula. *Proc. of Symposia in Pure Math. 33* (1979) 253–274.

[*B*] BOREL, A.: Automorphic L-functions. *Proc. of Symposia in Pure Math. 33* (1979) 27–63 (part II).

[*B-W*] BOREL and WALLACH, N.: Continuous cohomology, discrete groups, and representations of reductive groups. *Annals of Math Studies 94* (1980).

[*E*] EICHLER, M.: Quadratische Formen und orthogonale gruppen. *Grundlehren der Mathematischen Wissenschaften* 63 (1974).

[*H-M*] HOWE, R. and MOORE, C.: Asymptotic properties of unitary representations. *J. Func. Anal. 32* (1979) 72–96.

[*J-L*] JACQUET, H. and LANGLANDS, R.: Automorphic forms on GL(2). *Lecture Notes in Math. 114*, Springer (1970).

[*K-Z*] KOHNEN, W. and ZAGIER, D.: Values of L-series of modular forms in the middle of the critical strip. (1980) Preprint.

[*L*] LANG, S.: $SL_2(\mathbb{R})$. Addison-Wesley (1974).

[*La*] LANGLANDS, R.: Base change for $GL(2)$. *Annals of Math Studies 96* (1980).

[*M-R*] MILLSON, J. and RAGHUNATHAN, M.S.: Geometric construction of homology for arithmetic groups. Preprint.

[*P*] PIATETSKI-SHAPIRO: On the Saito-Kurokawa lifting. Preprint.

[*R*-1] RALLIS, S.: Langlands' functoriality and the Weil representation. *Amer. Jour. of Math. 104 (3)* (1982) 469–515.

[*R*-2] RALLIS, S.: On the Howe duality conjecture. *Comp. Math. 51 (3)* (1984) 333–399.

[*R*-3] RALLIS, S.: The Eichler commutation relation and the continuous spectrum of the Weil representation. Proc of Conference on Non Commutative Harmonic Analysis. *Lecture Notes in Math. 728* (1979) 211–244.

[*R-S*] RALLIS, S. and SCHIFFMANN, G.: Weil representation. I. Interwining distributions and discrete spectrum. *Memoirs of Amer. Math. Soc. 231* (1980).

[*W*] WALDSPURGER, J.L.: Correspondance de Shimura. *J. Math. pures et appl.* 59 (1980) 1–133.

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