BASE CURVES OF MULTICANONICAL SYSTEMS ON THREEFOLDS

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Introduction

Let $V$ be a smooth complex projective threefold of general type and $K_V$ the canonical divisor on $V$. In this paper we consider in detail the case when there exists an integer $m > 0$ such that the $m$-canonical system $|mK_V|$ has no fixed components, and the corresponding rational map $\phi_{mK_V}$ is generically finite; in particular we study the base locus of this linear system.

Using the theory of Hilbert schemes, it is easy to see that in this case $K_V$ is arithmetically effective (denoted a.e.); i.e. $K_V \cdot C \geq 0$ for all curves $C$ on $V$. In [12] (Theorem 6.2) we saw that in the case when $K_V$ is a.e., the canonical ring $R(V) = \oplus_{n \geq 0} H^0(V, nK_V)$ is finitely generated as an algebra over the complex numbers if and only if the linear system $|nK_V|$ has no fixed points for some $n > 0$.

It is easy to see that if $K_V \cdot C > 0$ for every base curve $C$ of $|mK_V|$, then such an $n$ does exist (1.2). Thus the interesting base curves $C$ are those with $K_V \cdot C = 0$. After this paper was written, the author was informed that Kawamata has now proved that such an $n$ exists in the case when $K_V$ is a.e. Thus we see that the curves $C$ with $K_V \cdot C = 0$ are precisely those curves that are contracted down on the canonical model (see [9]). We shall however not assume Kawamata’s result in the proofs of this paper.

In Section 2 we therefore study the base curves $C$ of $|mK_V|$ with $K_V \cdot C = 0$. We shall see that in this case $C$ must be isomorphic to $\mathbb{P}^1$ (2.3) and that (2.6) its normal bundle $N_{C/V}$ must be one of $\mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$, $\mathcal{O}_C(-2) \oplus \mathcal{O}_C$ or $\mathcal{O}_C(-3) \oplus \mathcal{O}_C(1)$.

The case of base curves $C$ with $K_V \cdot C = 0$ is closely connected with the problem of curves homologous to zero on an analytic threefold. Analogous results to (2.3) and (2.6) have been obtained by Pinkham in this case, using a different method [7].

In the cases $N_{C/V} = \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$ or $\mathcal{O}_C(-2) \oplus \mathcal{O}_C$, above, we essentially have all the information we want about $C$. In the remaining case, we need to know more about the infinitesimal neighbourhoods of $C$. 
Blowing $C$ up, say $f_1: V_1 \to V$ with exceptional surface $E_1$ (isomorphic to the ruled surface $\mathbb{F}_4$), we let $C_1$ denote the minimal section of $E_1$ (hence on $E_1$ we have $C_1^2 = -4$). Now blow $C_1$ up, obtaining $f_2, V_2, E_2$ and $C_2$. In this way we obtain a sequence of normal bundles $N_{C/V}, N_{C_1/V_1}, N_{C_2/V_2}, \ldots$. We discover that the sequence of normal bundles so obtained must be $(-3, 1), (-3, 0), \ldots, (-3, 0), (-2, -1), (-1, -1)$ (with the obvious notation and where there are a finite number, say $t \geq 0$, of $(-3, 0)$'s in the sequence).

As an illustration, we apply these results (in Section 3) to the case of a base curve $C$ with $K_V \cdot C = 0$ that is isolated in the base locus of $|mK_V|$. We discover here that $C$ is not then a base curve of $|(2m + 1)K_V|$. As a corollary, we can deduce for instance that the canonical ring is finitely generated in the case when for any two base curves $C, C'$ with $K_V \cdot C = 0 = K_V \cdot C'$, $C$ and $C'$ do not meet.

Let $Z_0$ denote the cycle of base curves $C$ of $|mK_V|$ with $K_V \cdot C = 0$. In Section 2 we obtained detailed information concerning the individual curves of $Z_0$; we now consider the question of the possible configurations of curves in $Z_0$.

This we consider in Section 4. We show that any two curves of $Z_0$ meet in at most one point (where they meet transversely), and that any given point of $V$ is contained in at most three curves of $Z_0$ (which meet normally there). Moreover we find that there are no “closed cycles” of curves in $Z_0$ (apart from three curves meeting at a point).

We then take into account the possible normal bundles (found in Section 2). We show that a curve $C$ in $Z_0$ with normal bundle $(-3, 1)$ cannot meet another such curve, and also cannot meet a triple point. Similarly, we find that a curve $C$ in $Z_0$ with normal bundle $(-2, 0)$ meets at most one $(-3, 1)$-curve, meets at most one triple point, and cannot meet both. Finally we consider the case of a curve in $Z_0$ with normal bundle $(-1, -1)$.

The results and methods of this paper are also closely connected with a conjecture of Reid. He conjectures that if we have a finite collection of curves $Z_0 = \cup C_i$ on a smooth projective threefold $V$ with $K_V \cdot C_i = 0$ for all $i$ and $H^1(\mathcal{O}_Z) = 0$ for all schemes $Z$ supported on $Z_0$, and such that $Z_0$ is isolated among cycles of this type, then $Z_0$ is contractible (by a morphism not contracting anything else). An easy modification of the methods of Section 2 shows that in this case also, any $C_i$ (trivially isomorphic to $\mathbb{P}^1$ here) has one of the above three normal bundles, and that on blowing up, the sequence of normal bundles obtained is as described above. The methods of sections 3 and 4 are therefore relevant in this case also.

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1. Preliminaries

Throughout this paper $V$ will denote a smooth complex projective threefold of general type, and $m$ a positive integer such that $|mK_V|$ has no fixed components and the rational map $\phi_{mK_V}$ is generically finite.

**Proposition 1.1:** $K_V$ is arithmetically effective.

**Proof:** If there exists a curve $C$ on $V$ with $K_V \cdot C < 0$, then by the theory of Hilbert schemes (see for instance [5] Section 1), we deduce that $C$ moves in an algebraic family. This then would yield a fixed component of $|mK_V|$ (one can of course say far more; see [6]). \( \square \)

**Proposition 1.2:** If $K_V \cdot C > 0$ for all base curves $C$ of $|mK_V|$, then for some $n$ the linear system $|nK_V|$ is without fixed points. If $C$ is a base curve with $K_V \cdot C = 0$, then $C$ has arithmetic genus $p_a(C) \leq 1$.

**Proof:** Let $D, D' \in |mK_V|$ be general elements, and $Z = D'|D$ denote the scheme theoretic intersection. For $n \geq 1$, we have the exact sequence of sheaves

$$0 \to \mathcal{O}_D(nD) \to \mathcal{O}_D((n+1)D) \to \mathcal{O}_Z((n+1)D) \to 0.$$

From the Kawamata-Viehweg form of Kodaira vanishing ([4] or [11]), we see that $h^i(V, nD) = 0$ for $i > 0$ and $n > 0$. Thus from the exact sequence

$$0 \to \mathcal{O}_V(nD) \to \mathcal{O}_V((n+1)D) \to \mathcal{O}_D((n+1)D) \to 0$$

we deduce that $h^i(D, \mathcal{O}_D(nD)) = 0$ for $i > 0$ and $n > 2$. Thus from the former exact sequence, we see that $h^1(Z, \mathcal{O}_Z(nD)) = 0$ for $n > 3$.

Suppose first that $C$ is a base curve with $K_V \cdot C = 0$. We have an epimorphism $\mathcal{O}_Z(nD) \to \mathcal{O}_C(nD)$ of sheaves supported on $Z$, and hence we deduce that $h^1(C, \mathcal{O}_C(nD)) = 0$ for $n > 3$. Since $D \cdot C = 0$, we see that $\chi(C, \mathcal{O}_C) = \chi(C, \mathcal{O}_C(nD)) \geq 0$. Thus $p_a(C) \leq 1$ as required.

Suppose therefore that $K_V \cdot C > 0$ for all base curves $C$ of $|mK_V|$. We see then that for all the curves $C$ underlying $Z$, we have $D \cdot C > 0$. Hence $D$ is ample on $Z$ (Propositions 4.2 and 4.3 of [2]). Thus for $n$ sufficiently large, $\mathcal{O}_Z(nD)$ is generated by its global sections.

From the first exact sequence and the fact that $h^1(\mathcal{O}_D(nD)) = 0$ for large $n$, we deduce that $\mathcal{O}_D(nD)$ is also generated by its global sections for $n$ sufficiently large. Hence, using the second exact sequence, we see that the same is also true of the sheaf $\mathcal{O}_V(nD)$. Thus the linear system $|nD|$ is fixed point free for large $n$. \( \square \)

We see therefore that it is vital to consider further those base curves $C$ with $K_V \cdot C = 0$. This we do in detail in the next section.
2. Base curves with $K_V \cdot C = 0$

Suppose that $C$ is a base curve of $|mK_V|$ with $K_V \cdot C = 0$, and let $f_1: V_1 \to V$ denote the blow up of $C$. Note here that in the case when $C$ is singular, it has at worst a node or a cusp (by (1.2)), and thus $V_1$ has only one singular point (cf. [8]), which in the terminology of [9] is a compound Du Val point (in particular it is Gorenstein and rational). Let $E_1$ denote the exceptional divisor on $V_1$; note that $E_1$ is a Cartier divisor. Now on $V_1$ we have $K_{V_1} = f_1^*K_V + E_1$.

**Proposition 2.1:**

\[
\chi(V, (sm + 1)K_V) = \chi(V_1, K_{V_1} + s(mf_1^*K_V - E_1))
\]

\[
= \frac{1}{2} s(s - 1)(2s - 1) E_1^3 \text{ for any integer } s \geq 1.
\]

**Proof:** First suppose that $C$ is smooth. By Riemann-Roch,

\[
\chi(V, (sm + 1)K_V)
\]

\[
= \frac{1}{2} s(sm + 1)(2sm + 1) K^3 - (1 + 2sm) \chi(\mathcal{O}_V)
\]

\[
\chi(V_1, K_{V_1} + s(mf_1^*K_V - E_1))
\]

\[
= \frac{1}{2}(f_1^*(sm + 1)K_V - (s - 1)E_1)(smf_1^*K_V - sE_1)
\]

\[
\times (f_1^*(2sm + 1)K_V - (2s - 1)E_1)
\]

\[
- (1 + 2sm) \chi(\mathcal{O}_{V_1}) - \frac{1}{s} sE_1 \cdot c_2(V_1).
\]

Now note that $\chi(2K_V) = \chi(K_{V_1} + f_1^*K_V)$ (using the Kawamata-Viehweg form of Kodaira vanishing and the birational invariance of plurigenera). But by Riemann-Roch

\[
\chi(2K_V) = \frac{1}{2} K^3 + \frac{1}{12} K_V \cdot c_2(V) - \chi(\mathcal{O}_V) \text{ and}
\]

\[
\chi(K_{V_1} + f_1^*K_V) = \frac{1}{2} K^3 + \frac{1}{12} f_1^*K_V \cdot c_2(V_1) - \chi(\mathcal{O}_{V_1}).
\]

Thus $f_1^*K_V \cdot c_2(V_1) = K_V \cdot c_2(V)$. In particular

\[
-24\chi(\mathcal{O}_V) = K_V \cdot c_2(V) = (K_{V_1} - E_1) \cdot c_2(V_1)
\]

\[
= -24\chi(\mathcal{O}_{V_1}) - E_1 \cdot c_2(V_1).
\]
Since the arithmetic genus is birationally invariant, we deduce that \( E_1 \cdot c_2(V) = 0 \). (The fact that \( E_1 \cdot c_2(V) = -K_V \cdot C \) for the blow up of a smooth curve \( C \) holds in general, and is a standard computation with Chern classes.)

Combining the above formulae now yields the result.

The case when \( C \) is singular is essentially the same: consider a desingularization \( h: \tilde{V} \to V \), and apply Riemann-Roch on \( \tilde{V} \). Since the singularity on \( V \) is rational, we know that \( R^i h_* \mathcal{O}_{\tilde{V}} = 0 \) for all \( i > 0 \), and that \( h_* \mathcal{O}_{\tilde{V}} = \mathcal{O}_V \).

Thus, considering \( \chi(\tilde{V}, h^*(K_V + f*K_V)) \) on \( \tilde{V} \), we deduce that \( h^*E_1 \cdot c_2(V) = 0 \), using essentially the same argument as above together with the Leray spectral sequence. We then deduce (again similarly to the case when \( C \) is smooth) that

\[
\chi(V, (sm + 1)K_V) - \chi(\tilde{V}, K_{\tilde{V}} + sh^*(m^*K_V - E_1)) = \frac{1}{12}s(1-1)(2s-1)E_1^3.
\]

However, using the fact that \( R^i h_* \mathcal{O}_{\tilde{V}} = 0 \) for \( i > 0 \), and hence that the Leray spectral sequence degenerates, we also have that

\[
\chi(\tilde{V}, K_{\tilde{V}} + h^*(m^*K_V - E_1)) = -\chi(\tilde{V}, -h^*(m^*K_V - E_1)) = -\chi(V, -(m^*K_V - E_1)) = \chi(V, K_V + s(m^*K_V - E_1)).
\]

Thus the result is now proved in general. \( \square \)

**Lemma 2.2:** Suppose that \( X \) is a Gorenstein threefold with only finitely many singularities, and \( \Delta \) is an effective Cartier divisor on \( X \) such that \( \Delta^3 > 0 \) and \( \Delta \cdot B \geq 0 \) for all but finitely many curves \( B \) on \( X \). Then \( h^i(X, K_X + \Delta) = 0 \) for \( i > 1 \).

**Proof:** Choose a smooth very ample divisor \( H \) on \( X \) not containing any of the finite number of curves or any of the finite number of singularities, such that \( \Delta + H \) is ample and \( \Delta^2 \cdot H > 0 \). In particular, \( h^1(H, K_H + \Delta|H) = 0 \). Since by Kodaira vanishing \( H^2(V, K_V + \Delta + H) = 0 \), the result follows using the exact sequence

\[
0 \to \mathcal{O}_X(K_X + \Delta) \to \mathcal{O}_X(K_X + \Delta + H) \to \mathcal{O}_H(K_H + \Delta) \to 0.
\]

By assumption \( C \) is fixed in \( |mK_V| \). Thus for some \( r > 0 \), \( |f^*mK_V| = |D_1| + rE_1 \), where \( |D_1| \) is the mobile part of the linear system. We say that
the base curve $C$ has multiplicity $r$ in $|mK_V|$. Clearly $\Delta = D_1$ satisfies the conditions of (2.2), and therefore so too does $r(mf_i^*K_V - E_1) \sim (r - 1)mf_i^*K_V + D_1$, and hence also $(mf_i^*K_V - E_1)$. (Throughout this paper, $\sim$ denotes linear equivalence.)

**PROPOSITION 2.3:** If $K_V \cdot C = 0$ for a base curve $C$ of $|mK_V|$, then $C$ is isomorphic to $\mathbb{P}^1$.

**PROOF:** From (1.2) we know that $p_\alpha(C) \leq 1$; let us assume that $p_\alpha(C) = 1$ and obtain a contradiction. Then, even if $C$ is singular, we know that it is regularly immersed in $V$, and that it is Gorenstein with dualizing line bundle $\omega_C \cong \mathcal{O}_C$. Thus, if $\mathcal{I}_C$ denotes the sheaf of ideals defining $C$, we have that $\mathcal{I}_C/\mathcal{I}_C^2$ on $C$ is locally free of rank 2. Using the generalized adjunction formula ([1], Chapter 1, Theorem 4.5) we know that the line bundle $\Lambda^2(\mathcal{I}_C/\mathcal{I}_C^2)$ on $C$ has degree 0.

With the notation above, we have $V_1 = \text{Proj}_V(\mathcal{O}_V \oplus \mathcal{I}_C \oplus \mathcal{I}_C^2 \oplus \ldots)$, and that $\mathcal{O}_{V_1}(-E_1) \cong \mathcal{O}_{V_1}(1)$ where the right hand side denotes the natural relatively ample bundle defined by $\mathcal{I}_C$ (see [3], Chapter II, Proposition 7.13). Let $Y$ denote the Cartier divisor on $E_1$ corresponding to the line bundle $\mathcal{O}_{E_1}(1)$ (which can be interpreted as either the restriction of $\mathcal{O}_{V_1}(1)$ to $E_1$, or else the natural relatively ample bundle on $E_1 = \mathbb{P}_C(\mathcal{I}_C/\mathcal{I}_C^2)$). Thus $E_1^3 = -Y^2$.

I claim that for any rank 2 bundle $\mathcal{F}$ on $C$ with $E = \mathbb{P}_C(\mathcal{F})$, and $Y$ the natural relatively ample divisor, we have $Y^2 = \text{deg}(\Lambda^2 \mathcal{F})$. We can assume without loss of generality that $Y$ is effective (by [3], Chapter II, Lemm 7.9 and Proposition 7.10). The claim now follows by [2], Chapter 1, §10 in the case when $C$ is smooth, and the same proof works also in the case when $C$ is singular.

Thus we have deduced that $E_1^3 = 0$ on $V_1$. Note however that for sufficiently large $s$,

$$h^0(V_1, K_{V_1} + s(mf_i^*K_V - E_1)) < h^0(V_1, (sm + 1)f_i^*K_V)$$

$$= h^0(V, (sm + 1)K_V)$$

since by Theorem 2.2 of [12], the multiplicity of $E_1$ in $|nf_i^*K_V|$ is bounded as $n \to \infty$.

Since however we have that $h^i(V, (sm + 1)K_V) = 0$ for $i > 0$, and by (2.2), that $h^i(V_1, K_{V_1} + s(mf_i^*K_V - E_1)) = 0$ for $i > 1$, we obtain an immediate contradiction from (2.1).

Thus $p_\alpha(C) = 0$ as required.

**REMARK 2.4:** The results (2.1) to (2.3) provide the prototype for a number of the proofs which follow. I shall therefore explain at this stage the ideas behind these proofs.
Let $Z_0$ denote the cycle of those base curves $C$ of $|mK_V|$ such that $K_V \cdot C = 0$. Suppose now that $\phi: \tilde{V} \to V$ is any birational morphism (with $\tilde{V}$ smooth) whose exceptional locus lies over $Z_0$, and let the effective divisor $\mathcal{E}$ on $\tilde{V}$ be defined by $K_V = \phi^* K_V + \mathcal{E}$. The proof of (2.1) goes over unchanged to show that

$$\chi(V, (sm + 1)K_V) - \chi(\tilde{V}, K_{\tilde{V}} + s(\phi^* mK_V - \mathcal{E}))$$

$$= \frac{1}{12}(s-1)(2s-1)(K_{\tilde{V}}^3 - K_V^3).$$

We note also in passing that if $B$ is a smooth rational curve on a threefold $W$, $h: \tilde{W} \to W$ the blow up of $B$, then

$$K_{\tilde{W}}^3 = K_W^3 - 2(K_W \cdot B - 1).$$

**Proposition 2.5:** With the notation above, suppose that $\mathcal{E}$ is fixed in $|m\phi^* K_V|$ (e.g. the case when $\phi$ is the composite of blow ups of base curves). Suppose also that $K_{\tilde{V}} \cdot B \leq 0$ for all but finitely many curves $B$ contained in $\mathcal{E}$ and that $K^3_{\tilde{V}} \leq K^3_V$. Then we have a contradiction.

**Proof:** Since $\mathcal{E}$ is fixed in $|m\phi^* K_V|$, $(m\phi^* K_V - \mathcal{E}) \cdot B \geq 0$ for all but finitely many curves $B$ not contained in $\mathcal{E}$. However since $(m\phi^* K_V - \mathcal{E})$ is numerically equivalent to $-K_{\tilde{V}}$ on $\tilde{V}$, the second assumption now implies that $(m\phi^* K_V - \mathcal{E}) \cdot B \geq 0$ for all but finitely many curves $B$ on $V$. Now since $\mathcal{E}^3 = K_{\tilde{V}}^3 - K_V^3 \leq 0$, we have that $(m\phi^* K_V - \mathcal{E})^3 = m^3 K_{\tilde{V}}^3 - \mathcal{E}^3 > 0$. Hence by (2.2), $h^i(\tilde{V}, K_{\tilde{V}} + s(m\phi^* K_V - \mathcal{E})) = 0$ for $i > 1$.

Moreover, as in (2.3) we know that

$$h^0(\tilde{V}, K_{\tilde{V}} + s(m\phi^* K_V - \mathcal{E})) \leq h^0(V, (sm + 1)K_V)$$

$$= h^0(V, (sm + 1)K_V)$$

for $s$ sufficiently large, and that $h^i(V, (sm + 1)K_V) = 0$ for $i > 0$. Hence

$$\chi(V, (sm + 1)K_V) - \chi(\tilde{V}, K_{\tilde{V}} + s(\phi^* mK_V - \mathcal{E})) > 0$$

for $s$ sufficiently large; this then provides the required contradiction using (2.4).

Returning now to our base curve $C$ of $|mK_V|$ with $K_V \cdot C = 0$, we know that $C$ is isomorphic to $\mathbb{P}^1$, and so the normal bundle is decomposable; say $N_{C/V} = \mathcal{O}_C(e - a) \oplus \mathcal{O}_C(-a)$ where $e \geq 0$. From the adjunction formula, $e - 2a = \deg(N_{C/V}) = -2$. In particular we note that $a > 0$. With this notation, $E_1$ is the ruled surface $F_e$. In this case when $e \neq 0$, let $C_1$ denote the minimal section of $E_1(C_1^2 = -e)$, and $f_2: V_2 \to V_1$ the blow up of $C_1$, with $E_2$ the corresponding exceptional surface. If the composite
\(f_1 \circ f_2\) is denoted \(f: V_2 \rightarrow V\), then \(|f^*mK_V| = |D_0| + rf_2^*E_1 + r_1E_2\), where \(|D_0|\) is the mobile part, and \(r_1 \leq r\) denotes the multiplicity of \(C_1\) in \(|D_0|\).

**Proposition 2.6:** The normal bundle \(N_{C/V}\) is one of \((-1, -1), (-2, 0)\) or \((-3, 1)\), where \((-1, -1)\) denotes \(\mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)\), etc.

**Proof:** We note first that \(D_1 \cdot C_1 = -rE_1 \cdot C_1 = -r(e - a) = -r(a - 2)\). Thus \(D_1 \cdot C_1 \geq 0\) if and only if \(N_{C/V} = (-1, -1)\) or \((-2, 0)\). Moreover, we note that the normal bundles listed in the Proposition correspond to \(a = 1, 2\) and 3 respectively.

Suppose therefore that \(D_1 \cdot C_1 < 0\) and hence \(r_1 > 0\). Consider \(f: V_2 \rightarrow V\) as described above. An easy calculation using (2.4) shows that \(K_{V_2} = K_V\). Thus clearly the conditions of (2.5) are satisfied, and a contradiction is obtained.

The curves \(N_{C/V} = (-1, -1)\) or \((-2, 0)\) are what Reid in [10] calls \((-2)\)-curves, and essentially we have all the information about their neighbourhoods in \(V\) that we want (see [10], §5). Let us now concentrate on the case when \(N_{C/V} = (-3, 1)\). Blowing up minimal sections of exceptional surfaces, we obtain \(V_i, E_i, C_i\) as before, and a sequence of normal bundles \(N_{C/V_i}, N_{C_i/V_i}\), etc.

**Proposition 2.7:** The only possible sequence of normal bundles in the case when \(N_{C/V} = (-3, 1)\) is:

\[(-3, 1), (-3, 0), \ldots, (-3, 0), (-2, -1), (-1, -1)\]

(where there are a finite number, say \(t \geq 0\), of \((-3, 0)\)'s in the sequence).

**Proof:** On \(C_1\) we have an exact sequence of sheaves

\[0 \rightarrow \mathcal{O}_{C_1}(-4) \rightarrow N_{C_1/V_1} \rightarrow \mathcal{O}_{C_1}(1) \rightarrow 0.\]

Hence \(N_{C_1/V_1} = (-4, 1), (-3, 0)\) or \((-2, -1)\). I claim that \((-4, 1)\) cannot occur.

To see this, blow up \(C_2\), giving \(f_3: V_3 \rightarrow V_2\), and let \(\phi: V_3 \rightarrow V\) denote the composite \(f_1 \circ f_2 \circ f_3\). If \(N_{C_1/V_3} = (-4, 1)\), we see that \(D_2 \cdot C_2 = -(r + r_1)\), and using (2.4) that \(K_{V_3}^3 = K_V^3\). The other conditions of (2.5) are now easily checked, and thus a contradiction is obtained.

Therefore \(N_{C_i/V_i} = (-3, 0)\) or \((-2, -1)\). If for some \(i\), \(N_{C_i/V_i} = (-2, -1)\), we deduce from the exact sequence on \(C_{i+1}\)

\[0 \rightarrow \mathcal{O}_{C_{i+1}}(-1) \rightarrow N_{C_{i+1}/V_{i+1}} \rightarrow \mathcal{O}_{C_{i+1}}(-1) \rightarrow 0\]

that \(N_{C_{i+1}/V_{i+1}} = (-1, -1)\). If however for some \(i\), \(N_{C_i/V_i} = (-3, 0)\), we
deduce from the exact sequence on \( C_{+1} \)

\[
0 \rightarrow \mathcal{O}_{C_{+1}}(-3) \rightarrow N_{C_{+1}V_{+1}} \rightarrow \mathcal{O}_{C_{+1}} \rightarrow 0
\]

that \( N_{C_{+1}V_{+1}} = (-3, 0) \) or \((-2, -1)\).

Finally we note that if \(|D_t|\) denotes the mobile part of \(|mK_V|\), and \( N_{C_j/V_j} = (-3, 0) \) for \( j = 1, \ldots, i - 1 \), an easy calculation shows that \( D_t \cdot C_t = -r \). Hence \( C_t \) is a base curve of \(|D_t|\). This then shows that an infinite sequence of \((-3, 0)\)'s cannot occur, since then \( C_t \) would be a base curve of \(|D_t|\) for all \( t > 0 \) (i.e. we cannot resolve the base locus), which is clearly in contradiction to the results say of \([13]\).

3. Isolated base curves

As an illustration of the above results, let us consider the case when \( C \) has \( K_V \cdot C = 0 \) and is isolated in the base locus of \(|mK_V|\) (we shall note later that a trivial modification of the argument then applies to a slightly more general case).

**Proposition 3.1:** If \( C \) is isolated in the base locus, and \( N_{C/V} = (-1, -1) \) or \((-2, 0)\), then \( C \) is not a base curve of \(|(2m + 1)K_V|\).

**Proof:** We noted in the proof of (2.6) that in this case \(-E_1|E_1\) is a.e. on \( E_1 \). Using the fact that \( C \) is isolated in the base locus of \(|mK_V|\), we deduce that \( mf_i^*K_V - E_1 \) is a.e. on \( V_1 \). The Kawamata-Viehweg form of Kodaira vanishing then gives

\[
h'(V_1, K_{V_1} + 2(mf_i^*K_V - E_1)) = 0 \quad \text{for} \quad i > 0.
\]

Thus (2.1) implies that

\[
h^0(V, (2m + 1)K_V) - h^0(V_1, (2m + 1)mf_i^*K_V - E_1) = 1.
\]

In particular we see that \( E_1 \) is not a fixed component of \(|(2m + 1)mf_i^*K_V|\) on \( V_1 \), and hence that \( C \) is not a base curve of \(|(2m + 1)K_V|\).

We consider therefore the case when \( N_{C/V} = (-3, 1) \). By (2.7), there exists an integer \( t \geq 0 \) such that \( N_{C_i/V_i} = (-3, 0) \) for \( 1 \leq i \leq t \), and \( N_{C_{i+1}/V_{i+1}} = (-2, -1) \).

**Proposition 3.2:** If \( C \) is isolated in the base locus, and \( N_{C/V} = (-3, 1) \), then \( C \) is not a base curve of \(|(2m + 1)K_V|\).

**Proof:** Suppose first that \( t = 0 \). Consider the divisor \( \mathcal{E} = f_2^*E_1 + E_2 \) on \( V_2 \). An easy check confirms that \(-\mathcal{E} \cdot B \geq 0 \) for all curves \( B \) on \( \mathcal{E} \). Since \( C \) is isolated in the base locus of \(|mK_V|\), we deduce that \( mf^*K_V - \mathcal{E} \) is a.e.
on $V_2$ (where as before $f$ denotes the composite $f_1 \circ f_2$). In particular we have that $h^i(V_2, K_{V_2} + 2(mf*K_V - \mathcal{E})) = 0$ for $i > 0$. Hence using Remark 2.4, we see that

$$h^0(V, (2m + 1)K_V) - h^0(V_2, (2m + 1)f*K_V - \mathcal{E}) = 1;$$

i.e. $\mathcal{E}$ is not fixed in $(2m + 1)f*K_V$.

If however $C$ is fixed in $(2m + 1)K_V$, then we note that $C_1$ is also fixed in $|D_1|$ on $V_1$ (since $-E_1 \cdot C_1 < 0$), and hence that $\mathcal{E}$ is fixed in $(2m + 1)f*K_V$; we conclude therefore that $C$ is not fixed in $(2m + 1)K_V$.

The case when $t > 0$ is slightly more complicated. We consider the threefold $V_{t+2}$. For $i < t + 2$, we shall denote by $h_i: V_{t+2} \to V_i$ the obvious composite morphism, and in particular $h: V_{t+2} \to V$. Let us consider first the case when $t = 1$ (where the salient features of the general case are already present). Here we have $N_{C/V} = (-3, 1), N_{C_1/V_1} = (-3, 0)$ and $N_{C_2/V_2} = (-2, -1)$.

If $E'_1$ denotes the proper transform of $E_1$ under $f_2$, an easy check shows that on $V_2$ the minimal section $C_2$ of $E_2$ meets $C_1 = E'_1 \cap E_2$ in one point. Thus when we blow up $C_2$, we obtain a configuration on $V_3$ as in Fig. 1.

In Fig. 1, $\ell$ is the curve on $V_3$ corresponding on $V_2$ to the fibre of the ruling on $E'_1$ containing the point $E'_1 \cap C_2$. As surfaces, $E''_1$ is $\mathbb{F}_4$ blown up in a point on the minimal section, $E'_2$ is $\mathbb{F}_3$ and $E_3$ is $\mathbb{F}_1$.

On $V_3$ therefore we consider the divisor $\mathcal{E} = h_1^*E_1 + h_2^*E_2 + E_3$. An easy check shows that $-\mathcal{E} \cdot B \geq 0$ for all curves $B$ on $\mathcal{E}$ except for $\ell$; also $-\mathcal{E} \cdot \ell = -1$. We note moreover that $N_{\mathcal{E}/V_3} = (-2, -1)$.

Figure 1.
We now blow up $\ell'$, obtaining $g: \tilde{V}_3 \to V_3$ and exceptional surface $E$ say. On $\tilde{V}_3$ we consider the divisor $\delta = g^*\delta + E$. It is now straightforward to check that $-\tilde{\delta} \cdot B \geq 0$ for all curves $B$ on $\tilde{\delta}$. Thus as before, using the fact that $C$ is isolated in the base locus of $|mK_V|$, we deduce that $h'(\tilde{V}_3, K_{\tilde{V}_3} + 2(m\tilde{h}^*K_V - \tilde{\delta})) = 0$ for $i > 0$ (where $\tilde{h}$ is the composite $h \circ g$).

Using Remark 2.4 however, we now have

$$h^0(V, (2m + 1)K_V) - h^0(\tilde{V}_3, (2m + 1)\tilde{h}^*K_V - \tilde{\delta}) = 1,$$

and hence that $\tilde{\delta}$ is not fixed in $|(2m + 1)\tilde{h}^*K_V|$. As before therefore, we discover that $C$ is not a base curve of $|(2m + 1)K_V|$ on $V$.

The case $t > 1$ is similar. Note first that for all $3 \leq i \leq t + 2$, an easy calculation shows that the minimal section $C_i$ of $E_i$ does not meet the proper transform $E_{i-1}'$ of $E_{i-1}$. Secondly, we note that in Fig. 1 (considered now for arbitrary $t > 1$), the minimal section $C_3$ of $E_3$ cannot meet $\ell$. For if it did meet $\ell$, then on $V_4$ we would have $K_{\tilde{V}_4} \cdot \ell' = 2$ (where $\ell'$ on $V_4$ corresponds to $\ell$ on $V_3$). Hence if we blow up $\ell'$ on $V_4$, obtaining $g': \tilde{V}_4 \to V_4$, we deduce via Remark 2.4 that $K_{\tilde{V}_4}^3 = K_V^3$. Letting $\phi: \tilde{V}_4 \to V$ denote the obvious composition of maps, we see that the conditions of (2.5) are satisfied, and hence a contradiction is obtained.

We now consider the divisor $\delta = h_1^*E_1 + h_2^*E_2 + \ldots + E_{t+2}$ on $V_{t+2}$; bearing in mind the above two observations, it is a straightforward check that $-\delta \cdot B \geq 0$ for all curves $B$ on $\delta$ except for the curve $\ell^*$ on $V_{t+2}$ corresponding to the curve $\ell$ on $V_3$; for this curve $-\delta \cdot \ell^* = -1$. Now blow up $\ell^*$ obtaining $g: \tilde{V}_{t+2} \to V_{t+2}$, and proceed precisely as in the $t = 1$ case.

We have thus seen that for any isolated base curve $C$ of $|mK_V|$ with $K_V \cdot C = 0$, $C$ is not a base curve of $|(2m + 1)K_V|$. However, the only place that we use the fact that $C$ is isolated in the above is that in order to check that $m\tilde{h}^*K_V - \delta$ is a.e., we need to check it not only on the curves in $\delta$, but also on the curves of $\tilde{V}_{t+2}$ correspond to the other base curves of $|mK_V|$. Clearly, the only such curves that we need to worry about are those that meet $C$. If however any other base curve $C'$ which meets $C$ has $K_V \cdot C' > 0$, we can then merely choose $N$ a sufficiently large multiple of $m$ so that, repeating the above argument for $|NK_V|$, the positivity is satisfied on $\tilde{V}_{t+2}$.

**Corollary 3.3:** If no two base curves $C, C'$ of $|mK_V|$ with $K_V \cdot C = 0 = K_V \cdot C'$ meet, then for some $n$ the linear system $|nK_V|$ is without fixed points.

**Proof:** As above, we choose $N$ sufficiently large so that we can then deduce that the base curves $C$ of $|NK_V|$ with $K_V \cdot C = 0$ are no longer base curves of $|(2N + 1)K_V|$. Hence we see that for some $n$, $|nK_V|$ has no
fixed components and the only base curves \( C' \) of \(|nK_V|\) have \( K_V \cdot C' > 0\). Now apply (1.2).

In Section 2, we obtained detailed information concerning the individual base curves \( C \) of \(|mK_V|\) with \( K_V \cdot C = 0\). We now wish to ask which configurations of such curves can occur. We consider this question in the next section.

4. Configuration of curves in the base locus

With the notation as before, let \( Z_0 \) denote the cycle of base curves of \(|mK_V|\) with \( K_V \cdot C = 0\); in particular any component of \( Z_0 \) is isomorphic to \( \mathbb{P}^1 \). We investigate the question of which configurations can appear in \( Z_0 \). In this section, we shall again make extensive use of (2.5).

**Theorem 4.1:**

(a) Any two curves in \( Z_0 \) meet in at most one point, where they meet transversely.

(b) At most three curves of \( Z_0 \) meet at any given point, where they meet normally.

(c) There are no “closed cycles” of curves in \( Z_0 \) (i.e. curves \( B_1, \ldots, B_k \) in \( Z_0 \) with \( B_i \) meeting \( B_{i+1} \) for each \( i \), where \( B_{k+1} \) is understood as \( B_1 \)) apart from three curves meeting at a point.

**Proof:**

(a) Let \( B_1 \) and \( B_2 \) be any two curves in \( Z_0 \). We blow up one of them, say \( B_2 \), obtaining a morphism \( h: V' \to V \) and exceptional divisor \( E \). If \( B'_1 \) denotes the curve on \( V' \) corresponding to \( B_1 \) on \( V \), suppose that \( E \cdot B'_1 = d \).

We need to show that \( d \leq 1 \).

We note however that \( K_{V'}^3 = K_V^3 + 2 \). Hence if we blow up \( B'_1 \) on \( V' \) and let \( \phi: \tilde{V} \to V \) denote the composite of this map and \( h \), then by (2.4) \( K_{\tilde{V}}^3 - K_V^3 = -2(d - 2) \). The result now follows easily using (2.5).

(b) Let \( B_1, \ldots, B_r \) be curves of \( Z_0 \) that meet at some point \( P \) on \( V \). Let us blow up one of the curves, say \( B_1 \). Let \( h: V' \to V \) denote this blow up and \( E \) the exceptional divisor. If \( B'_i \) \((1 \leq i \leq r)\) denotes the curve on \( V' \) corresponding to \( B_i \) on \( V \), then the \( B'_i \) all meet the same fibre \( F \) in the ruling of \( E \) over \( B_1 \). Note also that \( K_{V'}^3 = K_V^3 + 2 \), and that \( K_{V'} \cdot B'_i = 1 \) for \( 1 \leq i \leq r \).

Thus we note that blowing up a \( B'_i \) does not alter \( K^3 \). Suppose now that two of the \( B'_i \) meet, say \( B'_2 \) meets \( B'_3 \). Blowing up \( B'_2 \), say \( g: V^* \to V' \), we have that \( K_{V^*}^3 = K_{V'}^3 \) and that \( K_{V^*} \cdot B''_3 = 2 \) (where \( B''_3 \) on \( V^* \) corresponds to \( B'_3 \) on \( V' \)). Thus if we now blow up \( B''_3 \) and let \( \phi: \tilde{V} \to V \) be the composite of this morphism with the other two blow ups, then using (2.4) we have \( K_{V}^3 = K_{V'}^3 \). A contradiction now follows easily from (2.5).

Thus no two of \( B'_2, \ldots, B'_r \) meet on \( V' \). We need to show that \( r \leq 3 \).
Suppose not; let us then blow up the curves $B'_2$, $B'_3$ and $B'_4$; say $h: V^* \to V'$. Note that by (2.4) $K_{V^*}^3 = K_{V'}^3$.

If we denote by $F'$ the curve on $V^*$ corresponding to $F$ on $V'$, we observe that $K_{V^*} \cdot F' = 2$. Thus if we blow up $F'$ and let $\phi: \tilde{V} \to V$ be the composite of this morphism with the other blow ups, then using (2.4) we have $K_{\tilde{V}}^3 = K_{V'}^3$. A contradiction again follows easily using (2.5).

(c) The fact that one cannot have any "closed cycles" follows similarly; we blow up one of the curves, say $B_1$, and $K^3$ increases by 2. If we now blow up the base curve corresponding to $B_2$, $K^3$ will remain unchanged. We continue this procedure until we reach the curve corresponding to $B_k$. Blowing this curve up now decreases $K^3$ by 2, and so the total effect has been to leave $K^3$ unchanged. A contradiction now follows easily using (2.5).

We now need a couple of lemmas.

**Lemma 4.2:** Suppose that $B$ and $C$ are curves of $Z_0$ which meet, and that $C$ has normal bundle $(-3, 1)$. Let $f_1: V_1 \to C$ denote the blow up of $C$, $E_1$ the exception divisor (isomorphic to $\mathbb{F}_4$) and $C_1$ the minimal section. If we denote by $B'$ the curve on $V_1$ corresponding to $B$, then $B'$ does not meet $C_1$.

**Proof:** If $B'$ meets $C_1$, blow up $B'$ and then blow up the curve corresponding to $C_1$. If $\phi: \tilde{V} \to V$ denotes the composition of these morphisms, we find that $K_{\tilde{V}}^3 = K_{V'}^3$, and that a contradiction follows using (2.5). □

As in (4.2), suppose that $B$ and $C$ are curves of $Z_0$ which meet, and suppose that $C$ has normal bundle $(-3, 1)$. However, let us now blow up $B$ first, say $h: V'' \to V$ with exceptional divisor $E$. Then blow up the curve $C'$ on $V''$ corresponding to $C$, obtaining a threefold $V^*$ and an exceptional divisor $E^*$. We therefore obtain a configuration as in Fig. 2, where $E'$ is the proper transform of $E$, and $\ell$ is the curve on $V^*$ corresponding to the fibre of $E'$ meeting $C'$.

**Lemma 4.3:** $E^*$ is isomorphic to $\mathbb{F}_3$ and the minimal section $C^*$ meets $\ell$.

Figure 2.
PROOF: Note that $\ell$ has normal bundle $(-1, -1)$, and that blowing $B$ and $C$ up in the other order corresponds to making an elementary transformation on $\ell$. By an elementary transformation on $\ell$, we mean the operation of blowing up $\ell$ (obtaining an exceptional divisor isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$), and then contracting down along the other ruling; see [7] and [10].

If $E^*$ is isomorphic to $F$, then we deduce that $e' = 3$ or $5$ according to whether $C^*$ does, or does not, meet $\ell$ (since if we blow up $C$ on $V$, we obtain an exceptional divisor $E_1$ isomorphic to $F_4$). By inspection of the geometry however, we see that the case when $C^*$ does not meet $\ell$ corresponds (when we blow $B$ and $C$ up in the other order) to the case when $B'$ meets $C_1$ in (4.2). Hence the result follows.

REMARK 4.4: In (4.3) we note that $C'$ has normal bundle $(-3, 0)$. Thus we deduce via (2.4) that the operation of blowing up $C'$ and then blowing up $C^*$ does not alter the value of $K_3$. I shall refer to this operation as taking the double blow up of $C'$.

THEOREM 4.5:

(a) A curve of $Z_0$ with normal bundle $(-3, 1)$ cannot meet another such curve, and also cannot meet a triple point.

(b) A curve $Z_0$ with normal bundle $(-2, 0)$ meets at most one $(-3, 1)$-curve, meets at most one triple point, and cannot meet both.

PROOF: These results will all follow from (2.5). For brevity I shall therefore just give the sequence of blow ups needed, and leave the reader to verify that they work.

(a) Suppose $B$ and $C$ in $Z_0$ both have normal bundle $(-3, 1)$. Blow up $C$ obtaining exceptional divisor $E_1$ isomorphic to $F_4$ and minimal section $C_1$. If $B'$ is the curve corresponding to $B$ and $F$ is the fibre of $E_1$ meeting $B'$, then we take the double blow up of $B'$, blow up $C_1$ and then blow up the curve $F'$ corresponding to $F$. We then have a contradiction.

Suppose now that a curve $C$ in $Z_0$ has a point in common with two other curves $B_1$ and $B_2$ of $Z_0$. Blow up $C$ as before, obtaining $E_1$ with minimal section $C_1$. If $B'_i$ denotes the curve corresponding to $B_i$, there is a fibre $F$ of $E_1$ meeting both $B'_1$ and $B'_2$. Now blow up $C_1$, $B'_1$ and $B'_2$, and then blow up the curve $F'$ corresponding to $F$. A contradiction is obtained.

(b) Similar to (a); left as an exercise. 

Let us consider now a curve $B$ in $Z_0$ with normal bundle $(-1, -1)$. Suppose that $B$ meets a curve $C$ of $Z_0$ with normal bundle $(-3, 1)$. Let us make an elementary transformation on $B$; using (4.3), we discover that the curve $C$ corresponding to $C$ under this transformation has normal bundle $(-2, 0)$.

Moreover, it is not difficult to show using (2.5) that if $B$ meets a $(-3, 1)$-curve, then the transformed curve $B$ does not meet such a curve.
Thus by making an elementary transformation, we can assume that $\mathcal{B}$ does not meet any $(-3,1)$-curve. The question therefore arises as to whether, by also making the transformations described in [10] on $(-2,0)$-curves, we can eliminate all non-isolated $(-3,1)$-curves. The isolated $(-3,1)$-curves we know about from Section 3.

Finally, consider any curve $C$ of $\mathcal{Z}_0$, and let $\mathcal{B}_1, \ldots, \mathcal{B}_k$ be those curves of $\mathcal{Z}_0$ different from but meeting $C$. Let us blow up $C$, and then blow up all the $\mathcal{B}_i'$ in some order. If the resulting variety is $\tilde{\mathcal{V}}$, and $|\tilde{D}|$ is the mobile part of $|mK_{\tilde{\mathcal{V}}}|$, we know that the general element of $|\tilde{D}|$ cuts out an effective divisor on the proper transform of the exceptional divisor over $C$. This gives us an inequality relating the multiplicities in $|D| = |mK_{\mathcal{V}}|$ of $C$ and of the $\mathcal{B}_i$. We can now repeat this argument on the $\mathcal{B}_i'$.

We leave it as an exercise for the reader to show, using the above method, that $C$ meets at most nine other curves of $\mathcal{Z}_0$.

References


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