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ETALE p-COVERS IN CHARACTERISTIC p

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Introduction

Let X be a separated scheme of finite type over an algebraically closed field k of characteristic $p > 0$. If f: $Y \to X$ is a finite étale covering then the Euler-Poincaré formula states that

$$\chi_c(Y, \mathbf{Q}_l) = (\deg f)\chi_c(X, \mathbf{Q}_l) \tag{0.1}$$

for any prime $l \neq p$; here we have put, as usual

$$\chi_c(X, \mathbf{Q}_l) = \sum_{i=0}^{2 \dim X} (-1)^i \dim_{\mathbf{Q}_l} H_c^i(X, \mathbf{Q}_l)$$

(cohomology with supports) and similarly for Y. It is the purpose of this paper to inquire into case $l = p$; then 0.1 is no longer generally valid, though we shall show that 0.1 does hold with $l = p$ in one important special case: f: $Y \to X$ is a finite étale cover, *galois* and of degree a power of $p = \text{char } k$. (cf. 1.5 1.7 below).

If X is a complete nonsingular curve, then $\chi_c(X, \mathbf{Q}_p) = 1 - p_X$ where p_X is the p-rank of X, i.e. the number of independent Z/pZ-covers of X. Thus 0.1 gives the relation between the p-rank of a complete nonsingular curve and the p-rank of a finite étale galois p-cover. This relation was already known to Shafarevich [12], who used it to prove that the maximal pro-p-quotient of π_1 of a complete nonsingular curve is a *free* profinite p-group. We shall give another, independent proof of this last fact.

The proofs of the above statements and their corollaries forms the contents of §1 below. In §2 we give some applications to algebraic surfaces, and show in particular that the fundamental group of a weakly unirational surface has no p-torsion.

Let us now suppose that X/k is complete and nonsingular, so that the crystalline cohomology groups $H^{\cdot}_{\text{cris}}(X/W)$ are finitely generated modules over the ring W of Witt vectors of k. Let K be the fraction field of W, and denote by $H^{\cdot}_{\text{cris}}(X/W)_\lambda$ the part of $H^{\cdot}_{\text{cris}}(X/W) \otimes K$ where the geometric Frobenius acts with slope λ. Then we can define the quantities

$$\chi_\lambda(X) = \sum_{i=1}^{2 \dim X} (-1)^i \dim_K H^{\cdot}_{\text{cris}}(X/W)_\lambda \tag{0.2}$$

and for $\lambda = 0$ we have $\chi_0(X) = \chi(X, \mathbf{Q}_p)$. It would be interesting to know whether an analogue of 0.1 holds for the χ_λ; i.e. whether

$$\chi_\lambda(Y) = (\deg f)\chi_\lambda(X) \tag{0.3}$$

whenever f: $Y \to X$ is a finite etale galois p-covering. I hope to return to this question at a later time.

The paper concludes with two other proofs of 1.7 which were suggested to the author by L. Moret-Bailly and N.M. Katz.

Section 1

1.0. We fix once and for al a prime p and an algebraically closed field k of characteristic p.

The main result in this section will follow easily from a basic finiteness theorem in etale cohomology. To state it, we need to recall the notion of a *perfect complex:* if R is any ring with unit, then a perfect complex of R-modules is simply a complex of R-modules that is quasi-isomorphic to a bounded complex of finitely generated projective R-modules. For a thorough discussion of this notion the reader may consult SGA 6 I.

The finiteness result we need is a special case of SGA $4^{1/2}$ Rapport 4.9:

1.1. THEOREM: *Let X/k be a separated scheme over an algebraically closed field k, and let R be a finite ring and F a constructible sheaf of flat R-modules on X. Then $R\Gamma_c(X, F)$ is a perfect complex of R-modules.*

From now on we shall be concerned with the following situation: X is a separated scheme of finite type over k of characteristic p, and G is a finite group (a p-group, eventually) acting on X. We shall further assume that G acts *freely* on X, by which we mean that the quotient $Y = X/G$ exists and that the projection map f: $X \to Y$ is finite étale. In this situation, if R is any finite ring, then 1.1 says that the groups $H^{\cdot}_c(X, R)$ are finitely generated $R[G]$-modules. In fact, we have

1.2. PROPOSITION: *If Λ is a finite ring, X/k a separated scheme of finite type over an algebraically closed field, and G a finite group acting freely on X, then $R\Gamma_c(X, \Lambda)$ is a perfect complex of $\Lambda[G]$-modules.*

PROOF: Since f is finite étale, we have $Rf_* = f_*$ and the Leray spectral sequence

$$H_c^i(Y, R^j f_* \Lambda) \Rightarrow H_c^{i+j}(X, \Lambda)$$

degenerates to yield $H_c^i(Y, f_* \Lambda) = H_c^i(X, \Lambda)$. By 1.1 it is enough to show that $f_* \Lambda$ is a constructible sheaf of flat $\Lambda[G]$-modules on Y. Constructibility is trivial from the definition. As to flatness, we need only remark that if $\bar{y} \to Y$ is any geometric point, then $(f_* \Lambda)_{\bar{y}} = \Lambda[G]$.

We need analogues of 1.2 for the l-adic cohomology groups $H^{\cdot}(X, \mathbb{Z}_l) = \lim_n H^{\cdot}(X, \mathbb{Z}/l^n\mathbb{Z})$ and $H^{\cdot}(X, \mathbb{Q}_l) = H^{\cdot}(X, \mathbb{Z}_l) \# \mathbb{Q}_l$ for any prime l (even the characteristic!). The first step in the passage to the limit is

1.3. LEMMA: *Let R be a noetherian ring, I a two-sided ideal, $R_n = R/I^{n+l}R$, and suppose that R is I-adically separated and complete. Let $\{K_n^{\cdot}\}_n \in \{D(R_n)\}_n$ be an inverse system such that*

(a) $K_n^{\cdot}$ is a perfect complex of R_n-modulus;

(b) the transition maps $K_n^{\cdot} \to K_{n-l}^{\cdot}$ induce isomorphisms

$$K_n^{\cdot} \overset{L}{\underset{R_n}{\otimes}} R_{n-1} \tilde{\to} K_{n-1}^{\cdot}. \tag{1.3.1}$$

Then the inverse system $\{K_n^{\cdot}\}_n$ can be realized by an inverse system of perfect complexes, and $\lim_n K_n^{\cdot} = K^{\cdot}$ is a perfect complex of R-modules; furthermore the natural maps $K^{\cdot} \to K_n^{\cdot}$ induce quasi-isomorphisms

$$K^{\cdot} \overset{L}{\underset{R}{\otimes}} R_n \to K_n^{\cdot}. \tag{1.3.2}$$

A proof may be found in [3], B.11.

We now set $R = \mathbb{Z}_l[G]$, $I = (l)$, $K_n^{\cdot} = R\Gamma_c(X, \mathbb{Z}/l^{n+1}\mathbb{Z})$. To see that $K_n^{\cdot} \to K_{n-1}^{\cdot}$ induces an isomorphism as in 1.3.1. we need only compute

$$\begin{aligned} K_n^{\cdot} \overset{L}{\underset{R}{\otimes}} R_{n-1} &= R\Gamma_c(X, \mathbb{Z}/l^{n+1}\mathbb{Z}) \overset{L}{\underset{R_n}{\otimes}} R_{n-1} \\ &= R\Gamma_c(Y, f_*\mathbb{Z}/l^{n+l}\mathbb{Z}) \overset{L}{\underset{R_n}{\otimes}} R_{n-1} \\ &= R\Gamma_c\left(Y, (f_*\mathbb{Z}/l^{n+1}\mathbb{Z}) \overset{L}{\underset{R_n}{\otimes}} R_{n-1}\right) \\ &= R\Gamma_c(Y, f_*\mathbb{Z}/l^n\mathbb{Z}) \end{aligned}$$

since one readily verifies that

$$\left(f_*\mathbb{Z}/l^{n+1}\mathbb{Z}\right) \overset{L}{\underset{R}{\otimes}} R_{n-1} = \left(f_*\mathbb{Z}/l^{n+1}\mathbb{Z}\right) \underset{R_n}{\otimes} R_{n-1}$$

$$= f_*\mathbb{Z}/l^n\mathbb{Z}.$$

1.4. PROPOSITION: *There is perfect complex $K^\cdot$ of $\mathbb{Z}_l[G]$-modules such that $H^\cdot(K^\cdot) = H_c^\cdot(X, \mathbb{Z}_l)$.*

PROOF: We may apply 1.3 to the inverse system $R\Gamma_c(X, \mathbb{Z}/l^{n+1}\mathbb{Z})$, thanks to 1.2 and the preceding calculation. If $K_n^\cdot$ and $K^\cdot$ are as in 1.3.1 and 1.3.2, then we have

$$H^\cdot(K^\cdot) = H^\cdot\left(\lim_n K_n^\cdot\right)$$

$$= \lim_n H^\cdot(K_n^\cdot) \quad \text{since any inverse system of finite groups satisfies Mittag-Leffler.}$$

$$= \lim_n H_c^\cdot(X, \mathbb{Z}/l^n\mathbb{Z})$$

$$= H_c^\cdot(X, \mathbb{Z}_l)$$

We can now prove the main result of this section:

1.5. THEOREM: *Suppose that X/k is a separated scheme of finite type over an algebraically closed field k of characteristic p. Suppose that G is a finite p-group acting freely on X. Then the virtual representation of G on*

$$\sum_i (-1)^i H_c^i(X, \mathbb{Q}_p)$$

is a sum of regular representations.

PROOF: Let $K^\cdot$ be a bounded complex of projective $\mathbb{Z}_p[G]$-modules representing $R\Gamma_c(X, \mathbb{Z}_p)$. Then $H_c^\cdot(X, \mathbb{Q}_p) = H^\cdot(K^\cdot \otimes \mathbb{Q}_p)$ and therefore

$$\sum_i (-1)^i H_c^i(X, \mathbb{Q}_p) = \sum_i (-1)^i K^i \otimes \mathbb{Q}_p$$

as virtual representations. To conclude the proof we need only remark that $K^i \otimes \mathbb{Q}_p$ is a free $\mathbb{Q}_p[G]$-module; in fact each K^i is a free $\mathbb{Z}_p[G]$-module, as follows from the next lemma.

1.6. LEMMA: *If G is a finite p-group, then a finitely generated projective $\mathbb{Z}_p[G]$-module is free.*

If G is abelian, then $\mathbb{Z}_p[G]$ is a local ring and the result is standard. The general case offers no novelties; the reader may consult Serre [11], 14.4 Cor. 1 and 15.6.

If now S is any separated k-scheme of finite type, then we put, as usual,

$$\chi_c(S, \mathbb{Q}_p) = \sum_i (-1)^i \dim_{\mathbb{Q}_p} H_c^i(S, \mathbb{Q}_p).$$

We have

1.7. COROLLARY: *If X/k, G, Y are as in 1.5, then*

free action ⇒ étale map

$$\chi_c(X, \mathbb{Q}_p) = (\text{Card } G)\chi_c(Y, \mathbb{Q}_p).$$

PROOF: The Hochschild-Serre spectral sequence shows that

$$H_c^{\cdot}(Y, \mathbb{Q}_p) = H_c^{\cdot}(X, \mathbb{Q}_p)^G$$

and the corollary then follows from 1.5.

1.7.1. REMARK: 1.7 is of course true for $\chi_c(X, \mathbb{Q}_l)$ and G a finite group of any order, if $l \neq \text{char}(k)$; this is just the Euler-Poincaré formula in l-adic cohomology ([SGA 5]X). We shall see presently that 1.7 does not hold if G is not a p-group. ?

Let us now consider the case of a complete, connected, nonsingular curve X defined over an algebraically closed field k of characteristic $p > 0$. The p-rank p_X of X can be defined as

$$p_X = \dim_{\mathbb{Q}_p} H^1(X, \mathbb{Q}_p).$$

Now let G be a finite p-group acting on X, although we no longer make the assumption that G acts freely. Denote by Y the quotient of X by G and by X^{ram} the set of points of X that are ramified over Y. If $x \in X^{\text{ram}}$, let e_x denote the ramification index at x. With this notation 1.7 has the following

cf. Subrao, Manuscr. Math. 16 (1975), 169–193 Th. 4.1

1.8. COROLLARY: *With X, G, Y, and k as above, we have*

$$1 - p_X = (\text{Card } G)(1 - p_Y) - \sum_{x \in X^{\text{ram}}} (e_x - 1).$$

PROOF: Let $f: X \to Y$ denote the natural projection and let U be the open subscheme of Y over which f is étale. Let also $V = f^{-1}(U)$, $D = Y - U$;

then $f^{-1}(D)=X-V$ and $D, f^{-1}(D)$ are finite sets of points. The excision exact sequences

$$\to H_c^i(U,\mathbf{Q}_p)\to H_c^i(Y,\mathbf{Q}_p)\to H_c^i(D,\mathbf{Q}_p)$$
$$\to H_c^i(V,\mathbf{Q}_p)\to H_c^i(X,\mathbf{Q}_p)\to H_c^i(f^{-1}(D),\mathbf{Q}_p)\to$$

give

$$\chi_c(U,\mathbf{Q}_p)+(\text{Card } D)=\chi_c(Y,\mathbf{Q}_p)$$
$$\chi_c(V,\mathbf{Q}_p)+(\text{Card } f^{-1}(D))=\chi_c(X,\mathbf{Q}_p).$$

We can apply 1.7 to $V\to U$. Since

$$\chi(Y,\mathbf{Q}_p)=1-p_Y,\quad \chi(X,\mathbf{Q}_p)=1-p_X$$

we get

$$1-p_X=(\text{Card } G)(1-p_Y-\text{Card } D)+\text{Card } f^{-1}(D).$$

A simple calculation shows that

$$(\text{Card } G)(\text{Card } D)-\text{Card } f^{-1}(D)=\sum_{x\in X^{\text{ram}}}(e_x-1)$$

from which 1.8 follows immediately.

1.8.1. REMARK: It is interesting to note that 1.8 contains no terms depending on wild ramification, unlike the Hurwitz genus formula for this situation. On the other hand, simple examples show that there can be no formula such as 1.8 relating only the p-ranks, the degree, and the ramification indices in the case that G is not a p-group. If $p\neq 2$, for example, any elliptic curve can be represented as a double cover of $\mathbf{P}^1$ branched at four places, but the p-rank can be either zero or one.

In the case that $X\to Y$ is unramified, 1.8 says that

$$1-p_X=(\text{Card } G)(1-p_Y). \tag{1.8.2}$$

Recall that a complete connected nonsingular curve over an algebraically closed field is said to be *ordinary* if its p-rank is equal to its genus (i.e. the p-rank is the maximum possible). We have

1.8.3. COROLLARY: *Let Y be a complete nonsingular connected curve over an algebraically closed field of characteristic p and let $X\to Y$ be a finite étale galois covering of degree a power of p. Then X is ordinary if and only if Y is ordinary.*

PROOF: Follows from 1.8.2 and the definitions.

1.8.2 was first proven by Shafarevich [12]. He deduced from it the following interesting result:

1.9. THEOREM: *Let X be a complete, connected, nonsingular, curve over an algebraically closed field k of characteristic $p > 0$. Then the maximal pro-p-quotient of the fundamental group of X is a free pro-p-group on p_X generators.*

A free pro-p-group, the reader will recall, is the completion of a free group with respect to its set of normal subgroups of p-power index. The proof that 1.8.2 implies 1.9 can be found in Shafarevich's paper [12]. Nowadays, however, it is easier to prove 1.9 directly by using, for example,

1.9.1. LEMMA: *Let G be a pro-p-group*

1.9.1.1. The minimal number of generators of G is equal to

$$\dim_{\mathbb{F}_p} \mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})$$

1.9.1.2. G is free if and only if $H^2(G, \mathbb{Z}/p\mathbb{Z}) = 0$

PROOF: Cf. [13] Theorem 12 and Cor. 2 to Proposition 23.

We also need

1.9.2. LEMMA: *Let X be a complete irreducible curve over an algebraically closed field of characteristic p. Then $H^2(X, \mathbb{F}_p) = 0$.*

PROOF: Since X is irreducible and complete and k is algebraically closed, Artin-Schreier theory ([SGA 4]IX 3.5) tells us that

$$H^n(X, \mathbb{F}_p) = \mathrm{Ker}(1 - F: H^n(X, \mathcal{O}) \to H^n(X, \mathcal{O}))$$

Since X is a curve, $H^2(X, \mathcal{O}) = 0$, whence $H^2(X, \mathbb{F}_p) = 0$.

We now prove 1.9. By 1.9.1 it is enough to show that $H^1(\pi_1, \mathbb{F}_p)$ has dimension p_X and that $H^2(\pi_1, \mathbb{F}_p) = 0$. Now recall the equivalence of categories

$$\begin{pmatrix} \text{finite étale group} \\ \text{schemes on } X \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} \text{finite groups on which} \\ \pi_1(X, x) \text{ acts} \end{pmatrix}$$

$$F \longmapsto \begin{matrix} F_{\bar{x}} \text{ with its natural} \\ \text{action of } \pi_1(X, \bar{x}) \end{matrix} \qquad (1.9.2.1)$$

and the equality $H^1(X, F) = H^1(\pi_1(X, \bar{x}), F_{\bar{x}})$. If we put $F = \mathbb{F}_p$, we get

$$p_X = \dim_{\mathbb{Q}} H^1(X, \mathbb{Q}_p) = \dim_{\mathbb{F}} H^1(X, \mathbb{F}_p) \quad (\text{cf. } 3.1)$$
$$= \dim_{\mathbb{F}_p} H^1(\pi_1, \mathbb{F}_p) = \dim_{\mathbb{F}_p} \operatorname{Hom}(\pi_1, \mathbb{F}_p).$$

which proves the statement about H^1. We have now an exact sequence

$$0 \to \mathbb{F}_P \to A \to B \to 0$$

where A is the $\mathbb{F}_p$-module of continuous maps $\pi_1 \to \mathbb{F}_p$; so that A and B are ind-objects of the left hand side of 1.9.2.1. It is well known that A is cohomologically trivial. If $\bar{A}$, $\bar{B}$ are the sheaves corresponding to A, B, then we have a diagram

$$\begin{array}{ccccccc} H^1(\pi_1, A) & \to & H^1(\pi_1, B) & \to & H^2(\pi_1, \mathbb{F}_p) & \to & H^2(\pi_1, A) = 0 \\ \downarrow & & \downarrow & & & & \\ H^1(X, \bar{A}) & \to & H^1(X, \bar{B}) & \to & H^2(X, \mathbb{F}_p) = 0 & & \text{by } 1.9.2. \end{array}$$

Since the vertical arrows are isomorphisms, an easy diagram chase shows that $H^2(\pi_1, \mathbb{F}_p) = 0$, as was to be shown.

Section 2

In this section we shall apply 1.7 to some questions regarding algebraic surfaces. It will be necessary to prove that certain $H_c^i(X, \mathbb{Q}_p)$ vanish and for this we need some remarks.

Suppose that X/k is smooth and proper. As usual we denote by W the ring of Witt vectors $W(k)$ of k and by K the fraction field of W. The crystalline cohomology groups $H^{\cdot}_{\text{cris}}(X/W)$ are then finitely generated W-modules, and the K-space $H^{\cdot}_{\text{cris}}(X/W) \otimes_W K$ equipped with the semi-linear endomorphism arising from the Frobenius morphism is an F-isocrystal (cf. [2], [8]). If (M, F) is any F-isocrystal and λ is a positive rational number, then we shall denote by M_λ the part of M where F acts with slope λ (cf. [8] §2).

The next proposition, whose proof can be found in [6] (II Theorem 5.2) shows in particular that $H^{\cdot}(X, \mathbb{Q}_p) \otimes_W K$ is the "slope zero" part of $H^{\cdot}_{\text{cris}}(X/W) \otimes_W K$.

2.1. Proposition: *Suppose that X/k is smooth and proper. There are exact sequences*

$$0 \to H^{\cdot}(X, \mathbb{Q}_p) \to H^{\cdot}_{\text{cris}}(X/W) \underset{W}{\otimes} K \xrightarrow{1-F} H^{\cdot}_{\text{cris}}(X/W) \underset{W}{\otimes} K \to 0 \tag{2.1.1}$$

$$0 \to H^{\cdot}(X, \mathbb{Q}_p) \to H^{\cdot}(X, W\mathcal{O}_x) \underset{W}{\otimes} K \overset{1-F}{\to} H^{\cdot}(X, W\mathcal{O}_x) \underset{W}{\otimes} K \to 0 \tag{2.1.2}$$

where F is the map induced by the Frobenius morphism. (The $W\mathcal{O}_X$ are Serre's Witt vector sheaf *and the cohomology is that of the Zariski topology.)*

2.2. Corollary: *Suppose that X/k is smooth and proper and that $H^2_{\text{cris}}(X/W) \otimes_W K$ is generated by chern classes of algebraic cycles. Then $H^2(X, \mathbb{Q}_p) = 0$.*

Proof: If $x \in H^2_{\text{cris}}(X/W)$ is the chern class of an algebraic cycle, then $Fx = px$. Therefore $1 - F$ has no kernel on $H^2_{\text{cris}}(X/W) \otimes_W K$ if the latter is generated by chern classes, and so $H^2(X, \mathbb{Q}_p) = 0$ by 2.1.1.

2.3. Corollary: *Suppose that X/k is smooth and proper. Then $H^i(X, \mathbb{Q}_p) = 0$ if $i > \dim X$.*

Proof: Since the $W\mathcal{O}_X$ are inverse limits of successive extensions of coherent sheaves, we have $H^i(X, W\mathcal{O}_X) = 0$ if $i > \dim X$. The corollary is then an immediate consequence of 2.1.2.

We can now give the promised applications. Recall that a normal, complete connected variety X/k of dimension N is said to be weakly unirational if there exists a generically subjective, generically finite rational map $\mathbb{P}^N_k \to X$. X/k is said to be unirational if in addition the extension of function fields $k(\mathbb{P}^N_k/k(X)$ is separable. Lüroth's theorem says that any unirational surface is rational, but there are *weakly unirational* surfaces and unirational threefolds that are not rational (cf. [1], [14]). Let us recall some basic facts about weakly unirational varieties:

2.4.1. If X/k is weakly unirational, then $\pi_1(X)$ is finite. (For a proof, cf. SGA I XI 1.3)
2.4.2. An étale covering of a weakly unirational variety is weakly unirational. (Cf. [9] §1.)

In fact a unirational variety in characteristic zero is simply connected [9]. (cf. [10]) and a unirational threefold in any characteristic is simply connected. In a little while we shall describe an example due to Shioda [14] of a weakly unirational surface with nontrivial π_1.

2.4.3. If X/k is a smooth, weakly unirational surface then $H^2_{\text{cris}}(X/W) \otimes_W K$ is generated by the chern classes of algebraic cycles (cf. [9] §1).

The application we have in mind is

2.5. THEOREM: *Let X/k be a smooth weakly unirational surface over an algebraically closed field k of characteristic $p > 0$. Then $\pi_1(X)$ has no p-torsion.*

This was proven by Katsura in the case that k is the algebraic closure of a finite field [7]. We shall deduce 2.5 from

2.5.1. LEMMA: *If X/k is as in 1.11, then $\chi(X, \mathbb{Q}_p) = 1$.*

PROOF: We shall compute the $H^{\cdot}(X, \mathbb{Q}_p)$:
(1) $H^0(X, \mathbb{Q}_p) = \mathbb{Q}_p$ since X is connected
(2) $H^1(X, \mathbb{Q}_p) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p}(\lim_n H^1(X, \mathbb{Z}/p^n\mathbb{Z})$
$= \mathbb{Q}_p \otimes_{\mathbb{Z}_p}(\lim_n \mathrm{Hom}(\pi_1, \mathbb{Z}/p^n\mathbb{Z}) = 0$ by 2.4.1
(3) $H^2(X, \mathbb{Q}_p) = 0$ by 2.2 and 2.4.3.
(4) By 2.3, we have that $H^3(X, \mathbb{Q}_p) = H^4(X, \mathbb{Q}_p) = 0$.
Summing up, we find that $\chi(X, \mathbb{Q}_p) = 1$ for any weakly unirational surface X.

PROOF of 2.5: Since $\pi_1(X)$ is finite, there is a "universal cover" $X^{\mathrm{univ}} \to X$ of X which by 2.4.2 is also weakly unirational. Suppose now that $\pi_1(X)$ has a nontrivial subgroup G of order a power of p; then G corresponds to an étale map $X^{\mathrm{univ}} \to Y$ making X^{univ} an etale G-torsor over Y. We may therefore apply 1.7 to get

$$\chi\left(X^{\mathrm{univ}}, \mathbb{Q}_p\right) = (\mathrm{Card}\ G)\chi\left(Y, \mathbb{Q}_p\right). \tag{2.5.2}$$

On the other hand, if X^{univ} is weakly unirational, then Y must be so too; then 1.11.1 gives $\chi(Y, \mathbb{Q}_p) = \chi(X^{\mathrm{univ}}, \mathbb{Q}_p) = 1$, contradicting 2.5.2.

2.6. Here is Shioda's example of a non-simply-connected weakly unirational surface. Suppose that $p \neq 5$ and that $p \not\equiv 1 \bmod 5$. Let X be the surface in $\mathbb{P}^3$ defined by

$$X_0^5 + X_1^5 + X_2^5 + X_3^5 = 0.$$

Shioda [13] shows that X is weakly unirational. Now μ_5 acts on X by $X_i \to \zeta^i X_i$, $\zeta \in \mu_5$, and one easily checks that this action is free. The quotient X/μ_5, the "Godeaux surface", is then weakly unirational and has $\mathbb{Z}/5\mathbb{Z}$ as its fundamental group.

Here is another application of 1.7:

2.7. THEOREM: *Let X be a singular Enriques surface over an algebraically closed field of characteristic 2 and let X' be the K3 surface which is a double cover of X. Then X' is an ordinary K3 surface.*

The basic facts about Enriques surfaces in characteristic 2 can be found in Bombieri-Mumford [4]. There are three sorts of them, known as classical, singular, and supersingular accordingly as *Pic* is $\mathbb{Z}/2\mathbb{Z}$, μ_2, or α_2. Of these, only the singular ones have a double cover by a $K3$ surface. If X is any Enriques surface, then $H^2_{\text{cris}}(X/W)\otimes_W K$ is generated by the chern classes of algebraic cycles.

For our purposes an ordinary $K3$ surface is one for which $\dim_K H^2_{\text{cris}}(X/W)_0 = 1$, in which case we must also have $\dim^2_{\text{cris}}(X/W)_1 = 20$ and $\dim_K H^2_{\text{cris}}(X/W)_2 = 1$.

Proof of 2.7: We first show that if X is any Enriques surface, then $\chi(X, \mathbb{Q}_p) = 1$. In fact, we have that $H^0(X, \mathbb{Q}_p) = \mathbb{Q}_p$ and $H^1_{\text{cris}}(X/W) = 0$ (cf. [6] II 7.3.5). Since $H^2_{\text{cris}}(X/W)\otimes_W K$ is generated by algebraic cycles, we must have $H^2(X, \mathbb{Q}_p) = 0$ by Cor. 2.2. Finally 2.1 and 2.3 show that $H^1(X, \mathbb{Q}_p) = H^3(X, \mathbb{Q}_p) = H^4(X, \mathbb{Q}_p) = 0$, so that $\chi(X, \mathbb{Q}_p) = 1$.

Suppose now that X is a singular Enriques surface and that X' is its double cover. By 1.7 and the calculation in the previous paragraph, we must have that $\chi(X', \mathbb{Q}_p) = 2$. Since X' is a $K3$ we have $H^1_{\text{cris}}(X'/W) = 0$, and therefore $H^1(X', \mathbb{Q}_p) = 0$. We must therefore have that $\dim_K H^2(X'/W)_0 = \dim_{\mathbb{Q}_p} H^2(X', \mathbb{Q}_p) = 1$, showing that X' is ordinary.

2.8. **Remark**: This strengthens a result of Katsura [7]. One notes that in virtue of 2.5, a singular Enriques surface cannot be weakly unirational. On the other hand, P. Blass [5] has recently shown that the classical and supersingular Enriques surfaces are weakly unirational.

Section 3

We shall conclude by briefly describing two other proofs of 1.7, which were suggested to the author by L. Moret-Bailly and by N.M. Katz (respectively). Both proofs require

3.1. **Lemma**: *If X/k is a separated scheme of finite type over an algebraically closed field k, then*

$$\chi_c(X, \mathbb{Q}_p) = \chi_c(X, \mathbb{F}_p).$$

Proof: Let $K^{\cdot}$ be a perfect complex of $\mathbb{Z}_p$-modules representing $R\Gamma_c(X, \mathbb{Z}_p)$ (cf. 1.4). Then we have

$$R\Gamma_c(X, \mathbb{F}_p) = R\Gamma_c(X, \mathbb{Z}_p) \overset{L}{\underset{\mathbb{Z}_p}{\otimes}} \mathbb{F}_p$$

$$= \text{the mapping cone of } K^{\cdot} \xrightarrow{p} K^{\cdot}$$

If we let a_i be the number of torsion factors of $H^i(X, \mathbb{Z}_p)$, then a simple computation using the long exact sequence of the triangle

$$K^\cdot \to K^\cdot \to R\Gamma_c(X, \mathbb{F}_p) \to$$

gives

$$\dim_{\mathbb{F}_p} H_c^i(X, \mathbb{F}_p) = \dim_{\mathbb{Q}_p} H_c^i(X, \mathbb{Q}_p) + a_i + a_{i+1}.$$

We have $a_0 = 0$, and 3.1 follows on taking the alternating sum.

3.2. Remark: This is of course true for all primes p, not merely for $p = \mathrm{char}(k)$.

We must therefore show that

$$\chi_c(X, \mathbb{F}_p) = (\mathrm{Card}\ G)\chi_c(Y, \mathbb{F}_p)$$

where X, Y, G, are as in 1.5. Denote by $f\colon X \to Y$ the natural projection. The equivalence of categories 1.9.2.1 has a special case

$$\begin{pmatrix} \text{locally constant sheaves} \\ \text{of } \mathbb{F}_p\text{-vector spaces on } Y \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} \text{representations of } \pi_1(Y, \bar{y}) \\ \text{on finite-dimensional} \\ \mathbb{F}_p\text{-vector spaces} \end{pmatrix}.$$

The sheaf $f_*\mathbb{F}_p$ corresponds to a representation of $\pi_1(Y, \bar{y})$ in $GL_{\mathbb{F}_p}((f_*\mathbb{F}_p)_{\bar{y}})$ which must factor

$$\pi_1(Y, y) \to G \to GL_{\mathbb{F}_p}\big((f_*\mathbb{F}_p)_{\bar{y}}\big)$$

since $f_*\mathbb{F}_p$ becomes constant when pulled back to X. Since G is a p-group, however, the representation must be a successive extension of trivial representations, so that $f_*\mathbb{F}_p$ must be a successive extension of constant sheaves. An easy induction then shows that

$$\begin{aligned}\chi_c(X, \mathbb{F}_p) = \chi_c(Y, f_*\mathbb{F}_p) &= \mathrm{rank}_{\mathbb{F}_p}(f_*\mathbb{F}_p)\chi_c(Y, \mathbb{F}_p) \\ &= (\mathrm{Card}\ G)\chi_c(Y, \mathbb{F}_p).\end{aligned}$$

The second proof begins with a reduction to the case where k is the algebraic closure of a finite field. This is done as follows:

For any X, Y, G, k satisfying the hypotheses of 1.5 we may find a subring R of k that is finitely generated over $\mathbb{F}_p$, and separated schemes X_0, Y_0 of finite type over R such that

(a) G acts on X_0/R with quotient Y_0;

(b) the map η: Spec(k) $\to$ Spec(R) induces $X = X_0 \otimes_R k$,

$$Y = Y_0 \underset{R}{\otimes} k.$$

Denote by $g: X_0 \to R$, $h: Y_0 \to R$ the structure maps. By a basic finiteness theorem (e.g. [SGA $4\frac{1}{2}$] Rapport Thm. 4.9), the sheaves on Spec(R) are constructible, so that we may, at the expense of replacing Spec(R) by a nonvoid affine open subscheme, assume that

(c) the sheaves $R^i g_! \mathbb{F}_p$, $R^i h_! \mathbb{F}_p$ are locally constant.

Now let s be a closed point of Spec(R), and $\bar{s}$ a geometric point lying above S. By (b) and the proper base change theorem we have

$$H_c^i(X, \mathbb{F}_p) = H_c^i((X_0)_\eta, \mathbb{F}_p) = \left(R^i g_! \mathbb{F}_p\right)_\eta$$

and

$$H_c^i(X_{\bar{s}}, \mathbb{F}_p) = \left(R^i g_! \mathbb{F}_p\right)_{\bar{s}}$$

and similarly for Y. By (c) we have

$$\dim_{\mathbb{F}_p}\left(R^i g_! \mathbb{F}_p\right)_\eta = \dim_{\mathbb{F}_p}\left(R^i g_! \mathbb{F}_p\right)_{\bar{s}}$$

whence

$$\chi_c(X, \mathbb{F}_p) = \chi_c(X_{\bar{s}}, \mathbb{F}_p)$$

and similarly

$$\chi_c(Y, \mathbb{F}_p) = \chi_c(Y_{\bar{s}}, \mathbb{F}_p).$$

Since s is a closed point of a scheme finitely generated over $\mathbb{F}_p$, the residue field $\kappa(s)$ is a finite field. This completes the reduction to the case $k = \overline{\mathbb{F}}_p$.

To prove the formula in this case we shall use the well-known congruence formula, due to Katz, for the zeta-function of a variety over a finite field (cf. [SGA 7] XXII or [SGA$4\frac{1}{2}$] "Fonctions L..."). Let k_0 be a finite field over which X/k has a model X_0/k_0. The congruence formula is then

$$Z(X_0/k_0, T) = \prod_i \det\left(1 - TF^* | H_c^i(X, \mathbb{F}_p)\right)^{(-1)^{i+1}} \quad \text{in } \mathbb{F}_p(T).$$

Now given X, Y, G, and $k = \overline{\mathbb{F}}_p$ we can find models X_0, Y_0 over k_0 (at least after enlarging k_0 a bit) such that G acts on X_0/k_0 with quotient Y_0.

The congruence formula written above shows that the Euler characteristics are simply the total degree of the corresponding zeta-functions, so that it is enough to show that

$$Z(X_0/k_0, T) \equiv Z(Y_0/k_0, T)^{(\mathrm{Card}\, G)} \bmod p. \tag{3.3}$$

Since $Z(X_0/K_0, T) = L(Y_0/k_0, f_*\mathbb{F}_p, T)$, 3.3 will follow from the fact that $L(Y_0/k_0, f_*\mathbb{F}_p, T)$ and $Z(Y_0/k_0, T)^{(\mathrm{Card}\ G)}$ have the same Euler factors, i.e. that

$$\det\left(1 - TF^{\deg(y)}|(f_*\mathbb{F}_p)\right) \equiv \det\left(1 - TF^{\deg(y)}|\mathbb{F}_p\right)^{(\mathrm{Card}\ G)}$$

for all closed point y of Y_0. To check this last claim, it is enough to recall that since f is etale and G is a p-group, $(f_*\mathbb{F}_p)_{\bar{y}} = \mathbb{F}_p[G]$ is a successive extension of the trivial G-modules $\mathbb{F}_p$.

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