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ON THE MULTIPLICITIES OF BINARY COMPLEX RECURRENCES

F. Beukers and R. Tijdeman

A complex homogeneous binary recurrence with constant coefficients, shortly binary recurrence, is a sequence of complex numbers u_0, u_1, $u_2, \ldots$ such that

$$u_{n+2} = Mu_{n+1} - Nu_n \qquad \text{for} \quad n = 0, 1, 2, \ldots$$

where $M, N \in \mathbb{C}$ are fixed. Throughout the paper we assume $N \neq 0$ and $|u_0| + |u_1| > 0$. We study the number of times that u_n $(n = 0, 1, 2, \ldots)$ assumes a certain value ν. We call this number the ν-multiplicity of $\{u_n\}$ and denote it by $U(\nu)$. We define the multiplicity U of $\{u_n\}$ by $U = \sup_\nu U(\nu)$. In the study of multiplicities the associated polynomial $z^2 - Mz + N = (z - \alpha)(z - \beta)$ and its discriminant $\Delta = M^2 - 4N$ play an important role. It follows from the well known theorem of Lech-Mahler [7,8] that if $U = \infty$, then at least one of the numbers α, β, α/β is a root of unity. We call the recurrence $\{u_n\}$ non-degenerate, if none of these numbers is a root of unity.

Binary recurrences of rational integers occur frequently in discrete mathematics and number theory and have been treated by several authors. In 1977 K.K. Kubota [4] confirmed a conjecture of M. Ward by proving that $U \leqslant 4$ for every non-degenerate binary recurrence. A further improvement was obtained by F. Beukers [2] who proved that

$$\sup_\nu \left(U(\nu) + U(-\nu)\right) \leqslant 3$$

with only finitely many exceptions which were explicitly given.

In order to study the ν-multiplicity of arbitrary complex binary recurrences we write the equation $u_n = \nu$ in a different form as follows. If $\Delta = 0$, then there exist constants λ, μ such that $u_n = (\lambda + \nu n)\alpha^n$ for $n = 0, 1, 2, \ldots$. If $\Delta \neq 0$, there exist constants λ, μ such that $u_n = \lambda\alpha^n + \mu\beta^n$ for $n = 0, 1, 2, \ldots$. In case $\Delta = 0$ the equation $u_n = \nu$ implies $|\nu|^2 = |\lambda + \mu n|^2|\alpha|^{2n}$. The function $|\lambda + \mu x|^2|\alpha|^{2x}$ considered as a function of the real variable x has at most two stationary points if $\mu \neq 0$ or $|\alpha| \neq 1$ and is constant if $\mu = 0$, $|\alpha| = 1$. Therefore, in case $\Delta = 0$ the recurrence has

either multiplicity at most three or infinite multiplicity. The example $u_n = (1 - 3n)(-\frac{1}{2})^n$ with $u_0 = u_1 = u_3 = 1$ shows that the bound three is best possible. In case $\Delta \neq 0$ we distinguish the cases $\nu = 0$ and $\nu \neq 0$. If $\nu = 0$, our equation reads $\lambda\alpha^n + \nu\beta^n = 0$. Without loss of generality we may assume $\beta \neq 0$. Hence $\lambda(\alpha/\beta)^n = -\mu$. We see that there is at most one solution unless α/β is a root of unity, in which case we may have infinite multiplicity. The case $\nu \neq 0$ is by far the most important. After division by ν on both sides of $u_n = \nu$ we see that it sufficies to study the equation

$$\lambda\alpha^x + \mu\beta^x = 1 \tag{1}$$

in $x \in \mathbb{Z}_{\geqslant 0}$, or more generally, in $x \in \mathbb{Z}$. In our theorems we shall deal exclusively with Eqn. (1) and mention the applications to recurrent sequences in their corollaries.

The paper is divided into four sections. Section 1 provides some definitions and properties of valuations of algebraic number fields. We define the height $h(\alpha)$ of a non-zero algebraic number α, where $h(\alpha) = 1$ if and only if α is a root of unity and $h(\alpha) > 1$ for all other non-zero algebraic numbers. In Section 2 we give some properties and estimates of certain hypergeometric polynomials. Methods involving such polynomials essentially come from an explicit version of Thue's method which enables one to obtain bounds for the number of solutions of diophantine equations.

Section 3 deals with eqn. (1) for algebraic numbers α, β, λ, μ or equivalently, binary recurrences in algebraic numbers. It follows from Theorem 1 that the multiplicity of a non-degenerate binary recurrence of algebraic numbers is at most

$$\max\left(29, 20 + \frac{5.3}{\sqrt{\log H}}\right),$$

where $H = \max(h(\alpha), h(\beta), h(\alpha/\beta))$. In particular, the multiplicity of a non-degenerate recurrence of rational numbers is at most 29, since $H \geqslant \sqrt{2}$ in this case. By applying a recent result of Dobrowolski [3] we deduce in Corollary 1 that the multiplicity of a non-degenerate recurrence of numbers in an algebraic numberfield of degree d is bounded by $100 \max(d, 300)$. Qualitatively one might do better, but the results will look more awkward. The existence of a bound depending only on d for such sequences consisting of algebraic numbers had been shown by Kubota [5]. In principle Theorem 1 also enables us to estimate the zero-multiplicity of ternary recurrences $u_{n+3} = Pu_{n+2} + Qu_{n+1} + Ru_n$. However, this question will not be dealt with here, but in a forthcoming paper.

Theorem 2 implies that every non-degenerate binary recurrence of

multiplicity at least four consists of algebraic numbers. More precisely, if $\{u_n\}$ is a non-degenerate binary recurrence and $u_0 = u_p = u_q = u_r$ for rational integers p, q, r with $0 < p < q < r$, then the sequence u_n consists of elements from an algebraic number field of a degree depending only on p, q, r. By Corollary 1 of Theorem 1 we can derive an upper bound for the multiplicity depending only on r. It would be very interesting to know whether there exists an absolute constant which is an upper bound for the multiplicity of any non-degenerate binary recurrence, but we are not able to decide this.

Theorems 3 and 4 give bounds for the multiplicities of binary recurrences of real numbers. In this case M, N and Δ are real. If $\Delta > 0$, there exist real numbers $\lambda, \mu, \alpha, \beta$ such that

$$u_n = \lambda\alpha^n + \mu\beta^n \qquad \text{for} \quad n = 0, 1, 2, \ldots .$$

If $\lambda = 0$, $\beta \in \{-1, 1\}$ or $\mu = 0$, $\alpha \in \{-1, 1\}$ or $\alpha, \beta \in \{-1, 1\}$, then the multiplicity of the recurrence is infinite. Theorem 3 states that in all other cases the multiplicity of $\{u_n\}$ is at most three. This bound is best possible as the example $u_{n+2} = u_{n+1} + u_n$ with $u_0 = 1$, $u_1 = 0$, $u_2 = u_3 = 1$ shows. Theorem 3 is a direct consequence of results of Smiley [11] and Scott [10] on 0-multiplicities of ternary real recurrences. We give a simple straightfoward proof here. If $\Delta < 0$ there exist numbers $\lambda \in \mathbb{C} \setminus \{0\}$ and $\alpha \in \mathbb{C} \setminus \mathbb{R}$ such that

$$u_n = \lambda\alpha^n + \bar{\lambda}\bar{\alpha}^n \qquad \text{for} \quad n = 0, 1, 2, \ldots .$$

Thus we want to solve the equation

$$\lambda\alpha^x + \bar{\lambda}\bar{\alpha}^x = 1 \tag{1a}$$

in $x \in \mathbb{Z}$. The number of solutions of (1a) is finite, unless α is a root of unity. Theorem 4 gives an upper bound of seven solutions if $|\alpha| \geqslant 2$. It follows that for general α with $|\alpha| \neq 1$ there are at most $7 + 5/|\log|\alpha||$ solutions. We wonder whether there exists an absolute upper bound for the number of solutions of (1a).

There is a simple geometric interpretation of this problem. Let $\alpha \in \mathbb{C}$ and l a line in the complex plane not passing through the origin. Then, for a suitable choice of λ, (1a) represents the powers of α on l. Thus, if $|\alpha| \geqslant 2$, then the number of $x \in \mathbb{Z}$ with $\alpha^x \in l$ is at most seven. A weaker result can be obtained by using Baker's method instead of hypergeometric functions, see Tijdeman [12]. Part of this result was independently discovered by J. Oesterlé.

1. Valuations of algebraic number fields

For convenience we review a few facts about valuations of number fields

which we shall use in the sequel. For a complete treatment see [6] Ch. II.

Let K be a finite extension of $\mathbb{Q}$. In K we distinguish a set of finite valuations and a set of infinite valuations. The first set corresponds to the prime ideals in K, the second to the real and complex embeddings of K. We normalise these valuations as follows. Let $\alpha \in K$. If v is a finite valuation, let p be the rational prime above v and define

$$|p|_v = p^{-d_v/d},$$

where $d = [K:\mathbb{Q}]$, $d_v = [K_v:\mathbb{Q}_v]$, and where K_v, $\mathbb{Q}_v$ are the v-adic completions of K and $\mathbb{Q}$. If v is an inifite valuation, we define

$$|\alpha|_v = |\alpha|^{1/d} \quad \text{if } v \text{ is real}, \qquad |\alpha|_v = |\alpha|^{2/d} \quad \text{if } v \text{ is complex}. \tag{2}$$

Here $|\alpha|$ is the ordinary absolute value of α embedded in $\mathbb{C}$. With these definitions we have the *product formula*

$$\prod_{v \in K} |\alpha|_v = 1 \qquad \text{for all non-zero } \alpha \in K.$$

We define the function $r(v)$ by

$$r(v) = \begin{cases} 1 & \text{if } v \text{ is finite,} \\ 2^{1/d} & \text{if } d \text{ is real,} \\ 2^{2/d} & \text{if } v \text{ is complex.} \end{cases}$$

Then we have

$$|x+y|_v \leqslant r(v) \max(|x|_v, |y|_v)$$

for all $x, y \in K$ and $\prod_{v \in K} r(v) = 2$. The height of α is defined by

$$h(\alpha) = \prod_{v \in K} \max(1, |\alpha|_v) \quad \text{for all } \alpha \neq 0. \tag{3}$$

It has the following simple properties,

(1) $h(\alpha)$ is independent of the choice of K,
(2) $h(\alpha) = h(1/\alpha)$,
(3) $h(\alpha\beta) \leqslant h(\alpha)h(\beta)$,
(4) $h(\alpha+\beta) \leqslant 2h(\alpha)h(\beta)$,
(5) $h(\alpha) = 1 \Leftrightarrow \alpha$ is root of unity.

2. Some facts on hypergeometric polynomials

Let $F(\alpha, \beta, \lambda, z)$ be the hypergeometric function given by

$$F(\alpha, \beta, \lambda, z) = 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2\,\gamma(\gamma+1)} z^2 + \ldots .$$

LEMMA 1: *Let $k \in N$ be fixed and let $m, n \in N$ and $m, n < k$. Then*

$$\binom{m+n}{n} F(-m, -n-k, -m-n, t)$$

$$-\binom{m+n}{n}(1-t)^k F(k-m, -n, -m-n, t)$$

$$= (-1)^m \binom{k+n}{k-m-1} t^{m+n+1} F(n+1, m-k+1, m+n+2, t).$$

PROOF: The polynomial $F(-m, -n-k, -m-n, t)$ satisfies the hypergeometric differential equation

$$t(t-1)F'' + \{(1-m-n-k)t + m + n\}F' + m(n+k)F = 0.$$

The polynomials $(1-t)^k F(k-m, -n, -m-n, t)$ and $t^{m+n+1}F(n+1, m-k+1, m+n+2, t)$ also satisfy this equation. Hence there exists a linear relation between these functions. The coefficients of this linear relationship can be found by considering the constant term and the coefficient of the highest power of t.

LEMMA 2: *Let k, m, n be as in* Lemma 1. *Define*

$$p_{mn}(t) = \binom{m+n}{n} F(-m, -n-k, -m-n, t)$$

$$q_{mn}(t) = \binom{m+n}{n} F(k-m, -n, -m-n, t)$$

$$f_{mn}(t) = \binom{k+n}{k-m-1} F(n+1, m-k+1, m+n+2, t).$$

Then $p_{mn}(t)$, $q_{mn}(t)$, $f_{mn}(t)$ are polynomials of degree m, n and $k-m-1$ respectively and have coefficients in $\mathbb{Z}$.

PROOF: We have

$$p_{mn}(t) = \binom{m+n}{n} F(-m, -n-k, -m-n, t)$$

$$= \sum_{r=0}^{m} \binom{m+n}{n} \times \frac{(-m)\ldots(-m+r-1)(-n-k)\ldots(-n-k+r-1)}{r!(-m-n)\ldots(-m-n+r-1)} t^r$$

$$= \sum_{r=0}^{m} \frac{(m+n)!}{m!n!} \frac{m!}{r!(m-r)!} \frac{(n+k)!}{(n+k-r)!} \frac{(m+n-r)!}{(m+n)!} (-t)^r$$

$$= \sum_{r=0}^{m} \binom{m+n-r}{n}\binom{n+k}{r}(-t)^r.$$

In a completely analogous way we find that

$$q_{mn}(t) = \sum_{r=0}^{n} \binom{m+n-r}{m}\binom{k-m+r-1}{r}t^r$$

and

$$f_{mn}(t) = \sum_{r=0}^{k-m-1} \binom{n+r}{r}\binom{k+n}{m+n+r+1}(-t)^r.$$

LEMMA 3: *Let* p_{mn}, q_{mn}, f_{mn} *be as in* Lemma 2. *Then*

$$p_{mn}(t) = \frac{(k+n)!}{(k-m-1)!m!n!}\int_0^1 (1-x)^n x^{k-m-1}(1-x-t)^m \,\mathrm{d}x$$

$$q_{mn}(t) = \frac{(k+n)!}{(k-m-1)!m!n!}\int_0^1 (1-x)^m x^{k-m-1}(1-x+tx)^n \,\mathrm{d}x$$

$$f_{mn}(t) = \frac{(k+n)!}{(k-m-1)!m!n!}\int_0^1 x^n (1-x)^m (1-tx)^{k-m-1} \,\mathrm{d}x.$$

PROOF: These formulae can be checked by writing down the binomial expansion of $(1-x-t)^m$, $(1-x+tx)^n$ and $(1-tx)^{k-m-1}$ and then performing the integration, where we make use of the identity

$$\int_0^1 (1-x)^p x^q \,\mathrm{d}x = \frac{p!q!}{(p+q+1)!}.$$

LEMMA 4: *Let* $p_{mn}(t)$, $q_{mn}(t)$ *be as in* Lemma 2. *Then*

$$p_{mn}q_{m,n-1} - p_{m,n-1}q_{mn} \neq 0 \qquad \textit{for all } t \neq 0.$$

PROOF: We have by Lemma 1 and 2,

$$p_{mn}(t) - (1-t)^k q_{mn}(t) = (-1)^m t^{m+n+1} f_{mn}(t)$$

and

$$p_{m,n-1}(t) - (1-t)^k q_{m,n-1}(t) = (-1)^m t^{m+n} f_{m,n-1}(t).$$

Elimination of $(1-t)^k$ yields that

$$p_{mn}(t)q_{m,n-1}(t) - p_{m,n-1}(t)q_{m,n}(t)$$

equals t^{m+n} times some polynomial. However, $p_{mn}q_{m,n-1} - p_{m,n-1}q_{mn}$ is a polynomial of degree $m+n$ with a non-zero leading coefficient a_0. It is divisible by t^{m+n}, hence

$$p_{mn}q_{m,n-1} - p_{m,n-1}q_{mn} = a_0 t^{m+n}, \qquad \text{where } a_0 \neq 0.$$

Hence our Lemma follows for all $t \neq 0$.

LEMMA 5: *Let $r \in N$. Then*

$$\frac{(3r)!}{(r-1)!r!r!} < (0.28).27^r.$$

PROOF: We have

$$\frac{(3r)!}{(r-1)!r!r!} = 6 \prod_{k=2}^{r} \frac{3k(3k-1)(3k-2)}{(k-1)k.k}.$$

$$= 6.27^{r-1} \prod_{k=2}^{r} \left(1 + \frac{2/9}{k(k-1)}\right)$$

$$< 6.27^{r-1} \exp\left(\sum_{k=2}^{r} \frac{2}{9} \frac{1}{k(k-1)}\right)$$

$$< 6.27^{r-1} \exp(2/9) < (0.28)27^r.$$

LEMMA 6: *Let $r \in N$ and let γ, δ be fixed non-zero algebraic numbers in an algebraic number field K. Then there exist polynomials A_r, B_r, $C_r \in \mathbb{Z}[z]$ of degree at most r, such that*

$$A_r(z) - (1-z)^{2r} B_r(z) = z^{2r} C_r(z), \qquad \forall z \in \mathbb{C}. \tag{4}$$

$$A_r(\gamma) - \delta B_r(\gamma) \neq 0 \tag{5}$$

$$\max(|A_r(\zeta)|_v, |B_r(\zeta)|_v, |C_r(\zeta)|_v)$$

$$\leqslant (6\sqrt{3})^{r(\log r(v)/\log 2)} \max(1, |\zeta|_v^r) \tag{6}$$

for any $\zeta \in K$ and any valuation v of K.

PROOF: Put $k = 2r$. Let m, n, $p_{m,n}$,... be as in the preceding lemmas. Suppose that $p_{r,r}(\gamma) - \delta q_{r,r}(\gamma) \neq 0$. Then we take $A_r(z) = p_{rr}(z)$, $B_r(z) = q_{rr}(z)$, $C_r(z) = (-1)^r z f_{rr}(z)$. By Lemma 1, condition (4) is satisfied and (5) is automatically satisfied. If $p_{rr}(\gamma) - \delta q_{rr}(\gamma) = 0$ then, by Lemma 4, $p_{r,r-1}(\gamma) - \delta q_{r,r-1}(\gamma) \neq 0$. Then we take $A_r(z) = p_{r,r-1}(z)$, $B_r = q_{r,r-1}(z)$

and $C_r = (-1)^r f_{r,r-1}(z)$. By Lemma 1, condition (4) is again satisfied together with condition (5).

It remains to show the estimates (6). Since q_{mn} has positive coefficients and f_{mn}, p_{mn} have alternating coefficients, we have

$$|q_{mn}(z)| \leqslant q_{mn}(1) \max(1, |z|^n),$$

$$|p_{mn}(z)| \leqslant p_{mn}(-1) \max(1, |z|^m),$$

$$|f_{mn}(z)| \leqslant f_{mn}(-1) \max(1, |z|k - m - 1).$$

We assert that

$$|q_{rr}(1)|, |q_{r,r-1}(1)|, |p_{rr}(-1)|, |p_{r,r-1}(-1)|, |f_{rr}(-1)|, |f_{r,r-1}(-1)| < (6\sqrt{3})^r.$$

This suffices for the proof of (6) for archimedean v. For non-archimedean v estimate (6) is trivial, since A_r, B_r, $C_r \in \mathbb{Z}[z]$. We now show $|p_{rr}(-1)| < (6\sqrt{3})^r$ and leave the proof of the other estimates to the reader since they are entirely similar.

On using the integral expression for p_{rr} in Lemma 3, we find

$$p_{rr}(-1) = \frac{(3r)!}{(r-1)!r!r!} \int_0^1 (1-x)^r x^{r-1} (2-x)^r \, dx.$$

It is straightforward to see that $|x(1-x)(2-x)| \leqslant 2/3\sqrt{3}$ for all $0 \leqslant x \leqslant 1$. Together with the estimate from Lemma 5 this yields

$$p_{rr}(1) < (0.28)\, 27^r \left(\frac{2}{3\sqrt{3}}\right)^{r-1} \int_0^1 (1-x)(2-x)\, dx < (6\sqrt{3})^r.$$

One cannot expect to improve on this estimate asymptotically, since it is not hard to show that $r^{-1} \log p_{rr}(-1) \to \log 6\sqrt{3}$ as $r \to \infty$.

3. Binary recurrences of algebraic numbers

Let α, β, λ, $\mu \in K$, where $[K:\mathbb{Q}] < \infty$. We suppose throughout this section that $\lambda\mu\alpha\beta \neq 0$. Consider the equation

$$\lambda\alpha^x + \mu\beta^x = 1 \tag{7}$$

in the rational integer x. Lemmas 7, 8, 9 provide an upper bound for some of the larger solutions of (7).

LEMMA 7: *Suppose* $\lambda\alpha^x + \mu\beta^x = 1$ *has the solutions* $x = 0 < p < l < m$ *with* $m \geqslant 10l$. *Let* $H = \max(h(\alpha), h(\beta), h(\beta/\alpha))$ *and suppose* $H > 1$. *Then*

$$l \leqslant 27\frac{\log 2}{\log H} + \frac{50}{3}p.$$

PROOF: Put $m = 2lr + \delta l$ with $r \in N$, δ real and $0 \leqslant \delta < 2$. Without loss of generality we may assume $h(\alpha) = H$. Apply Lemma 6(4) with $z = \lambda\alpha^l$,

$$A_r(\lambda\alpha^l) - (\mu\beta^l)^{2r}B_r(\lambda\alpha^l) = (\lambda\alpha^l)^{2r}C_r(\lambda\alpha^l).$$

On using $m = 2lr + \delta l$ in the exponent of β and $\mu\beta^m = 1 - \lambda\alpha^m$ we find

$$\Delta \overset{\text{def}}{=} A_r(\lambda\alpha^l) - \mu^{2r-1}\beta^{-\delta l}B_r(\lambda\alpha^l)$$

$$= (\lambda\alpha^l)^{2r}\left\{C_r(\lambda\alpha^l) - \left(\frac{\mu}{\lambda}\right)^{2r-1}\left(\frac{\alpha}{\beta}\right)^{\delta l}B_r(\lambda\alpha^l)\right\}.$$

For all valuations v of K with $|\lambda\alpha^l|_v < 1$ we estimate Δ by

$$|\Delta|_v \leqslant |\lambda\alpha^l|_v^{2r} r(v)(6\sqrt{3})^{r(\log r(v)/\log 2)} \max\left(1, \left|\frac{\mu}{\lambda}\right|_v^{2r-1+\delta}\left|\frac{\lambda\alpha^l}{\mu\beta^l}\right|_v^{\delta}\right)$$

$$\leqslant |\lambda\alpha^l|_v^{2r} r(v)(6\sqrt{3})^{r(\log r(v)/\log 2)} \max\left(1, \left|\frac{1-\lambda}{\lambda}\right|_v^{2r-1+\delta}\right)$$

$$\times \max\left(1, \frac{1}{|\mu\beta^l|_v^{\delta}}\right)$$

$$\leqslant |\lambda\alpha^l|_v^{2r} r(v)^{2r+\delta}(6\sqrt{3})^{r(\log r(v)/\log 2)} \max\left(1, \frac{1}{|\lambda|_v^{2r-1+\delta}}\right)$$

$$\times \max\left(1, \frac{1}{|\mu\beta^l|_v^{\delta}}\right). \tag{8}$$

If $|\lambda\alpha^l|_v \geqslant 1$, we estimate $|\Delta|_v$ by

$$|\Delta|_v \leqslant r(v)(6\sqrt{3})^{r(\log r(v)/\log 2)}|\lambda\alpha^l|_v^r \max\left(1, |\mu|_v^{2r-1+\delta}|\mu\beta^l|_v^{-\delta}\right)$$

$$\leqslant r(v)^{2r+\delta}(6\sqrt{3})^{r(\log r(v)/\log 2)}|\lambda\alpha^l|_v^r \max\left(1, |\lambda|_v^{2r-1+\delta}\right)$$

$$\times \max\left(1, \frac{1}{|\mu\beta^l|_v^{\delta}}\right).$$

Using the product-formula $\Pi_v |\Delta|_v = 1$ and the estimates (8), (9) we find, by $\Pi_v r(v) = 2$,

$$1 \leqslant \frac{2^{2r+\delta}}{h(\lambda\alpha^l)^r}(6\sqrt{3})^r h(\lambda)^{2(2r-1+\delta)} h(\mu\beta^l)\delta,$$

hence

$$1 \leqslant \frac{4^{r+\delta}}{h(\lambda\alpha^l)^r}(6\sqrt{3})^r h(\lambda)^{2(2r-1+\delta)} h(\lambda\alpha^l)^\delta.$$

Using $h(\lambda\alpha^l \geqslant h(\alpha^l)/h(\lambda)$ we find

$$h(\alpha^l)^{r-\delta} \leqslant 4^{r+\delta}(6\sqrt{3})^r h(\lambda)^{5r-2+\delta}. \tag{10}$$

From $\lambda + \mu = 1$ and $\lambda\alpha^p + \mu\beta^p = 1$ it follows that $\lambda = (\beta^p - 1)/(\beta^p - \alpha^p)$ and hence

$$h(\lambda) = h\left(\frac{1-\beta^{-p}}{1-(\alpha/\beta)^p}\right) \leqslant h(1-\beta^{-p})h\left(1-(\alpha/\beta)^p\right) \leqslant 4h(\alpha)^{2p}.$$

Thus we deduce from (10),

$$h(\alpha^l)^{r-\delta} \leqslant 4^{6r-2+2\delta}(6\sqrt{3})^r h(\alpha)^{2(5r-2+\delta)p}.$$

Finally, using $r \geqslant 5$, $0 \leqslant \delta < 2$ we obtain

$$h(\alpha)^l \leqslant 4^{32/3}(6\sqrt{3})^{5/3} h(\alpha)^{50p/3}$$

from which our Lemma follows.

LEMMA 8: *Suppose* $\lambda\alpha^x + \mu\beta^x = 1$ *has the solutions* $x = 0 < p < l < m$ *with* $m \leqslant \frac{3}{2}l$. *Let* $H = \max(h(\alpha), h(\beta), h(\beta/\alpha))$ *and suppose* $H > 1$. *Then*

$$l \leqslant 10\frac{\log 2}{\log H} + 6p.$$

PROOF: As in the proof of Lemma 7, we assume $H = h(\alpha)$. Put $m = l + \delta l$ with $0 \leqslant \delta \leqslant \frac{1}{2}$. From $1 - \mu\beta^l = \lambda\alpha^l$ and $\mu\beta^m = 1 - \lambda\alpha^m$ it is easily follows that

$$1 - \beta^{l-m} = \lambda\alpha^l\left(1 - \left(\frac{\alpha}{\beta}\right)^{m-l}\right).$$

If $|\lambda\alpha^l|_v < 1$, then we estimate as follows,

$$|1-\beta^{-\delta l}|_v \leqslant r(v)|\lambda\alpha^l| \max\left(1, \left|\frac{\mu}{\lambda}\right|_v^{\delta} \left|\frac{\lambda\alpha^l}{\mu\beta^l}\right|_v^{\delta}\right)$$

$$\leqslant r(v)^{1+\delta}|\lambda\alpha^l|_v \max\left(1, \left|\frac{1}{\lambda}\right|_v^{\delta}\right) \max\left(1, \frac{1}{|\mu\beta^l|_v^{\delta}}\right). \tag{11}$$

If $|\lambda\alpha^l|_v \geqslant 1$, then we estimate

$$|1-\beta^{-\delta l}|_v \leqslant r(v) \max\left(1, |\mu|_v^{\delta}|\mu\beta^l|_v^{-\delta}\right)$$

$$\leqslant r(v)^{1+\delta} \max\left(1, |\lambda|_v^{\delta}\right) \max\left(1, |\mu\beta^l|_v^{-\delta}\right). \tag{12}$$

Notice that β^{m-l} cannot be 1, since this would imply $\alpha^{m-l}=1$, and hence $H=1$. Thus we can apply the product formula $\Pi_v|1-\beta^{-\delta l}|_v = 1$. Together with (11) and (12) we obtain

$$1 \leqslant \frac{2^{1+\delta}}{h(\lambda\alpha^l)} h(\lambda)^{2\delta} h(\mu\beta^l)^{\delta}.$$

By using $h(\mu\beta^l) \leqslant 2h(\lambda\alpha^l)$, $h(\lambda\alpha^l) \geqslant h(\alpha^l)/h(\lambda)$ and $h(\lambda) \leqslant 4h(\alpha)^{2p}$ successively, we find

$$h(\alpha^l)^{1-\delta} \leqslant 2^{3+4\delta} h(\alpha)^{2(1+\delta)p}$$

and our Lemma follows.

LEMMA 9: *Let* 0, p, l *be solutions of* (7) *with* $0 < p < l$. *Suppose there are nine solutions* $x_1 < x_2 < \ldots < x_9$ *larger than* l. *Then*

$$l \leqslant 10\frac{\log 2}{\log H} + 6p.$$

PROOF: Put $x_0 = l$. Suppose $x_{r+1}/x_r \leqslant \frac{3}{2}$ for at least one r. Then application of Lemma 8 yields

$$l \leqslant x_r \leqslant 10\frac{\log 2}{\log H} + 6p,$$

as desired. Now suppose $x_{r+1}/x_r > \frac{3}{2}$ for $0 \leqslant r \leqslant 8$. Then

$$x_9/x_3 > (3/2)^6 > 11.$$

Application of Lemma 8 yields

$$x_3 < 27\frac{\log 2}{\log H} + \frac{50}{3}p$$

and since $x_3 > (3/2)^3 x_0 = (3/2)^3 l$, we find that

$$l < 8\frac{\log 2}{\log H} + 5p.$$

which proves our Lemma.

The following Lemma shows that a given difference between solutions of (7) can occur at most once.

LEMMA 10: *Suppose* $\lambda\alpha^x + \mu\beta^x = 1$ *has the solutions* q, $q+d$, r, $r+d$ *with* $r > q$. *Then* α *and* β *are roots of unity.*

PROOF: From $1 = \lambda\alpha^q + \mu\beta^q = \lambda\alpha^{q+d} + \mu\beta^{q+d}$ and $1 = \lambda\alpha^r + \mu\beta^r = \lambda\alpha^{r+d} + \mu\beta^{r+d}$ it follows for the coefficient determinant of λ, μ that

$$\begin{vmatrix} \alpha^{q+d} - \alpha^q & \beta^{q+d} - \beta^q \\ \alpha^{r+d} - \alpha^r & \beta^{r+d} - \beta^r \end{vmatrix} = 0.$$

On the other hand, this determinant is equal to $(\alpha^d - 1)(\beta^d - 1)\beta^q\alpha^r((\beta/\alpha)^{r-q} - 1)$. The vanishing of the determinant implies the vanishing of at least one of the factors. Thus α or β is a d-th root of unity or β/α is an $(r-q)$th root of unity. If α is a d-th root of unity, then it follows from $\lambda\alpha^q + \mu\beta^q = \lambda\alpha^{q+d} + \mu\beta^{q+d}$ and $\mu \neq 0$ that β is a d-th root of unity and vice versa. If β/α is a $(r-q)$th root of unity, we have $\alpha^{-q} = \lambda + \mu(\beta/\alpha)^q$, $\alpha^{-r} = \lambda + \mu(\beta/\alpha)^r$ and since $(\beta/\alpha)^r = (\beta/\alpha)^q$ it follows that $\alpha^{-q} = \alpha^{-r}$. Hence α is an $(r-q)$th root of unity, and similarly for β.

THEOREM 1: *Let* α, β, λ μ *be non-zero algebraic numbers. Let* $H = \max(h(\alpha), h(\beta), h(\alpha/\beta))$ *and suppose* $H > 1$. *Then the equation*

$$\lambda\alpha^x + \mu\beta^x = 1 \tag{7}$$

in the rational integer x *has at most*

$$\max\left(29, 20 + \frac{5.3}{\sqrt{\log H}}\right)$$

solutions.

PROOF: Write the sequence of solutions as $\ldots < x_{-1} < x_0 < x_1 < \ldots$ where x_0 is chosen in such a way that $x_1 - x_0 = \min_r(x_{r+1} - x_r)$. Put $p = x_1 - x_0$. Instead of our original equation we might as well consider $\lambda\alpha^{x_0}\alpha^x + \mu\beta^{x_0}\beta^x = 1$, which has $x_t - x_0$ as solutions. Hence there is no loss of generality in assuming that the sequence of solutions is given by $\ldots < x_{-1} < 0 < p < x_2 < \ldots$ and $p \leqslant x_{r+1} - x_r$ for all r.

By Lemma 9 there are at most nine solutions larger than $10 \log 2/\log H + 6p$. On applying the same argument to the equation $\lambda\alpha^p(\alpha^{-1})^x + \mu\beta^p(\beta^{-1})^x = 1$ we obtain that there are at most nine solutions smaller than $-10 \log 2/\log \mathrm{H} - 5p$. Let N be the total number of solutions of (7). Then at least $N - 18$ solutions are located in an interval of length $20 \log 2/\log H + 11p$. The differences between consecutive solutions must all be different according to Lemma 1. Hence

$$p + (p+1) + (p+2) + \ldots + (p+N-20) \leqslant 20\frac{\log 2}{\log H} + 11p$$

which amounts to

$$(N-30)p + \tfrac{1}{2}(N-19)(N-20) \leqslant 20\frac{\log 2}{\log H}.$$

If $N > 29$, then

$$\tfrac{1}{2}(N-20)^2 \leqslant 20\frac{\log 2}{\log H},$$

hence

$$N \leqslant 20 + \sqrt{\frac{40 \log 2}{\log H}}.$$

Thus we have

$$N \leqslant \max\left(29, 20 + \frac{5.3}{\sqrt{\log H}}\right).$$

COROLLARY 1: *Let K be an algebraic number field of degree d. Let $u_n \in K$ $(n = 0, 1, 2, \ldots)$ be a binary recurrence given by $u_{n+2} = Mu_{n+1} - Nu_n$ with $M, N, u_0, u_1 \in K$ such that $M^2 - 4N \neq 0$, $N \neq 0$ and $u_1^2 - Mu_1u_0 + Nu_0^2 \neq 0$. Suppose that $x^2 - Mx + N = 0$ has at most one root of unity as a solution. Let $a \in K$, $a \neq 0$. Then the equation $u_n = a$ has at most*

$$100 \max(d, 300) \text{ solutions } n \geqslant 0.$$

PROOF: Let α, β be the roots of $x^2 - Mx + N = 0$. Since $M^2 - 4N \neq 0$, we have $\alpha \neq \beta$ and by $N \neq 0$, $\alpha\beta \neq 0$. The recurrence can be written as $u_n = \lambda\alpha^n$ or $u_n = \mu\beta^n$ or $u_n = \lambda\alpha^n + \mu\beta^n$ ($\lambda\mu \neq 0$), where λ, $\mu \in K(\alpha, \beta)$. In the first two cases we have $u_1^2 - Mu_1u_0 + Nu_0^2 = 0$ which we have excluded from our considerations. Thus we have

$$u_n = \lambda\alpha^n + \mu\beta^n.$$

The multiplicity of this sequence is bounded by

$$\max\left(29, 20 + \frac{5.3}{\sqrt{\log H}}\right),$$

where $H = \max(h(\alpha), h(\beta), h(\alpha/\beta))$, according to Theorem 1. The numbers α, β, α/β lie in a number field of degree $2d$ over $\mathbb{Q}$.

Let θ be a number in a number field of degree D. If θ is not a root of unity, then we have according to Dobrowolski [3],

$$h(\theta) = M(\theta)^{1/D} > \left(1 + \frac{1}{1200}\left(\frac{\text{loglog } D}{\log D}\right)^3\right)^{1/D}.$$

Taking $D = \max(2d, 600)$ we find a lower bound for H, from which our Corollary follows.

4. The equation $\lambda\alpha^x + \mu\beta^x = 1$ for complex numbers α,β,λ,μ

As before, we assume $\alpha\beta\lambda\mu \neq 0$. The following theorem implies that if the above equation has at least four integer solutions, then α, β are algebraic.

THEOREM 2: *Let α, β, λ μ be non-zero complex numbers such that the equation $\lambda\alpha^x + \mu\beta^x = 1$ in rational integers x has the solutions $x = 0, p, q, r$ with $0 < p < q < r$. Then either α, β, λ, μ are algebraic numbers in a field of degree at most $(p + q - 3)(p + r - 3)$, or α and β are d-th roots of unity, where $d = (p, q, r)$.*

PROOF: Let $d = (p, q, r)$. Suppose that one of α, β, α/β is a d-th root of unity, α say. Then we have

$$\lambda + \mu = \lambda + \mu\beta^p = \lambda + \mu\beta^q = \lambda + \mu\beta^r = 1.$$

Hence $1 = \beta^p = \beta^q = \beta^r$ and thus β is also a d-th root of unity. From now on we assume that none of α, β, α/β is a d-th root of unity.

Suppose $d = 1$. Eliminate λ and μ from the three equations obtained

by putting $x = 0$, p, q in $\lambda\alpha^x + \mu\beta^x = 1$. We find that

$$(\beta^p - 1)(\alpha^q - 1) - (\alpha^p - 1)(\beta^q - 1) = 0,$$

that is, the point (α, β) lies on the algebraic curve

$$F(X, Y) = (X^q - 1)(Y^p - 1) - (Y^q - 1)(X^p - 1) = 0.$$

By taking r instead of q we also see that (α, β) lies on the curve

$$G(X, Y) = (X^r - 1)(Y^p - 1) - (Y^r - 1)(X^p - 1) = 0.$$

Notice that

$$F(X, Y) = (X - 1)(Y - 1)(Y - X)\, F_1(X, Y),$$

$$G(X, Y) = (X - 1)(Y - 1)(Y - X)\, G_1(X, Y)$$

for certain polynomials F_1, G_1. Since 1, α, β are distinct, the point (α, β) lies in the intersection of $F_1(X, Y) = 0$ and $G_1(X, Y) = 0$. It suffices to show that F_1 and G_1 have no factors in common, since then, by Bezout's theorem, the number of points in the intersection of $F_1 = 0$ and $G_1 = 0$ is bounded by $(p + q - 3)(p + r - 3)$. Let D be the degree of $K = \mathbb{Q}(\alpha, \beta)$ over $\mathbb{Q}$. Then the D points $(\sigma\alpha, \sigma\beta)$, $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ are all distinct and lie also in the intersection of $F_1 = G_1 = 0$. Hence $D \leqslant (p + q - 3)(p + r - 3)$.

Assume $F_1(X, Y) = P(X, Y)F_2(X, Y)$ and $G_1(X, Y) = P(X, Y)G_2(X, Y)$. Let ζ be a primitive r-th root of unity. Consider $G(\zeta, Y) = (1 - \zeta^p)(Y^r - 1)$. We see that $\deg_y G(\zeta, Y) = \deg_y G(X, Y)$ and hence $\deg_y P(\zeta, Y) = \deg_y P(X, Y)$. Furthermore, the fact that $(1 - \zeta^p)(Y^r - 1) = (\zeta - 1)(Y - 1)(Y - \zeta)P(\zeta, Y)\, G_2(\zeta, Y)$ implies that every root $Y = \rho$ of $P(\zeta, Y)$ is an r-th root of unity with $\rho \neq 1, \zeta$. Rewrite $F(\zeta, \rho) = 0$ as

$$(\zeta^q - 1)\rho^p - (\zeta^p - 1)\rho^q + \zeta^p - \zeta^q = 0.$$

The solution of the equation $(\zeta^q - 1)t - (\zeta^p - 1)s + \zeta^p - \zeta^q = 0$ in s, $t \in \mathbb{C}$ with $|s| = |t| = 1$ comes down to determining all triangles in $\mathbb{C}$ with sides of lengths $|\zeta^q - 1|$, $|\zeta^p - 1|$, $|\zeta^p - \zeta^q|$ and having the vector $\zeta^p - \zeta^q$ as basis. It is clear that there are no more than two solutions. The pairs $(t, s) = (1, 1)$ and (ζ^p, ζ^q) provide two different solutions. Because of $(p, q, r) = 1$ we deduce that $\rho = 1$ or $\rho = \zeta$, which is a contradiction. Hence $P(\zeta, Y)$ is a constant. Since $\deg_y P(\zeta, Y) = \deg_y P(X, Y)$, we have $P(X, Y) \in \mathbb{C}[X]$. Similarly we deduce that $P(X, Y) \in \mathbb{C}[Y]$. Thus $P(X, Y)$ is a constant. This proves the assertion for $d = 1$.

Suppose $d > 1$. Consider the equation

$$\lambda(\alpha^d)^x + \mu(\beta^d)^x = 1$$

which has solutions $x = 0$, p/d, q/d, r/d. Notice that α^d, β^d, 1 are distinct and $(p/d, q/d, r/d) = 1$. We obtain

$$[\mathbb{Q}(\lambda, \mu, \alpha^d, \beta^d):\mathbb{Q}] \leqslant \left(\frac{p}{d} + \frac{q}{d} - 3\right)\left(\frac{p}{d} + \frac{r}{d} - 3\right)$$

and hence

$$[\mathbb{Q}(\lambda, \mu, \alpha, \beta):\mathbb{Q}] \leqslant d^2\left(\frac{p}{d} + \frac{q}{d} - 3\right)\left(\frac{p}{d} + \frac{r}{d} - 3\right)$$

$$< (p + q - 3)(p + r - 3).$$

The following theorem gives an upper bound for the number of solutions in the real case.

THEOREM 3 (Smiley-Scott [11,10,9]): *Let α, β, λ, μ be non-zero real numbers such that α, β are not both in $\{-1, 1\}$. Let $\nu \in \mathbb{R}$. Then the equation $\lambda\alpha^x + \mu\beta^x = \nu$ has at most three solutions $x \in \mathbb{Z}$.*

PROOF: Let γ, δ, ξ, η be real numbers with $\xi\eta \neq 0$, $\gamma > 0$, $\delta > 0$, $|\gamma - 1| + |\delta - 1| > 0$ and s a real variable. Then the function $\xi\gamma^s + \eta\delta^s$ has a stationary point if $\xi\eta \log \gamma \log \delta < 0$ and none otherwise. Hence it assumes a certain value at most twice if $\xi\eta \log \gamma \log \delta < 0$ and at most once otherwise. On applying this to $\lambda\alpha^s + \mu\beta^s$ we see that there are at most two solutions x if $\alpha > 0$, $\beta > 0$. An application to $\lambda(\alpha^2)^x + \mu(\beta^2)^s$ reveals that there are at most two even solutions if $\lambda\mu \log(\alpha^2) \log(\beta^2) < 0$ and at most one otherwise, and to $\lambda\alpha(\alpha^2)^s + \mu\beta(\beta^2)^s$ that there are at most two odd solutions if $\lambda\mu\alpha\beta \log(\alpha^2) \log(\beta^2) < 0$ and at most one otherwise. Suppose there are four integer solutions x of $\lambda\alpha^x + \mu\beta^x = \nu$. Without loss of generality we may assume that they are given by 0, p, q, r with $p > 0$, $q > 0$, $r > 0$. It follows that $\alpha < 0$, $\beta < 0$, $\alpha \neq -1$, $\beta \neq -1$ and exactly two of the solutions, q and r say, are odd. By eliminating λ, μ, ν we obtain $(1 - \alpha^q)/(1 - \beta^q) = (1 - \alpha^r)/(1 - \beta^r)$. Since the function $(1 + \gamma^s)/(1 + \delta^s)$ in the variable $s > 0$ is strictly monotonic for $\gamma \neq \delta$, this is a contradiction.

COROLLARY: *Let $\{u_n\}$ be a recurrence of real numbers given by $u_{n+2} + Mu_{n+1} - Nu_n$ with M, $N \in \mathbb{R}$, $N \neq 0$, $(M, N) \neq (0, -1)$, $M^2 > 4N$ and u_0, u_1 not both zero. Let $\nu \in \mathbb{R}$. Then the equation $u_n = \nu$ has at most three solutions $n \in \mathbb{Z}$.*

The next theorem gives an upper bound for the number of solutions in the so-called complex case: $\alpha = \bar{\beta}$, $\lambda = \bar{\mu}$, $\alpha \neq \beta$.

THEOREM 4: *Let* $\alpha, \lambda \in \mathbb{C}$ *and* $|\alpha| \geqslant 2$. *Then the equation* $\lambda\alpha^x + \bar{\lambda}\bar{\alpha}^x = 1$ *has at most seven solutions* $x \in \mathbb{Z}$.

Obviously we may assume that $\alpha \notin \mathbb{R}$. For the proof of Theorem 4 we need the following lemma.

LEMMA 11: *Let* $\alpha, \lambda \in \mathbb{C}$ *with* $|\alpha| \geqslant 2$, $\alpha \in \mathbb{R}$. *Suppose* $\lambda\alpha^x + \bar{\lambda}\bar{\alpha}^x = 1$ *has integer solutions* $x = 0, k, l, m$ *with* $2 \leqslant k < l < m$. *Then*

$$m - k > 2|\alpha|^k.$$

PROOF: Eliminate $\lambda, \bar{\lambda}$ from the expressions obtained by putting $x = 0, k, l, m$ in $\lambda\alpha^x + \bar{\lambda}\bar{\alpha}^x = 1$. We find

$$\frac{\bar{\alpha}^k - 1}{\alpha^k - 1} = \frac{\bar{\alpha}^l - 1}{\alpha^l - 1} = \frac{\bar{\alpha}^m - 1}{\alpha^m - 1}.$$

If these quotients equal one, then α^k, α^l, α^m are real and hence $(\lambda + \bar{\lambda})\alpha^k = (\lambda + \bar{\lambda})\alpha^l = (\lambda + \bar{\lambda})\alpha^m = 1$ which is clearly impossible, since $|\alpha| \neq 1$. Put

$$\eta = \frac{\bar{\alpha}^k - 1}{\alpha^k - 1} - 1.$$

Thus, $\eta \neq 0$. Notice that

$$\frac{\bar{\alpha}^k - 1}{\alpha^k - 1} = \left(\frac{\bar{\alpha}}{\alpha}\right)^k + \frac{\eta}{\alpha^k}, \quad \frac{\bar{\alpha}^l - 1}{\alpha^l - 1} = \left(\frac{\bar{\alpha}}{\alpha}\right)^l + \frac{\eta}{\alpha^l}, \quad \frac{\bar{\alpha}^m - 1}{\alpha^m - 1} = \left(\frac{\bar{\alpha}}{\alpha}\right)^m + \frac{\eta}{\alpha^m}.$$

Hence

$$\left(\frac{\bar{\alpha}}{\alpha}\right)^{l-k} - 1 = \eta\left(\frac{\alpha}{\bar{\alpha}}\right)^k \left(\frac{1}{\alpha^k} - \frac{1}{\alpha^l}\right),$$

$$\left(\frac{\bar{\alpha}}{\alpha}\right)^{m-l} - 1 = \eta\left(\frac{\alpha}{\bar{\alpha}}\right)^l \left(\frac{1}{\alpha^l} - \frac{1}{\alpha^m}\right). \tag{13}$$

Observe that $|\eta| \leqslant 2$ and $|\eta(\alpha^{-k} - \alpha^{-l})| \leqslant 2|\alpha|^{-k}(1 + \frac{1}{2}) \leqslant 1$. For any complex number w with $|1 + w| = 1$ and $|w| \leqslant 1$ the inequalities $|w| \leqslant |\mathrm{Arg}(1 + w)| \leqslant \pi/3|w|$ hold. Hence, by (13),

$$(l - k)\,\mathrm{Arg}\frac{\bar{\alpha}}{\alpha} + 2\pi r$$

$$= \operatorname{Arg}\left[1 + \eta\left(\frac{\alpha}{\bar{\alpha}}\right)^k \left(\frac{1}{\alpha^k} - \frac{1}{\alpha^l}\right)\right] = \mu\left|\frac{1}{\alpha k} - \frac{1}{\alpha^l}\right||\eta|,$$

$$(m-l)\operatorname{Arg}\frac{\bar{\alpha}}{\alpha} + 2\pi s$$

$$= \operatorname{Arg}\left[1 + \eta\left(\frac{\alpha}{\bar{\alpha}}\right)^l \left(\frac{1}{\alpha^l} - \frac{1}{\alpha^m}\right)\right] = \nu\left|\frac{1}{\alpha^l} - \frac{1}{\alpha^m}\right||\eta|,$$

where r and s are integers and μ and ν are real numbers with $1 \leqslant |\mu| \leqslant \pi/3$ and $1 \leqslant |\nu| \leqslant \pi/3$. We eliminate $\operatorname{Arg}(\bar{\alpha}/\alpha)$ from these equalities and find that

$$E := (m-l)\mu\left|\frac{1}{\alpha^k} - \frac{1}{\alpha^l}\right||\eta| - (l-k)\nu\left|\frac{1}{\alpha^l} - \frac{1}{\alpha^m}\right||\eta| \tag{14}$$

is either zero or larger than 2π in absolute value.

We first show that $E \neq 0$. Suppose $E = 0$. Then, by $\eta \neq 0$,

$$\frac{m-l}{l-k} = \left|\frac{\nu}{\mu}\right| \frac{|1 - \alpha^{l-m}|}{\alpha^{l-k} - 1|}.$$

Hence

$$\frac{m-l}{l-k} \leqslant \frac{\pi}{3} \frac{1 + |\alpha|^{-1}}{|\alpha|^{l-k} - 1},$$

and therefore

$$|\alpha|^{l-k} \leqslant 1 + \frac{\pi}{3}\left(1 + \frac{1}{|\alpha|}\right)\frac{l-k}{m-l}.$$

If $m - l \geqslant 2$, then, by $|\alpha| \leqslant 2$,

$$1 + \frac{\pi}{3}\left(1 + \frac{1}{|\alpha|}\right)\frac{l-k}{m-l} \leqslant 1 + \frac{\pi}{3}\cdot\frac{3}{2}\cdot\frac{l-k}{2}$$

$$< 1 + l - k \leqslant 2^{l-k} \leqslant |\alpha|^{l-k},$$

a contradiction. If $m - l = 1$ and $l - k \geqslant 3$ then similarly

$$1 + \frac{\pi}{3}\left(1 + \frac{1}{|\alpha|}\right)\frac{l-k}{m-l} \leqslant 1 + \frac{\pi}{3}\cdot\frac{3}{2}(l-k) < 2^{l-k} \leqslant |\alpha|^{l-k}.$$

If $m^{-1} = 1$ and $l - k = 2$, then, by (13),

$$\left(\frac{\bar{\alpha}}{\alpha}\right)^2 - 1 = \eta\left(\frac{\alpha}{\bar{\alpha}}\right)^k \left(\frac{1}{\alpha^k} - \frac{1}{\alpha^{k+2}}\right),$$

$$\left(\frac{\bar{\alpha}}{\alpha}\right) - 1 = \eta\left(\frac{\alpha}{\bar{\alpha}}\right)^{k+2} \left(\frac{1}{\alpha^{k+2}} - \frac{1}{\alpha^{k+3}}\right).$$

Hence, by division,

$$\frac{\bar{\alpha}}{\alpha} + 1 = \left(\frac{\bar{\alpha}}{\alpha}\right)^2 \frac{\alpha^3 - \alpha}{\alpha - 1} = \frac{\bar{\alpha}^2}{\alpha}(\alpha + 1).$$

This implies $(\bar{\alpha} - 1)(\alpha + \bar{\alpha} + \alpha\bar{\alpha}) = 0$. Since $\alpha \neq 1$, we infer $(1 + \alpha)(1 + \bar{\alpha}) = 1$, hence $|1 + \alpha| = 1$, a contradiction. If $m - l = l - k = 1$, then we conclude from (9) that

$$\frac{1}{\alpha^k} - \frac{1}{\alpha^{k+1}} = \left(\frac{1}{\alpha^{k+1}} - \frac{1}{\alpha^{k+2}}\right)\frac{\alpha}{\bar{\alpha}}$$

and hence $\alpha = 1$, again a contradiction. Thus we have $E \neq 0$ and hence $|E| \geqslant 2\pi$. By $|\eta| \leqslant 2$ we deduce from (14)

$$2\pi \leqslant (m - l) \cdot \frac{\pi}{3} \frac{1}{|\alpha|^k}(1 + \tfrac{1}{2})2 + (l - k)\frac{\pi}{3}\frac{1}{|\alpha|^k}(1 + \tfrac{1}{2})2$$

and hence

$$m - k \geqslant 2|\alpha|^k.$$

PROOF OF THEOREM 4: Suppose that the equation $\lambda\alpha^x + \bar{\lambda}\bar{\alpha}^x = 1$ has at least eight solutions. Without loss of generality we may assume these solutions to be $x = 0, p, q, r, s, t, u, v$ with $0 < p < q < r < s < t < u < v$. As a consequence of Theorem 2 the number α is algebraic of degree at most $(p + q - 3)(p + r - 3)$. By Lemma 11 we have

$$v - t > 2^{t+1}, \qquad t - r > 2^{r+1}, \tag{15}$$

and

$$r - p > 2^{p+1} \qquad \text{if} \quad p \geqslant 2. \tag{16}$$

From $t \geqslant 5$ and $v - t > 2^{t+1}$ we deduce $v > 10t$. By applying Lemma 7 with $p = p$, $l = t$, $m = v$ we find that

$$t \leqslant 27\frac{\log 2}{\log H} + \frac{50}{3}p,$$

where $H = h(\alpha) \geqslant 2^{2^{(p+q-3)(p+r-3)}}$ in view of (3) and (2). Thus

$$t < \frac{27}{2}(p+q-3)(p+r-3) + 17p. \tag{17}$$

First assume $p \geqslant 2$. On one hand we have, by (16) and (15), $r \geqslant 11$ and hence $t > 2^{r+1}$. On the other hand, $q \leqslant r-1$, $p \leqslant r-2$, and therefore by (17), $t \leqslant 27/2\ (2r-6)(2r-5) + 17r$. It is obvious that these inequalities are contradictory of $r \geqslant 11$.

Next assume $p = 1$. By Lemma 10 we obtain $q \geqslant 3$, $r \geqslant 6$ and $q \leqslant r-2$. On one hand, by (15), $t > 2^{r+1}$. On the other hand, by (17), $t < \frac{27}{2}(r-4)(r-2)+17$. This leads also to a contradiction, which completes the proof of the theorem.

COROLLARY: *Let* $\alpha, \lambda \in \mathbb{C}$ *with* α *not a root of unity of* $\alpha \neq 0$. *Then the equation* $\lambda\alpha^x + \bar{\lambda}\bar{\alpha}^x = 1$ *in* $x \in \mathbb{Z}$ *has at most two solutions if* $|\alpha| = 1$ *and at most* $7 + 5/|\log|\alpha||$ *solutions if* $|\alpha| \neq 1$.

PROOF: We may assume $\lambda \neq 0$. Suppose $|\alpha| = 1$. If $\lambda\alpha^x + \bar{\lambda}\bar{\alpha}^x = 1$, then $Re\ \lambda\alpha^x = \frac{1}{2}$, $|\lambda\alpha^x| = |\lambda|$. We can therefore have at most two possible values for $\lambda\alpha^x$. If, for $x_1, x_2 \in \mathbb{Z}$, $x_1 \neq x_2$, we have $\lambda\alpha^{x_1} = \lambda\alpha^{x_2}$, then α is a root of unity, which we had excluded. Thus there are at most two solutions.

Suppose $|\alpha| \neq 1$. Since $\lambda\alpha^x + \bar{\lambda}\bar{\alpha}^x = 1$ is equivalent to $\lambda(1/\alpha)^{-x} + \bar{\lambda}(1/\bar{\alpha})^{-x} = 1$, we may assume $|\alpha| > 1$. Split the integers in residue classes mod. g, where g is the smallest integer with $|\alpha|^g \geqslant 2$. In each residue class there are at most seven solutions because of Theorem 3 and Theorem 4. The number of residue classes is at most $1 + \log 2/\log|\alpha|$. Hence the number of solutions is at most $7 + 7 \log 2/\log|\alpha| < 7 + 5/\log|\alpha|$.

COROLLARY 2: *Let* $\alpha \in \mathbb{C}$, $|\alpha| \geqslant 2$. *Let* l *be a line in the complex plane not passing through the origin. Then there are at most seven integral powers of* α *on* l.

PROOF: Let z be the complex coordinate of $\mathbb{C}$. It is straightforward to see that there exists a complex number λ such that $z \in l$ if and only if $\lambda z + \bar{\lambda}\bar{z} = 1$. Hence, if $\alpha^x \in l$, then $\lambda\alpha^x + \bar{\lambda}\bar{\alpha}^x = 1$. According to Theorem 4 there exist at most seven integral powers of α on l.

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