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U. N. BHOSLE

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MODULI OF ORTHOGONAL AND SPIN BUNDLES OVER HYPERELLIPTIC CURVES

Mrs U.N. Bhosle (Desale)

To a nondegenerate pencil ϕ of quadrics in a $(2g+1)$-dimensional projective space one can associate a hyperelliptic curve X of genus g. It has been known for over a decade that the Jacobian of X is isomorphic to the space of $(g-1)$-dimensional linear spaces contained in the intersection of quadrics of ϕ (see [7,5]). This kind of relationship between the moduli spaces of bundles on X and space of linear subspaces of $\mathbb{P}^{2g+1}$ related to a family of quadrics parametrised by $\mathbb{P}^1$ has further been studied in [5], Theorem 4, [4] and [2] for vector bundles and in [6] for stable orthogonal and spin bundles. In this paper I study the semistable orthogonal bundles on X generalizing Theorem 3 of [2] (see Theorem 4). The main result is the following:

THEOREM: *Let M be the space of equivalence classes of semistable orthogonal bundles F of rank n with i-action (compatible with the orthogonal structure on it) of a fixed allowable local type τ over a hyperelliptic curve X of genus $\geqslant 2$. Let W be the set of Weierstrass points of X. Let $Q_1 = \Sigma_{w \in W} Q_w$, $Q_2 = \Sigma_{w \in W} k_w Q_w$ be quadrics on an even dimensional vector space $\Sigma_{w \in W} C_w$, where C_w is a vector space of dimension r_w carrying a nondegenerate quadratic form Q_w, k_w are mutually distinct scalars corresponding to the Weierstrass points of X and r_w = dimension of F_w^-. Let R' denote the space of subspaces of ΣC_w of dimension $\frac{1}{2}\Sigma r_w$ which are maximal isotropic for Q_1, and have rank $\leqslant n$ with respect to Q_2. Then the quotient of R', in the sense of geometric invariant theory, by $\Pi_w O(Q_w)$ is isomorphic to M.*

This theorem is proved in §2. In §4, I prove similar results for Clifford bundles. The idea of the proofs is similar to that in [2]. However, in [2] one uses explicitly the special properties of the i-invariant orthogonal bundles of forms $E \oplus i^*E$ and $E \otimes i^*E$. Such special forms are possible only for orthogonal bundles of very low ranks. To overcome this difficulty the definition of "irreducible bundles" is introduced (Definition 1.1). The theorem is in fact proved first for irreducible bundles and then generalised by induction.

In §3, properties of some spaces of linear subspaces related to a pencil of quadrics (including those involved in the above theorems) are studied. This study can be hoped to have good applications in future. For example, Corollary 3.5 applied to results of Professor Ramanan [6] shows that the moduli space of stable spin bundles F with $\mathbb{Z}/2$-action on a hyperelliptic curve is nonsingular if the dimension of the (-1)-eigenspace of F_w for $\mathbb{Z}/2$-action is one for all Weierstrass points w.

§1. Notations and preliminaries

We will assume that characteristic of the base field is zero.

Let X be an irreducible nonsingular hyperelliptic curve. Let i denote the hyperelliptic involution on X. We denote by W the set of Weierstrass points of X as well as its image on $\mathbb{P}^1$. We use h to denote both the line bundle $\theta_{\mathbb{P}^1}(1)$ on $\mathbb{P}^1$ as well as its pull back to X.

DEFINITION 1.1: An i-action on a bundle E over X is a map $j: E \to i^*E$ such that $i^*j \circ j = Id$.

If E is a vector bundle with i-action, then for w in W, the i-action induces an involution on the fibre E_w of E at w. We denote by E_w^+ (resp. E_w^-) the eigenspace corresponding to eigenvalue $+1$(resp. -1) for this involution. The i-action induces one on the cohomology groups $H^k(X, E)$ too. Let $H^k(X, E)^+$ and $H^k(X, E)^-$ denote the eigenspaces corresponding to eigenvalues $+1$ and -1 respectively. We use similar notations for $H^k(X, E)$ replaced by the Euler characteristic $\chi(E)$ of E.

Henceforth, in this section, E will denote an $0(n)$-or $\mathrm{SO}(n)$-bundle with i-action.

DEFINITION 1.2: Let τ be a fixed topological $0(n)$ (or $\mathrm{SO}(n)$)-bundle with i-action. Then E is said to be of local type τ if E is topologically i-isomorphic to τ.

DEFINITION 1.3: An i-invariant subbundle of E is a subbundle of E invariant under the given i-action on E.

DEFINITION 1.4: A bundle E is semistable (resp. stable) if every isotropic i-invariant (resp. proper) subbundle F of E has degree less than or equal to (resp. less than) zero.

We remark that an $0(n)$ (or $\mathrm{SO}(n)$)-bundle with i-action is semistable if and only if it is semistable as an orthogonal (or special orthogonal)bundle (Proposition 4.6, [6]). However, there do exist orthogonal bundles with i-action which are stable as orthogonal bundles with i-action but not stable as orthogonal bundles (Example 1.35, [1], Lemma 1.8).

If E is semistable but not stable (as a bundle with i-action), then by induction we can find a flag

$$0 = N_0 \subset N_1 \subset \ldots \subset N_r \subset N_r^{\perp} \subset \ldots \subset N_r^{\perp} \subset E$$

such that N_j are isotropic i-invariant subbundles of E of degree zero, N_{j+1}/N_j and $N_j^{\perp}/N_{j+1}^{\perp}$ are stable vector bundles with i-action for $j = 0,\ldots,r-1$ and $N_r^{\perp}/N_r$ is a stable orthogonal (or SO(n))-bundle with i-action. Then grE, the associated graded of E, is defined as the bundle $N_1 \oplus \ldots \oplus N_r/N_{r-1} \oplus N_r^{\perp}/N_r \oplus N_{r-1}^{\perp}/N_r^{\perp} \oplus \ldots \oplus E/N_1^{\perp}$. The bundle grE gets an orthogonal structure as follows: $N_j^{\perp}/N_{j+1}^{\perp}$ and N_{j+1}/N_j are dual to each other and hence their direct sum carries a non-degenerate quadratic form with both these direct summands as isotropic subbundles. On $N_r^{\perp}/N_r$ there is a nondegenerate quadratic form induced from that on E. On grE we put the quadratic form which is a direct sum of these forms. The bundle grE is unique upto quadratic i-isomorphisms.

DEFINITION 1.5: Two semistable orthogonal bundles with i-action are *equivalent* if their associated gradeds are isomorphic (as orthogonal bundles with i-action).

REMARK: The local type of an O(n) (or SO(n))-bundle with i-action is completely determined by its topological type as an O(n) (or SO(n))-bundle (without i-action) and the integers ($r_w = \dim E_w^-$), w in W, E being the associated vector bundle (see Proposition 1.48, pp. 76–87 [1]).

DEFINITION 1.6: An orthogonal bundle (or special orthogonal bundle) with i-action is *irreducible* if it has no trivial sub-bundle with induced trivial i-action.

LEMMA 1.7: *Let F be a semistable orthogonal (special orthogonal) bundle of rank n with i-action. Then*

(*i*) *F is irreducible if and only if $H^0(F)^+ = 0$,*

(*ii*) *if F is not irreducible, then F is equivalent to $I_{n-m} \oplus F_m$, F_m being an irreducible orthogonal bundle with an i-action and I_{n-m} a trivial bundle of rank $n-m$ with trivial i-action, $m < n$.*

PROOF: (i) Suppose F is reducible, i.e. F contains a trivial bundle N with trivial i-action; then $H^0(F)^+ \supseteq H^0(N)^+ \neq 0$. Conversely, suppose that $H^0(F)^+ \neq 0$. We shall show that F is reducible. Let N be the subbundle of F generated by $H^0(F)^+$. Then N is i-invariant being generated by invariant sections. We claim that N is trivial. Since F is a semistable bundle of degree zero, degree $N \leqslant 0$. On the other hand, as N is generated by sections, degree $N \geqslant 0$. It follows that degree $N = 0$ and hence N is generated by nowhere vanishing sections. Therefore N is trivial. The i-action on N is trivial as it is generated by i-invariant sections. Thus F is reducible.

(ii) Let F be a reducible semistable orthogonal bundle with an i-action. By part (i), $H^0(F)^+ \neq 0$ and $H^0(F)^+$ generates a trivial subbundle N of F with trivial i-action. If N is non-isotropic in the sense that the subbundle N' generated by $N \cap N^\perp$ is zero, then $F = N \oplus N^\perp$ (see the proof of Proposition 4.2, [6]). If $N' \neq 0$, N' being isotropic i-invariant we get the flag $0 \subset N' \subset N'^\perp \subset F$ showing that F is equivalent to $N' \oplus N'^* \oplus N'^\perp/N'$ as an orthogonal bundle with i-action. We claim that N' is trivial. We first show that N' has degree zero. Consider the exact sequence $0 \to N' \to N \oplus N^\perp \to M \to 0$ where M is the subbundle generated by $N + N^\perp$. In view of the semistability of F, degree $N' \leqslant 0$, degree $M \leqslant 0$. But, from the exact sequence, degree N' + degree M = deg $\cdot N + \deg \cdot N^\perp = 2(\deg N)$ as $F/N^\perp \approx N^*$, so that degree N' = degree $M = 0$. We will now show that N' is generated by sections so that N' will be trivial being of degree zero. Notice that the evaluation map $X \times H^0(N') \to N'$ is injective as $H^0(N') \subset H^0(N)$ and $X \times H^0(N) \to N$ is an injection (in fact an isomorphism). Thus dim $H^0(N') \leqslant$ rank N'. We only have to show that this is an equality. Consider $0 \to N' \to N \to L \to 0$. Note that L is semistable as N is so and $\mu(N') = \mu(N) = \mu(L)$. The map $X \times H^0(L) \to L$ is an injection, for if a section of L vanishes at a point, it will generate a line subbundle of L of positive degree contradicting the semistability of L. Thus dim $H^0(L) \leqslant$ rank L. Thus,

$$\begin{aligned}\dim H^0(N') &\geqslant \dim H^0(N) - \dim H^0(L)\\ &\geqslant \operatorname{rank} N - \operatorname{rank} L\\ &= \operatorname{rank} N'.\end{aligned}$$

This finishes the proof of the claim that N' is trivial. Thus F is equivalent to $(N' \oplus N'^*) \oplus F'$, where $(N' \oplus (N')^*)$ is a trivial bundle with trivial i-action, $F' \approx N'^\perp/N'$ is an orthogonal bundle with an i-action, $\oplus$ being an orthogonal direct sum. If F' is irreducible, we are through; otherwise by inducation on rank applied to F' we get the result.

LEMMA 1.8: *For every integer n, there exists a stable special orthogonal bundle with i-action, of rank n.*

PROOF: The case $n = 1$ is trivial. For $n = 2$, though there does not exist a stable special orthogonal bundle, there do exist stable special orthogonal bundles with i-action. For example, take $E = L \oplus L^*$ with $L \not\approx i^*L$, deg $L = 0$. Since deg $L = 0$, $L^* \approx i^*L$ so that there is an i-action on E obtained by switching the direct summands. E is clearly semistable. E is stable since any i-invariant subbundle of E of degree zero has to be L or L^* and the latter are not i-invariant. Thus we may assume that $n \geqslant 3$.

Let $\tilde{X}$ be the universal covering of X. Let Γ be the group for the composite covering $\tilde{X} \to X \to \mathbb{P}^1$. It is known that Γ is generated by

$(2g+2)$ elements $x_1,\ldots,x_{2g+2}$, say, with the only relations $x_i^2=1$ for all i and $x_1x_2,\ldots,x_{2g+2}=1$. The fundamental group π of X is a normal subgroup of Γ with quotient $\approx \mathbb{Z}/2$, generated by X_i, Y_i, $i=1,\ldots,g$ where $X_i=x_1x_{2i}$, $Y_i=x_{2i+1}x_1$ with the only relations $X_1Y_1X_2Y_2\ldots X_gX_g = X_gX_g\ldots Y_1X_1$. The induced involution on π is given by $X_i\to X_i^{-1}$, $Y_i\to Y_i^{-1}$ for all $i=1,\ldots,g$. The bundles associated to irreducible unitary representations of Γ in SO(n) are stable special orthogonal bundles of rank n with i-action, hence it suffices to construct irreducible unitary representations ρ of Γ in SO(n) for $n\geqslant 3$. Let V be a vector space of dimension n and B a symmetric bilinear form on V.

Case (i): $n=2m$, m even. Let $(e_1,\ldots,e_m,f_1,\ldots,f_m)$ be a basis of V (taken in this order) such that $B(e_i,f_j)=\delta_{ij}$ for all i,j. Let M, $N\in$ End V be defined by $M(e_i)=f_i$, $M(f_i)=e_i$, $N(e_i)=\lambda_i^{-1}f_i$, $N(f_i)=\lambda_ie_i$, $i=1,\ldots,m$ where λ_i are non zero real numbers such that $\lambda_i\neq\lambda_j$ for $i\neq j$ and $\lambda_i\lambda_j\neq 1$ for all i,j. Since m is even, M, $N\in$ SO(n). Define $\rho(x_1)=M$, $\rho(x_2)=M$, $\rho(x_3)=N$, $\rho(x_4)=N$ and $\rho(x_i)=\mathrm{Id}$ for $i>4$. Then ρ gives a unitary representation of Γ in SO(n). We have to check that ρ is irreducible i.e. $\{M, N\}$ is an irreducible subset i.e. no nonzero element in the Lie algebra of SO(n) commutes with both M and N. Let $\mathfrak{J}$ be the Lie algebra of SO(n). Then $A\in\mathfrak{J}$ iff $AM+M^tA=0$ i.e. iff

$$A=\begin{bmatrix}\alpha & \beta\\ \gamma & -{}^t\alpha\end{bmatrix}$$

with β and γ skew symmetric matrices. Take

$$D=\begin{bmatrix}0 & \mu\\ \mu^{-1} & 0\end{bmatrix}$$

where $\mu=(\mu_1,\ldots,\mu_m)$ is a diagonal matrix. Then A commutes with D iff $\beta\mu^{-1}=\mu\gamma$ and $\alpha\mu=-\mu{}^t\alpha$. Taking $\mu=Id$, i.e. $D=M$, we get $\beta=\gamma$, $\alpha=-{}^t\alpha$ i.e. α is skew symmetric. Taking $D=N$, we get $\lambda_i\lambda_j\beta_{ij}=\beta_{ij}$, $\lambda_i^{-1}\lambda_j\alpha_{ij}=\alpha_{ij}$ so that $\alpha_{ij}=0\forall i\neq j$ and $\beta_{ij}=0\forall i,j$. Since α is skew symmetric, we have $\alpha=0=\beta$. Thus $A=0$.

Case (ii): $n=2m+1$, m even. Take an ordered basis of V, $(e_0,e_1,\ldots,e_m,f_1,\ldots,f_m)$, with $B(e_0,e_0)=1$, $B(e_i,f_j)=\delta_{ij}$ $i,j=1,\ldots,m$. The Lie algebra consists of matrices of type

$$\begin{bmatrix}0 & b & c\\ -{}^tc & \alpha & \beta\\ -{}^tb & \gamma & -{}^t\alpha\end{bmatrix}$$

with β, γ skew symmetric $m\times m$ matrices. Let

$$M_1=\begin{bmatrix}1 & 0\\ 0 & M\end{bmatrix},\qquad N_1=\begin{bmatrix}1 & 0\\ 0 & N\end{bmatrix}.$$

As in the case i), one can check that $\{M_1, N_1\}$ is an irreducible set. Define the representation ρ by $\rho(x_1)=M_1$, $\rho(x_2)=M_1$, $\rho(x_3)=N_1$, $\rho(x_4)=N_1$, $\rho(x_i)=Id\forall i>4$.

Case (iii): $n=2m$, m odd. Let $(e_1,\dots,e_{m-1}, f_1,\dots,f_{m-1}, e_m, f_m)$ be an ordered basis of V s.t. $B(e_i, f_j)=\delta_{ij}$, $i,j=1,\dots,m$. Let

$$J_2=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

Then the Lie algebra $\mathfrak{F}_1$ consists of matrices of the form

$$\begin{bmatrix}A & B\\ C & D\end{bmatrix}$$

with $AM+M^tA=0$, $BJ_2+M^tC=0$, $CM+J_2^tB=0$ and

$$D=\begin{pmatrix}b & 0\\ 0 & -b\end{pmatrix},$$

b being a scalar. Let I_q denote identity matrix of rank q. Let

$$M_2=\begin{bmatrix}M & 0\\ 0 & -I_2\end{bmatrix},\qquad N_2=\begin{bmatrix}N & 0\\ 0 & I_2\end{bmatrix}$$

and let P_2 be defined by $P_2(e_1)=f_1$, $P_2(f_1)=e_1$, $P_2(e_m)=f_m$, $P_2(f_m)=e_m$, $P_2(e_i)=e_i$, $P_2(f_i)=f_i$ for $i\neq 1, m$. Then an element in $\mathfrak{F}_1$ commutes with M_2 and N_2 iff it is of the form

$$\begin{bmatrix}0 & 0\\ 0 & D\end{bmatrix}\quad\text{with}\quad D=\begin{bmatrix}b & 0\\ 0 & -b\end{bmatrix}.$$

An element of this form commutes with P_2 iff $J_2D=DJ_2$ i.e. iff $D=0$. Thus $\{M_2, N_2, P_2\}$ is an irreducible set for $\mathfrak{F}_1$. Hence the representation ρ defined by $\rho(x_1)=M_2$, $\rho(x_2)=M_2$, $\rho(x_3)=(N_2)$, $\rho(x_4)=N_2$, $\rho(x_5)=P_2$, $\rho(x_6)=P_2$ and $\rho(x_i)=Id$ for $i>6$ gives an irreducible representation.

Case (iv) (a): $n=3$. Let (e_0, e_1, f_1) be an ordered basis of V such that the matrix of the quadratic form with respect to this basis is

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}.$$

The Lie algebra $\mathfrak{F}_2$ consists of matrices of the form

$$\begin{bmatrix}0 & b & c\\ -c & e & 0\\ -b & 0 & -e\end{bmatrix}.$$

Let

$$M_0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & \lambda \\ 0 & \lambda^{-1} & 0 \end{bmatrix}, \qquad \lambda^2 \neq 1, \qquad N_0 = \tfrac{1}{2}\begin{bmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -1 & 1 \\ \sqrt{2} & 1 & -1 \end{bmatrix}.$$

It is easy to check that M_0, N_0 form an irreducible set for $\mathfrak{F}_2$. Define a representation ρ by $\rho(x_1) = \rho(x_2) = M_0$, $\rho(x_3) = \rho(x_4) = N_0$, $\rho(x_i) = Id \forall i > 4$.

Case (iv) (b): $n = 2m + 1$, m odd $\geqslant 3$. Let $(e_0, e_1, \ldots, e_{m-1}, f_1, \ldots, f_{m-1}, e_m, f_m)$ be an ordered basis of V such that $B(e_0, e_i) = \delta_{i,0}$, $B(e_0, f_i) = 0$, $B(e_i, e_j) = 0 = B(f_i, f_j) = 0$, $B(e_i, f_j) = \delta_{ij}$, $i, j = 1, \ldots, m$. The Lie algebra $\mathfrak{F}_2$ consists of matrices of the form

$$\begin{bmatrix} 0 & B \\ -C^tB & A \end{bmatrix}$$

where

$$C = \begin{bmatrix} 0 & I_{m-1} & 0 \\ I_{m-1} & 0 & 0 \\ 0 & 0 & J_2 \end{bmatrix} \qquad \text{and} \qquad A \in \mathfrak{F}_1.$$

$$\text{Let } M_3 = \begin{bmatrix} 1 & 0 \\ 0 & M_2 \end{bmatrix}, \qquad N_3 = \begin{bmatrix} 1 & 0 \\ 0 & N_2 \end{bmatrix}, \qquad P_3 = \begin{bmatrix} -1 & 0 \\ 0 & P_2 \end{bmatrix},$$

where $P_2(e_i) = f_i$, $i = 1, 2, m$; $P_2(f_i) = e_i$, $i = 1, 2, m$ and $P_2(e_i) = e_i$, $P_2(f_i) = f_i$ for $i \neq 1, 2, m$. As before, it can be checked that M_3, N_3, P_3 form an irreducible set for $\mathfrak{F}_2$. Define a representation ρ by $\rho(x_1) = \rho(x_2) = M_3$, $\rho(x_3) = \rho(x_4) = N_3$, $\rho(x_5) = \rho(x_6) = P_3$, $\rho(x_i) = Id$ for $i > 6$.

LEMMA 1.9: *For every integer n, there exists an irreducible stable special orthogonal bundle of rank n with i-action.*

PROOF: We shall show that the stable SO(n)-bundles F associated to the irreducible unitary representations of Γ in SO(n) constructed in Lemma 1.8 (with a minor change in case ii) are irreducible. By Lemma 1.7, F is irreducible iff $H^0(F)^+ = 0$ i.e. iff V^Γ, the subspace of V on which Γ acts trivially, is zero. Hence it suffices to show that the representation $\tilde{\rho}$ obtained by composing ρ with the inclusion SO(n) $\to$ GL(n) contains no trivial representation.

Case (i): $n = 2m$, m even. Since N and M both keep the subspaces $(e_i) \oplus (f_i)$, $i = 1, \ldots, m$, invariant and each of these subspaces is irreducible for their action as $\lambda_i \neq \pm 1$, it follows that $\tilde{\rho}$ has irreducible components $(e_i) \oplus (f_i)$, $i = 1, \ldots, m$. Moreover M, N act notrivially on any component, showing that $\tilde{\rho}$ has no trivial subrepresentation.

Case (ii): $n = 2m + 1$, m even. Define P_1 by $P_1(e_0) = -e_0$, $P_1(e_1) = f_1$, $P_1(f_1) = e_1$ and $P_1(e_i$, $P(f_i) = f_i$ for $i \neq 1$. Define ρ by $\rho(x_1) = \rho(x_2) = M_1$, $\rho(x_3) = \rho(x_4) = N_1$, $\rho(x_5) = \rho(x_6) = P_1$ and $\rho(x_i) = Id$ for $i > 6$. Then $\tilde{\rho}$ has irreducible components (e_0) and $(e_i) \oplus (f_i)$, $i = 1, \ldots, m$. P_1 acts nontrivially on (e_0). It follows that $\tilde{\rho}$ contains no trivial subrepresentation.

Case (iii): $n = 2m$, m odd. In this case $\tilde{\rho}$ has irreducible components $(e_i) \oplus (f_i)$, $i = 1, \ldots, m-1$, $(e_n + f_n)$, $(e_n - f_n)$. Since M_2 acts nontrivially on these components, it follows that $\tilde{\rho}$ contains no trivial subrepresentation.

Case (iv): $n = 2m + 1$, m odd. (a) M_0 acts trivially on an element $v = a_0 e_0 + a_1 e_1 + b_1 f_1$ iff $a_1 = b_1 \lambda$, $a_0 = 0$; and N_0 acts trivially on $a_1 e_1 + b_1 f_1$ iff $a_1 + b_1 = 0$. Since $\lambda \neq -1$, the result follows.

(b) In this case $\tilde{\rho}$ has irreducible components (e_0), $(e_i) \oplus (f_i)$, $i = 1, \ldots, m-1$, $(e_m + f_m)$, $(e_m - f_m)$. It is easy to see that $\tilde{\rho}$ contains no trivial subrepresentation.

Note that the SO(2)-bundle given in the proof of Lemma 1.8 is an irreducible stable SO(2)-bundle with i-action.

DEFINITION 1.10: A local type τ of an orthogonal (or SO(n))-bundle with i-action is *allowable* if there exists an irreducible stable orthogonal (or SO(n)) bundle with i-action of local type τ.

The above lemma gives an example of an orthogonal bundle with i-action of an allowable local type for every n. I do not know a complete classification of allowable local types. It is easy to see that in case $r_w = 0$ for at least $2g$ of the Weierstrass points, there exist no irreducible unitary representations. In case $n = 2$, using the matrices

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad N = \begin{bmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{bmatrix}$$

with $\lambda^2 \neq 1$, it is easy to see that provided Σr_w is even and the i-action is not $\pm Id$ for at least four Weierstrass points, there exist irreducible stable orthogonal bundles with given r_w's.

It is not difficult to construct examples of allowable local types in low ranks.

§2. The main theorem

This section is devoted to the proof of the following theorem.

THEOREM 1: *Let M be the space of equivalence classes of semistable orthogonal bundles F of rank n with an i-action and of an allowable fixed* *local type τ on a hyperelliptic curve X of genus $g \geqslant 2$. For each Weierstrass point w, let C_w be a vector space of dimension r_w equal to the dimension of F_w^-, carrying a nondegenerate quadratic form Q_w. We construct a quadratic form on $\mathbb{P}^1 \times \Sigma C_w$ with values in $\mathcal{O}_{\mathbb{P}^1}(2g+1)$ (see the proof). This can be regarded as a family of quadratic forms on ΣC_w parametrised by $\mathbb{P}^1$. Fix two distinct non-Weierstrass points a and x. Let $p(a)$ and $p(x)$ be their projections on $\mathbb{P}^1$ and let $Q_1 = \Sigma Q_w$ and $Q_2 = \Sigma k_w Q_w$ be the quadratic forms in the family corresponding to points $p(a)$ and $p(x)$ respectively. Let R' denote the space of subspaces of ΣC_w of dimension $\frac{1}{2}\Sigma r_w$ which are maximum isotropic for Q_1 and have rank less than or equal to n for Q_2. Then the quotient R of R'_{ss} (the set of semistable points of R'), in the sense of geometric invariant theory, by $\Pi O(Q_w)$ is isomorphic to the moduli space M.*

Let I be the set of isomorphism classes of semistable orthogonal bundles F of rank n with i-action and quadratic isomorphisms $n_w : (F \otimes h^g)_w^- \to C_w$ for all w in W. We first give a map f from I to R'. Write $E = F \otimes h^g$. We claim that the evaluation map $e: H^0(E)^- \to \Sigma E_w^-$ is an injection. From the cohomology exact sequence associated to the exact sequence $0 \to E(-W) \to E \to E \otimes O_W \to 0$, it follows that the kernel of the evaluation map is $H^0(E(-W))^-$. The latter is zero as $E(-W)$ is semistable of negative degree. Composing e with the isomorphisms (n_w), we get an injective map from $H^0(E)^-$ into ΣC_w. We define $f(F, (n_w))$ to be the image of $H^0(E)^-$ in ΣC_w under this composite. We check below that $f(F, (n_w)) \in R'$ (see Proposition 2.1 below).

For the sake of convenience, we interpret the quadratic form Q_w on C_w as having values in the one dimensional vector spaces h_w^{2g} for all w in W. Let $p: X \to \mathbb{P}^1$ be the canonical projection. Fix a in $X - W$. Using $p(a)$ we get an isomorphism

$$\sum_w h_w^{2g} \underset{p(a)}{\to} \sum_w h_w^{2g+1}.$$

Now there is an isomorphism $\Sigma_w h_w^{2g+1} \approx H^0(\mathbb{P}^1, h^{2g+1})$ induced by the evaluation map, since both the spaces are of dimension $2g+2$ and the kernel of the evaluation map is $H^0(\mathbb{P}^1, h^{-1}) = 0$. Define q to be the composite

$$q: \mathbb{P}^1 \times \sum C_w \overset{\Sigma Q_w}{\to} \mathbb{P}^1 \times \sum h_w^{2g} \underset{\approx}{\overset{p(a)}{\to}} \mathbb{P}^1 \times \sum h_w^{2g+1}$$

$$\approx \mathbb{P}^1 \times H^0(\mathbb{P}^1, h^{2g+1}) \overset{e}{\to} h^{2g+1}$$

where e is the evaluation map. Then q is a quadratic form on ΣC_w parametrised by $\mathbb{P}^1$ and with values in h^{2g+1}. For x in $\mathbb{P}^1$ the quadratic form q_x corresponding to x is obtained simply by replacing e above by e_x, the evaluation at x. The pullback of q to X is again denoted by q.

PROPOSITION 2.1: *(a) $f(F, (n_w))$ is isotropic with respect to the quadratic forms q_q, q_{ia} at a and ia respectively.*

(b) The space $H^0(X, E \otimes h^{-1})^-$ embedded in $f(F, (n_w))$ via the divisor $(x \cup ix)$ is orthogonal to $f(F, (n_w))$ with respect to q_x for x in $X - (W \cup a \cup ia)$.

If F is irreducible, then $H^0(X, E \otimes h^{-1})^-$ embedded in $f(F, (n_w))$ via $x \cup ix$ is the orthogonal complement of $f(F, (n_w))$ in $f(F, (n_w))$ for q_x, for x in $X - (W \cup a \cup ia)$.

(c) (i) $\dim f(F, (n_w)) = \frac{1}{2}\Sigma_w \dim C_w$,

(ii) rank $q_x/f(F, (n_w)) \leqslant n$, $\forall F$ in I,

(iii) rank $q_x/f(F, (n_w)) = n$ if and only if F is irreducible.

PROOF: (a) Follows from the definition of q_a and the following commutative diagram

$$\begin{array}{ccc} H^0(X, E)^- & \rightarrow & \Sigma C_w \\ \downarrow & & \downarrow \Sigma Q_w \\ H^0(\mathbb{P}^1, h^{2g}) = H^0(X, h^{2g})^+ & \rightarrow & \Sigma h_w^{2g} \\ \downarrow p(a) & & \downarrow p(a) \\ H^0(\mathbb{P}^1, h^{2g+1}) & \rightarrow & \Sigma h_w^{2g+1} \\ \downarrow e_{p(a)} & & \\ h_{p(a)}^{2g+1} & & \end{array}$$

The composite $e_{p(a)} \circ p(a) : H^0(\mathbb{P}^1, h^{2g}) \rightarrow h_{p(a)}^{2g+1}$ is zero in view of the cohomology exact sequence associated to the exact sequence $0 \rightarrow h^{2g} \rightarrow h^{2g+1} \rightarrow h^{2g+1} \otimes O_{p(a)} \rightarrow 0$.

(b) Consider the commutative diagram

$$\begin{array}{ccccc} H^0(E)^- \times H^0(E \otimes L_{x+ix}^{-1})^- & \rightarrow & \Sigma C_w \times \Sigma C_w & & \\ \downarrow & & \downarrow & & \\ H^0(X, h^{2g} \otimes L_{x+ix}^{-1})^+ & & & & \\ = H^0(P^1, h^{2g} \otimes L_{p(x)}^{-1}) & \rightarrow & \Sigma h_w^{2g} & & \\ \downarrow & & \downarrow & & \\ H^0(\mathbb{P}^1, h^{2g+1} \otimes L_{p(x)}^{-1}) & \underset{p(x)}{\rightarrow} & H^0(h^{2g+1}) & \underset{e_{p(x)}}{\rightarrow} & h_{p(x)}^{2g+1}. \end{array}$$

The lower composite is zero in view of the exact sequence

$$0 \to h^{2g+1} \otimes L_{p(x)}^{-1} \to h^{2g+1} + h^{2g+1} \otimes O_{p(x)} \to 0.$$

Thus $q_x(H^0(E)^-,\ H^0(E \otimes L_{x+ix}^{-1}) = 0 \forall x \in X$, which shows that $H^0(E \otimes L_{x+ix}^{-1})^- \subset$ orthogonal complement of $H^0(X, E)^-$ in $H^0(X, E)^-$ with respect to the quadric q_x at $x \in X - (W \cup a \cup ia)$.

For the second part, let $s \in H^0(X, E)^-$ be orthogonal to $H^0(X, E)^-$ for q_x. This means $q'_x(s(x), t(x)) = 0$ for all t in $H^0(X, E)^-$ where q'_x is the quadratic form on E_x given by the orthogonal structure of E. We claim that it suffices to show that the evaluation map $e_x : H^0(X, E)^- \to E_x$ is onto. For, then we shall have $q'_x(s(x), E_x) = 0$, which implies $s(x) = 0$ as q'_x is nondegenerate on E_x. Since s is i-antiinvariant, we have also $s(ix) = 0$. Thus $s \in H^0(X, E \otimes L_x^{-1} \otimes L_{ix}^{-1}) = H^0(X, E \otimes h^{-1})^-$. It remains to show that the evaluation map e_x is onto. Consider the exact sequence

$$0 \to E \otimes h^{-1} \to E \to E_x \oplus E_{ix} \to 0.$$

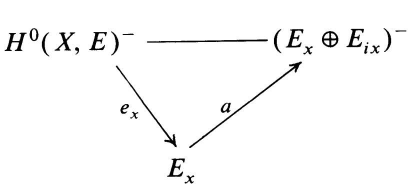
The associated cohomology exact sequence gives

$$H^0(X, E)^- \to (E_x \oplus E_{ix})^- \to H^1(X, E \otimes h^{-1})^- \text{ exact.}$$

By Serre' duality $H^0(X, F)^+ \approx H^1(X, F^* \otimes K)^+)^* \approx (H^1(X, F \otimes h^{g-1})^-)^*$ as $F \approx F^*$ and $K \approx h^{g-1}$, the latter isomorphism being noncompatible with i-actions as i-acts by (-1) on K_w, while it acts trivially on h_w for all w in W. Since F is irreducible, $H^0(X, F)^+ = 0$ so that $H^1(X, E \otimes h^{-1})^- = 0$ as $E = F \otimes h^g$. Thus $H^0(X, E)^- \to (E_x \oplus E_{ix})^-$ is onto. Now

$$E_x \underset{a}{\overset{\approx}{\to}} (E_x \oplus E_{ix})^-$$

under the map $y \to (y, -iy)$ and the following diagram commutes, showing that e_x is onto.



(c) The assertion (i) follows from Proposition 2.2 [2]. We similarly have

$$h^0(X, E)^- - h^0(X, E \otimes h^{-1})^- = n - h^1(X, E \otimes h^{-1})^-.$$

The assertions (ii) and (iii) now follow from this and the part (b).

We recall that a subspace V of ΣC_w is semistable (respectively properly stable) for the action of $\Pi O(Q_w)$ on ΣC_w if and only if for every proper family $(N_w)_w$ where N_w is an isotropic subspace of C_w, we have

$$\dim V\cap\left(\sum N_w\right)+\dim V\cap\left(\sum N_w^{\perp}\right)$$
$$\leqslant (\text{resp.} <) \dim V \text{ (see Proposition 5.3 [6])}.$$

PROPOSITION 2.2: *Let F be an orthogonal bundle with i-action, $E=F\otimes h^g$ and $H^1(E)^-=0$. If the space $H^0(E)^-$ embedded in ΣC_w via quadratic isomorphisms $(n_w)_w$ is semistable, then F is a semistable orthogonal bundle.*

PROOF: Follows similarly as Proposition 5.6 [6] by taking $\alpha=h^g$.

Let $(R_n')_{ss}$ be the set of semistable subspaces V of ΣC_w of dimension $\frac{1}{2}\Sigma \dim C_w$ which are maximum isotropic for Q_1 and have rank exactly n for Q_2. From Propositions 2.1 (C) and 2.2, it follows that $f^{-1}((R_n')_{ss})=I_n\cap f^{-1}(R_{ss}')$ where I_n is the subset of I consisting of irreducible bundles.

PROPOSITION 2.3: *The map f induces a bijection from*

$$I_n'=I_n\cap f^{-1}(R_{ss}') \text{ onto } (R_n')_{ss}.$$

PROOF: We shall now construct a map $f':(R_n')_{ss}\to I_n'$ which will be shown to be the inverse of f. Let $V\in(R_n')_{ss}$. We claim that the composite $\Sigma C_w\otimes L_w^{-1}\to X\times\Sigma C_w\to X\times\Sigma C_w/V$ is a surjection. We have only to check that at $w_0\in W$, Image$(\Sigma C_w\otimes L_w^{-1})+V=\Sigma C_w$ i.e. $\Sigma_{w'\neq w_0}C_{w'}+V=\Sigma C_w$. By Lemma 5.5, [6] we have $V\cap C_{w_0}=0$. Taking orthogonal complements for Q_1 and noting that V is a maximal isotropic space for Q_1 so that $V^{\perp}=V$, we have $V+\Sigma_{w'\neq w_0}C_{w'}=\Sigma C_w$. Define V' by

$$\begin{array}{ccccccccc}
0 & \to & X\times V & \to & X\times\sum C_w & \to & X\times\sum C_w/V & \to & 0\\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \to & V' & \to & \sum C_w\otimes L_w^{-1} & \to & X\times\sum C_w/V & \to & 0
\end{array}$$

Now, the quadratic form q on $X\times\Sigma C_w$ induces a quadratic form on $\Sigma(C_w\otimes L_w^{-1})$ which vanishes at each w and hence factors through a quadratic form with values in $h^{2g+1}\otimes L_W^{-2}=h^{-1}$. Its restriction to V' vanishes completely on V' at a and ia and hence induces an h^{-2}-valued quadratic form Q_a on V'. We claim that Q_a has constant rank on X. The adjoint form (§2.5 [2]) of the h^{-1}-valued form has values in h and its restriction to the polar V_0 vanishes at a, ia and factors through a quadratic form with values in the trivial line bundle. Since V_0 is the trivial bundle, this form has constant rank on V_0. Our claim now follows in the same way as Proposition 2.6 [2]. Define $f'(V) = (F \equiv V'/V'^{\perp} \otimes h, (n_w))$, where $V'^{\perp}$ denotes the orthogonal complement of V' in V' for the quadratic form Q_a and $n_w : E_w^- \to C_w$ are isomorphisms obtained from the fact that V_w' contains $C_w \otimes L_w^{-1}$.

We now claim that $H^1(F \otimes h^g)^- = 0$. Since i acts on the canonical bundle by -1 at each Weierstrass point, using Serre' duality and $(V'/V'^{\perp})^* \approx V'/V'^{\perp} \otimes h^2$, we have $h^1(F \otimes h^g)^- = h^0(V'/V'^{\perp})^+$. Writing down the cohomology exact sequence associated to the exact sequence

$$0 \to V' \to \Sigma C_w \otimes L_w^{-1} \to X \times \sum C_w/V \to 0$$

and taking invariants and anti-invariants we get $H^0(V') = 0$, $h^1(V')^- = g(\Sigma r_w/2)$, $h^1(V')^+ = \Sigma r_w/2$. Since $V'^{\perp} \subset V'$, $H^0(V'^{\perp}) = 0$. Using the Riemann-Roch theorem and Proposition 2.2, [2], since $(V_w'^{\perp})^- = 0$, we get $h^1(V'^{\perp})^+ = 0$. From the cohomology exact sequence associated to the exact sequence $0 \to V'^{\perp} \to V' \to V'/V'^{\perp} \to 0$, on taking invariants, we have $H^0(V'/V'^{\perp})^+ = 0$, i.e. $H^1(F \otimes h^g)^- = 0$.

Lemma 2.4: *There is a canonical isomorphism of V onto $H^0(F \otimes h^g)^-$ such that the following diagram commutes.*

$$\begin{array}{ccc} V & \to & C_w \\ \downarrow & & \downarrow n_w \\ H^0(F \otimes h^g)^- & \to & (F \otimes h^g)_w^- \end{array}$$

The lower horizontal map here is the evaluation at w.

Proof: See Lemma 3.6, [2]. Note that on $F \otimes h^g = V'/V'^{\perp} \otimes h^{g+1}$, the i-action is the same as that on $V'/V'^{\perp}$. Since i-acts by -1 on $L_{W,w}$ for all w, we have $(F \otimes h^g)_w^- = (V'/V'^{\perp} \otimes L_W)_w^+$ and $H^0(F \otimes h^g)^- = H^0(V'/V'^{\perp} \otimes L_W)^+$. As $H^1(F \otimes h^g)^- = 0$, from Proposition 2.2, [2] and Lemma 3.6, [2], it follows that there is a canonical isomorphism from V onto $H^0(F \otimes h^g)^-$ making the above diagram commutative.

Lemma 2.4 shows that the evaluation map $H^0(F \otimes h^g)^- \to \Sigma_w (F \otimes h^g)_w^-$ is an injection and the space $H^0(F \otimes h^g)^-$ embedded in ΣC_w via $\Sigma(n_w)$ is in fact V. Thus $f \circ f' = Id$. Also, since V is semi-stable, it follows from Proposition 2.2 that F is semistable. Finally, Proposition 2.1, (c), (iii) shows that F is irreducible.

We now proceed to show that $f' \circ f = Id$. Let $(F, (n_w)) \in I_n'$, $f(F) = H^0(F \otimes h^g)^- = V$, say. From the commutative diagram

$$\begin{array}{ccc} X \times V & \to & X \times \Sigma C_w \\ \uparrow & & \uparrow \\ V' & \to & \Sigma C_w \otimes L_w^{-1} \end{array}$$

it follows that the natural map $V' \to V$ composed with the evaluation map $V \to E = F \otimes h^g$ is zero at all w in W and at x not in W, $V^{\perp} = V'^{\perp}_x = H^0(E \otimes h^{-1})^-$ embedded in $H^0(X, E)^-$ via the divisor $x \cup ix$ (see Proposition 2.1). Hence the composite induces a map $V'/V'^{\perp} \otimes L_w \to E$. This is a map of vector bundles of the same rank n and same degree 2 gn, (as $V'/V'^{\perp}$ has a nondegenerate form with values in h^{-2}), so to show that it is an isomorphism, it is enough to show that it is a generic isomorphism. We claim that it is an isomorphism on $X - (W \cup a \cup ia)$ i.e. the map $H^0(X, E)^-/H^0(X, E \otimes h^{-1})^- \to E_x$ induced by the evaluation is an isomorphism. From the proof of Proposition 2.1 (b) we have

$$H^0(X, E)^- \xrightarrow{\text{evalua.}} (E_x \oplus E_{ix})^- \approx E_x \text{ with kernel}$$

$$H^0\left(X, E \otimes L_x^{-1} \otimes L_{ix}^{-1}\right)^- = H^0(X, E \otimes h^{-1})^-,$$

hence the claim follows.

This completes the proof of Proposition 2.3. Let $\tilde{\varepsilon}_1 \to \tilde{R}_1^f \times X$ be the universal family for semistable orthogonal bundles with i-action of fixed type τ (Theorem**, p. 75[1]). (This can be thought of as a suitable open subset in the quot. scheme for orthogonal bundles consisting of semistable bundles only.) Define a functor F from (schemes over $\tilde{R}_1^f$) to (sets) by $F(\bar{f}: S \to \tilde{R}_1^f) =$ Set of quadratic isomorphisms in $\mathrm{Hom}_{R_1^f \times W}(S \times W, (((\tilde{\varepsilon}_1 \otimes p_X^* h^g)/\tilde{R}_1^f \times W)^-)^* \otimes (\tilde{R}_1^f \times \Sigma w \times C_w)) = H^0(S \times W, (\bar{f} \times Id_W)^*(((\tilde{\varepsilon}_1 \otimes p_X^* h^g)/\tilde{R}_1^f \times W)^-)^* \otimes (\tilde{R}_1^f \times \Sigma w \times C_w))$. This functor is representable by a scheme R_2^f over $\tilde{R}_1^f$ (Proposition 1.39, p. 62, [1]). The map $f: I \to R'$ clearly induces a morphism $f_2: R_2^f \to R'$. Let $P' = f_2^{-1}(R'_{ss})$, R'_{ss} being the set of semistable points of R'. We will show below (Proposition 2.7) that P' is saturated for the action of $H \times \Pi O(Q_w)$ so that the image P of P' in the moduli space is a good quotient of P'. Here H denotes the reductive subgroup of $GL(N) = GL(H^0(X, E))$ consisting of elements commuting with i-action. Being invariant under the action of $H \times \Pi O(Q_w)$, the morphism $f_{2/P'}: P' \to R'_{ss}$ goes down to a morphism $\bar{f}: P \to R$.

LEMMA 2.5: *Let $E = F \otimes h^g$, F being a semistable orthogonal bundle. The evaluation map embeds $H^0(E)^-$ in ΣE_w^-. Let E_1 be a subbundle of E with $\mu(E_1) = 2g$. Then*

(i) $H^0(E_1)^- = H^0(E)^- \cap \sum (E_1)_w^-$ in E_w^-,
(ii) $H^0(E_1^{\perp})^- = H^0(E)^- \cap \Sigma(E_1^{\perp})_w^-$ in E_w^-,
(iii) $\dim H^0(E)^- \cap \Sigma(E_1)_w^- + \dim H^0(E)^- \cap \Sigma(E_1^{\perp})_w^- = \dim H^0(E)^-$.

PROOF: (i) Clearly, $H^0(E_1)^- \subset H^0(E)^- \cap \Sigma(E_1)_w^-$, so we have only to prove the opposite inclusion. First notice that E_1, E and E/E_1 are all semistable vector bundles with $\mu = 2g$, therefore we have the exact commutative diagram.

$$\begin{array}{ccccccccc} 0 \to & H^0(E_1)^- & \to & H^0(E)^- & \underset{p'}{\to} & H^0(E/E_1)^- & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \Sigma(E_1)_w^- & \to & \Sigma E_w^- & \to & \Sigma(E/E_1)_w^- & \to 0 \end{array}$$

Let $p : E \to E/E_1$ be the canonical surjection. We have to show that for $s \in H^0(E)^-$ if $p \circ s(w) = 0$ for all w, then $p \circ s = 0$ i.e. $p'(s) = 0$. Now, $p \circ s(w) = 0$ for all w implies that $p \circ s$ belongs to $H^0((E/E_1) \otimes L_W^{-1})$. Since E/E_1 is a semistable vector bundle with $\mu = 2g$, we have $H^0(E/E_1 \otimes L_W^{-1}) = 0$. Thus $p \circ s = 0$.

(ii) The bundle $E_1^\perp$ is defined by the exact sequence $0 \to E_1^\perp \to E \to E_1^* \otimes h^{2g} \to 0$, where the latter map is the composite of the isomorphism $E \to E^* \otimes h^{2g}$ given by the quadratic form with the canonical surjection $E^* \otimes h^{2g} \to E_1^* \otimes h^{2g}$. It follows that $\mu(E_1^\perp) = 2g$ and hence $E_1^\perp$ is semistable. The proof of (ii) is now identical to that of (i).

(iii) From the exact sequence in the above proof of the part (ii), we have

$$0 \to H^0\left(E_1^\perp\right)^- \to H^0(E)^- \to H^0\left(E_1^* \otimes h^{2g}\right)^- \to 0.$$

Now, E_1 and $E_1^* \otimes h^{2g}$ are both semistable vector bundles with i-action of the same rank, degree and $\dim(E_1)_w^- = \dim(E_1^* \otimes h^{2g})_w^-$ for all w. Therefore, by Proposition 2.2 [2], we have $\dim H^0(E_1)^- = \dim H^0(E_1^* \otimes h^{2g})^-$. Thus, $\dim H^0(E_1)^- + \dim H^0(E_1^\perp)^- = \dim H^0(E)^-$. Using (i) and (ii), part (iii) now follows.

Proposition 2.6: *Let $E = F \otimes h^g$ with F a semistable orthogonal bundle with i-action such that $f(E)$ is a semistable subspace of ΣC_w. Let E_1 be an isotropic subbundle of E with $\mu(E_1) = 2g$. Then $f(E_1 \oplus E/E_1^\perp \oplus E_1^\perp/E_1)$ is also a semistable subspace of ΣC_w.*

Proof: Let $N = \Sigma(E_1)_w^-$, $N^\perp = \Sigma(E_1^\perp)_w)^-$, $V = f(E)$. We claim that $f(E_1 \oplus E/E_1^\perp \oplus E_1^\perp/E_1)$ belongs to the $\prod O(Q_w)$ orbit of $V \cap N \oplus V/V \cap N^\perp \oplus V \cap N^\perp/V \cap N$ in ΣC_w. We have

$$\begin{aligned} & f\left(E_1 \oplus E/E_1^\perp \oplus E_1^\perp/E_1\right) \\ & = H^0(E_1)^- \oplus H^0\left(E/E_1^\perp\right)^- \oplus H^0\left(E_1^\perp/E_1\right)^- \\ & \hookrightarrow \sum(E_1)_w^- + \sum\left(E/E_1^\perp\right)_w^- + \sum\left(E_1^\perp/E_1\right)_w^-. \end{aligned}$$

The exact sequence $0 \to E_1 \to E_1^{\perp} \to E_1^{\perp}/E_1 \to 0$ gives the diagram

$$\begin{array}{ccccccc} 0 \to & H^0(E_1)^- & \to & H^0(E_1^{\perp})^- & \to & H^0(E_1^{\perp}/E_1)^- & \to 0 \\ & \updownarrow & & \updownarrow & & \updownarrow & \\ 0 \to & \sum (E_1)_w^- & \to & \sum (E_1^{\perp})_w^- & \to & \sum (E_1^{\perp}/E_1)_w^- & \to 0 \end{array}$$

and hence by Lemma 2.5 the diagram

$$\begin{array}{ccccc} H^0(E_1^{\perp}/E_1)^- & \overset{\approx}{\leftarrow} & \dfrac{H^0(E_1^{\perp})^-}{H^0(E_1)^-} & = & \dfrac{V \cap N^{\perp}}{V \cap N} \\ \updownarrow & & \updownarrow & & \updownarrow \\ \sum (E_1^{\perp}/E_1)_w^- & \overset{\approx}{\leftarrow} & \sum \dfrac{(E_1^{\perp})_w^-}{(E_1)_w^-} & = & \dfrac{N^{\perp}}{N}. \end{array}$$

Similarly, we have another commutative diagram

$$\begin{array}{ccccc} H^0(E/E_1^{\perp})^- & \overset{\approx}{\leftarrow} & \dfrac{H^0(E)^-}{H^0(E_1^{\perp})^-} & = & \dfrac{V}{V \cap N^{\perp}} \\ \updownarrow & & \updownarrow & & \updownarrow \\ \sum (E/E_1^{\perp})_w^- & \overset{\approx}{\leftarrow} & \sum \dfrac{E_w^-}{(E_1^{\perp})_w^-} & = & \dfrac{\Sigma C_w}{N^{\perp}}. \end{array}$$

Thus $f(E_1 \oplus E_1^{\perp}/E_1 \oplus E/E_1^{\perp})$ is the element $V \cap N \oplus [(N^{\perp} \cap V)/(N \cap V)] \oplus [V/(V \cap N^{\perp})]$ in $N \oplus N^{\perp}/N \oplus \Sigma C_w/N^{\perp}$, the latter being isomorphic to ΣC_w under $\prod O(Q_w)$. By fixing a $\prod O(Q_w)$-isomorphism φ of the two spaces, we can identify them.

Let P be the parabolic subgroup in $\prod O(Q_w)$ which fixes the flag $0 \subset N \subset N^{\perp} \subset \Sigma C_w$. It is well-known that there exists a one-parameter subgroup k such that $P = P_k$, the parabolic associated to k; such a k is not unique. The 1-parameter subgroup k keeps N and $N^{\perp}$ invariant, it acts on N and $\Sigma C_w/N^{\perp}$ by the characters $t \mapsto t^a$ and $t \mapsto t^{-a}$ respectively, the action on $N^{\perp}/N$ being trivial (see the proof of Proposition 5.3 [6]). On the space $(\lim_{t \to 0} k_t(V))$, k acts as on the associated graded space $V \cap N \oplus [(V \cap N^{\perp})/(V \cap N)] \oplus [V/(V \cap N^{\perp})]$, the two spaces being isomorphic under φ. Hence, on the line in $\Lambda^{\frac{1}{2}\Sigma C_w}(\Sigma C_w) = F$ which represents this limit, k acts by the character $t \mapsto t^{s(V)}$, where

$$\begin{aligned} s(V) &= a \dim V \cap N + (-a) \dim V/V \cap N^{\perp} \\ &= 0, \qquad \text{by Lemma 2.5.} \end{aligned}$$

The result now follows from the following by taking x to be the space $\mathrm{P}(V)$ considered as a point of $\mathrm{P}(F)$.

Claim: Let x be a semistable point of $\mathrm{P}(F)$. Let k be a 1-parameter subgroup such that $s(x)=0$. Then $x_0=\lim_{t\to 0}k_t(x)$ is semistable.

PROOF: Let $(e_i)_{i\in I}$ be a basis of F consisting of eigenvectors for k and $\hat{x}=\Sigma_i x_i e_i$. Let $(e_j)_{j=1,\dots,m}$ be the set of those e_j for which $x_j\neq 0$. Let a_j be the eigenvalue corresponding to e_j. We can assume that $a_1\leqslant a_2\leqslant \dots\leqslant a_m$. Now, $s(x)=0$ iff $\min_j a_j=0$. Hence $s(x)=0$ implies $a_1=0$, $a_j\geqslant 0$ for $j>1$. Let n be the integer such that $a_1=\dots=a_n=0$, $a_{n+1}\neq 0$. Then $\hat{x}_0=\lim_{t\to 0}k_t(\hat{x})=\Sigma_{j=1}^n x_j e_j$ exists. Any $G\equiv\prod O(Q_w)$-invariant function f which is nonzero at $\hat{x}$ is nonzero and constant on $G\hat{x}$ and hence so at $\hat{x}_0\in\overline{G\hat{x}}$. This proves that x_0 is semistable.

PROPOSITION 2.7: *The set $P'=f^{-1}(R'_{ss})$ is saturated under the equivalence of orthogonal bundles.*

PROOF: By induction, it follows from the above proposition that $f(gr\ E)$ is a semistable point if $f(E)$ is so, $gr\ E$ denoting the associated graded of E. Now, let E_1 and E_2 be two semistable orthogonal bundles such that $gr\ E_1\approx gr\ E_2$, $E_1\in f^{-1}(R'_{ss})$. We have to show that $E_2\in f^{-1}(R'_{ss})$ i.e. $f(E_2)$ is a semistable point. We know that $f(gr\ E_2)$ is a semistable point. From the proof of Proposition 2.6, we know that $gr\ f(E_2)$ and $F(gr\ E_2)$ are in the same $\prod O(Q_w)$-orbit. Thus $gr\ f(E_2)$ is a semistable point and hence $f(E_2)$ is a semistable point.

We now proceed to show that $\bar{f}\colon P\to R$ is an isomorphism.

Let $P'_m\subset P'$ be defined as the subset corresponding to bundles F with $F=I_{n-m}\oplus F_m$, where I_{n-m} is a trivial bundle with trivial i-action and F_m is an irreducible orthogonal bundle of rank m with i-action. Let $P_m=$ image of P'_m in P. Let $R'_m\subset R'_{ss}$ be the subset consisting of elements V such that Q_2 on V has rank m. We claim that P'_m maps into R'_m. We have $H^0(F)^+=H^0(I_{n-m})^+\oplus H^0(F_m^+)=H^0(I_{n-m})^+$ so that for F in P'_m, $\dim H^0(F)^+=n-m$. By the proof of Proposition 2.1 (b) and (c), it follows that the rank of Q_2 restricted to $H^0(F\otimes h^g)^-$ is m, noting that $h^1(F\otimes h^{(g-1)})^-=h^0(F)^+$. Also, $H^0(F\otimes h^g)^-=H^0(F_m\otimes h^g)^-$ so that we can as well replace F by the irreducible bundle F_m to get an induction on m. Note that $P=\bigcup_{m\leqslant n}P_m$ in view of Lemma 1.7, and $R=\bigcup_{m\leqslant n}R_m$, R_m being the image of R'_m in R. By induction and Proposition 2.3 we in fact have a bijection from I'_m onto R'_m given by restriction of f to I'_m, where I'_m is the subset of $f^{-1}(R'_{ss})\cap I$ consisting of bundles of the form $I_{n-m}\oplus F_m$, F_m irreducible, of rank m.

PROPOSITION 2.8: *$\bar{f}$ is birational.*

PROOF: The bijection $f: I'_n \to R'_n$ induces an isomorphism of the set of stable points in R_n onto an open subset of P_n. Therefore to show the birationality of $\bar{f}$, it suffices to show that P_n contains a subset which is open in M. We claim that the set U of stable irreducible bundles in M is such a set. The subset $\bar{U}$ of $\tilde{R}_1^f$ corresponding to stable orthogonal bundles F with i-action which are irreducible i.e. satisfy $H^0(F)^+ = 0$ is an open subset of $\tilde{R}_1^f$. The set U is the image of $\bar{U}$ in M. Since the quotient on stable points is a geometric quotient, it follows that U is open in M.

We now proceed to show that $\bar{f}: P_m \to R_m$ is bijective. As remarked earlier, f restricted to I'_m is a bijection onto R'_m. This shows that $\bar{f}$ is a surjection. In fact the inverse of $\bar{f}$ restricted to P_m can be constructed as follows. We first construct a map $f': (R'_m)_{ss} \to I'_m$ which goes down to the inverse $\bar{f}'$ of $\bar{f}$.

On $(R'_m)_{ss}$ there is a universal bundle $\mathbb{V}$ which associates to V in $(R'_m)_{ss}$ the space V contained in ΣC_w. $\mathbb{V}$ carries a quadratic form q with values in h^{2g+1} which, by definition of $(R'_m)_{ss}$ has the property that its restriction to $\mathbb{V}/(R'_m)_{ss} \times (X - (W \cup a \cup ia))$ has rank exactly m. As in the proof of Proposition 2.3, we have a diagram

$$\begin{array}{ccccccc} 0 \to \mathbb{V} & \to & (R'_m)_{ss} \times X \times \sum C_w & \to & (R'_m)_{ss} \times X \times \sum C_w/\mathbb{V} & \to 0 \\ \uparrow & & \uparrow & & \| & \\ 0 \to \mathbb{V}' & \to & (R'_m)_{ss} \times X \times \sum C_w \otimes L_w^{-1} & \to & (R'_m)_{ss} \times X \times \sum C_w/\mathbb{V} & \to 0 \end{array}$$

with $\mathbb{V}'$ carrying a nondegenerate quadratic form Q_a of rank m on $(R'_m)_{ss} \times X$. Let $\mathbb{F}_m = \mathbb{V}'/\mathbb{V}'^{\perp} \otimes p_2^*(h)$ and $\eta_w: \mathbb{F}_m^-/(R'_m)_{ss} \times (w) \to (R'_m)_{ss} \times C_w$ isomorphisms obtained in the construction (as in Proposition 2.3). Define $\mathbb{F} = I_{n-m} \oplus \mathbb{F}_m$ where I_{n-m} is a trivial orthogonal bundle with a trivial i-action, $\oplus$ denoting an orthogonal direct sum. For $V \in (R'_m)_{ss}$ define $f'(V) = (\mathbb{F}/V \times X,\ \eta_w$ restricted to $\mathbb{F}/V \times X)$. Then f' clearly defines a map from $(R'_m)_{ss}$ to I'_m which is the inverse of f, it is defined as a morphism into P'_m locally. This map is equivariant for the actions of $\prod O(Q_w)$ on $(R'_m)_{ss}$ and I'_m (if $V \overset{f'}{\mapsto} (F, n_w)$, $gV \mapsto (F, g \circ n_w)$) and hence goes down to a morphism $\bar{f}'_m: R_m \to P_m$. We check that $\bar{f}' \circ \bar{f} = Id$. Denoting the equivalence class of an element by a bar above it since for $\bar{E}'$ in P_m, we have a bundle E in P'_m mapping into $\bar{E}'$, i.e. an E in P'_m with $\bar{E} = \bar{E}'$ we get

$$\bar{f}' \circ \bar{f}(\bar{E}') = \bar{f}\left(\overline{f(E, n_w)}\right) = \left(\overline{f' \circ f(E, n_w)}\right) = \left(\overline{E, n_w}\right) = \bar{E} = \bar{E}'$$

Thus we have proved

PROPOSITION 2.9: *$\bar{f}$ restricted to P_m is an isomorphism.*

PROPOSITION 2.10: *$\bar{f}$ has finite fibres.*

PROOF: Let $V \in R$. Then V belongs to finitely many R_i, say, $R_{i_1}, \ldots, R_{i_k}$. Let $\bar{f}_j$ denote the restriction of $\bar{f}$ to P_{i_j}. Then $(\bar{f})^{-1}(V) = \bigcup_{j=1}^{k} (\bar{f}_j)^{-1}(V)$. Since $\bar{f}_j$ is an injection by Proposition 2.9, the result follows.

Thus we have a surjective birational morphism $\bar{f}: P \to R$ with finite fibres. We will show in the next section that R is normal (Proposition 3.8). Since P (and hence R) is irreducible as M is so (Proposition 3.9. [1]), it follows by Zariski's main theorem that $\bar{f}$ is an isomorphism of P onto R. So, since R is complete, it follows that P is complete and hence a closed and open subset of M. Irreducibility of M then implies $P = M$. Thus $\bar{f}$ is an isomorphism of the moduli space M onto R.

§3. Spaces related to a pencil of quadrics

Let V be a vector space of dimension $2N$ with non-degenerate quadratic forms Q_1 and Q_2. For a subspace V_1 of V, $V_1^{\perp 1}$ and $V_1^{\perp 2}$ denote the orthogonal complements of V_1 with respect to Q_1 and Q_2 respectively. For $k \leqslant N$, let S_k be the subvariety of $\mathrm{Grass}_n(V)$ (= the grassmannian of n-dimensional subspaces of V) consisting of those subspaces V_1 of V such that rank of Q_2 restricted to V_1 is exactly k. Let $S^k = \bigcup_{1 \leqslant k} S_1$ and let S'_k, S'^k denote the corresponding varieties for Q_1. By Witt's theorem, S_k (resp. S'_k) is an orbit under $O(Q_2)$ and hence is non-singular.

PROPOSITION 3.1: *Let $n = N$. Then*

$$\dim S_k - \dim S_{k-1} = N - k + 1.$$

PROOF: Denote by U the null space for Q_2 restricted to V_1, i.e., $U = V_1 \cap V_1^{\perp 2}$. Let $m = N - k$, $P =$ the Lie algebra of the parabolic subgroup of $O(Q_2)$ keeping the following flag invariant:

$$0 \subset U \subset U^{\perp 2} \subset V.$$

With respect to a suitable basis $e_1, \ldots, e_{2N}$ of V where $e_1, \ldots, e_m$ is a basis of U and $e_1, \ldots, e_{2N-m}$ is a basis of $U^{\perp 2}$, the matrix of Q_2 can be written in the form

$$Q_2 = \begin{bmatrix} 0 & 0 & I_m \\ 0 & I_{2N-2m} & 0 \\ I_m & 0 & 0 \end{bmatrix},$$

where I_m and I_{2N-2m} are unit matrices of rank m and $2N-2m$ respectively. A matrix M belongs to the Lie algebra of $O(Q_2)$ if and only if $MQ_2 + Q_2^t M = 0$. Taking M in the form

$$M = \begin{bmatrix} A_{mxm} & B_{mx(2N-2m)} & 2_{mxm} \\ D_{(2N-2m)\times m} & E_{(2N-m)^2} & F_{(2N-2m)\times m} \\ G_{mxm} & H_{mx(2N-2m)} & J_{mxm} \end{bmatrix}$$

the above condition reduces to : E, C and G are skew symmetric, $F = -{}^tB$, $J = -{}^tA$, and $H = -{}^tD$, where ${}^tA$ denotes the transpose of A, etc. Since for M in P, exp M keeps the flag $0 \subset U \subset U^{\perp 2} \subset V$ invariant, $H = D = G = 0$. Hence dimension of P is given by

$$\dim P = m^2 + m(2N-2m) + \frac{m(m-1)}{1} + \frac{(2N-2m)(2N-2m-1)}{2}.$$

If P'_m denotes the Lie algebra of the parabolic fixing the flag

$$0 \subset U \subset V_1 \subset U^{\perp 2} \subset V, \qquad \dim U = m, \qquad U = V_1 \cap V_1^{\perp 2},$$

then $M \in P'_m$ satisfies (in addition to above conditions) the condition that

$$E = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}$$

where E_1, E_2 are skew symmetric matrices of rank $N-m$. This follows from the fact that V_1/U is a nondegenerate subspace of $[U^{\perp 2} = V_1 + V_1^{\perp 2}]/U$ so the basis $(e_i)_i$ of V can be so chosen that $e_1, \ldots, e_N$ form a basis of V_1. Hence

$$\dim P'_m = m^2 + m(2N-2m) + \frac{m(m-1)}{2} + (N-m)(N-m-1)$$

$$= m^2 + \frac{m(m-1)}{2} + (N-m)(N+m-1).$$

Thus,

$$\dim S_k - \dim S_{k-1} = \dim P'_{m+1} - \dim P'_m$$

$$= m+1$$

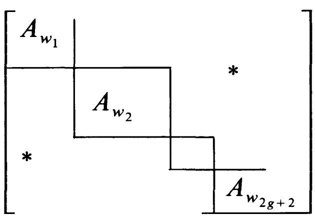
$$= N-k+1.$$

Proposition 3.2: *Let* $Q_1 = \Sigma Q_w$, $Q_2 = \Sigma k_w Q_w$, Q_w *being non-degenerate quadratic forms on a space* C_w, $V = \Sigma C_w$ *and* k_w*'s mutually distinct scalars. Suppose* V_1 *in* $S'_{k'} \cap S_k$ *is such that either* $V_1 \cap C_w = 0$ *or the projection* $V_1 \to C_w$ *is onto. Then* $S'_{k'}$ *and* S_k *intersect transversally at* V_1.

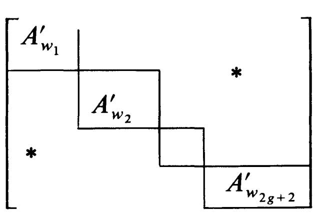
Proof: A neighbourhood of V_1 in $\mathrm{Grass}_n(V)$ is given by $\mathrm{Hom}(V_1, V/V_1)$. Hence the tangent space to $\mathrm{Grass}_n(V)$ at V_1 can be identified with $\mathrm{Hom}(V_1, V/V_1)$. We have an exact sequence

$$0 \to P \to \mathrm{Hom}(V, V) \to \mathrm{Hom}\left(V_1, \frac{V}{V_1}\right) \to 0$$

where P is the parabolic fixing the flag $0 \subset V_1 \subset V$. Since S_k (resp. $S'_{k'}$) is an orbit under $O(Q_2)$ (resp. $O(Q_1)$), the space H_1(resp. H_2) $= (A \in \mathrm{Hom}(V, V)$ which are skew symmetric for Q_2 (respectively Q_1)) maps onto the tangent space of S_k (respectively $S'_{k'}$) at V_1, the latter being contained in $\mathrm{Hom}(V_1, V/V_1)$. Hence, to show that $S_k S'_k$, intersect transversally at V_1, it suffices to show that H_1, H_2 and P together span $\mathrm{Hom}(V, V)$. Clearly the span of H_1 and H_2 is the set of matrices of the form



where A_{w_i} are skew symmetric matrices for Q_1/C_{w_i} i.e. elements of $\mathrm{Hom}(V, V)$ which when mapped to elements of $\mathrm{Hom}(C_w, C_w)$ are skew symmetric for $Q_1/C_w = Q_2/C_w$ for all w. This follows from the fact that H_1 = set of Skew symmetric matrices, $H_2 = \{(s_{ij}) \mid (s_{ij}k_j + s_{ji}k_i = 0), k_i = k_w$ on $c_w\}$. Hence to prove the proposition we have only to prove that P contains matrices of the form



where A'_{w_i} are arbitrary square matrices. This follows from the following lemma.

LEMMA 3.3: *Let* $V = C_w \oplus V_2$, $V_2 = \Sigma_{w' \neq w} C_{w'}$. *Suppose that either* $V_1 \cap C_w = 0$ *or* $V_1 \to C_w$ *is onto. Then any homomorphism* $f\colon C_w \to C_w$ *can be extended to a homomorphism* $\Psi : V \to V$ *such that* Ψ *keeps* V_1 *invariant,* $P_1 \circ \Psi \circ j_1 = f$ *and* $P_2 \circ \Psi \circ j_2 = 0$ *where* $j_1 : C_2 \hookrightarrow V$, $j_2 : V_2 \hookrightarrow V$ *are canonical inclusions and* P_i, $i = 1, 2$ *are canonical projections.*

PROOF: *Case (i).* Suppose $V_1 \to C_w$ is a surjection. Let s be a section of this surjective homomorphism. Let $f_1 = s \circ f$. Define ψ by $\psi/C_w = f_1$, $\psi/V_2 = 0$. Then $P_1 \circ \psi \circ j_1 = P_1 \circ f_1 = P_1 \circ s \circ f = f$ and $\psi \circ j_2 = 0$ by definition. Moreover ψ keeps V_1 invariant, for if $v = (v_1, v_2)$, $v_1 \in C_w$, $v_2 \in V_2$. $\Psi(v) = f_1(v_1) \in V_1$.

Case (ii). Suppose $V_1 \cap C_w = 0$. Then the projection $V_1 \overset{p_2}{\to} P_2(V_1)$ is an isomorphism, we will denote its inverse by P_2^{-1}. If $P_1(V_1) = 0$ i.e. $V_1 \subset V_2$ we can extend f to ψ by defining $\psi(V_2) = 0$. So we may assume $P_1(V_1) \neq 0$.

Let s be a fixed section of the surjection $V_1 \to P_1(V_1)$. Let V_4 and V_3 denote respectively complements of $P_1(V_1)$ and $P_2(V_1)$ in C_w and V_2. Define $f_1 : P_1(V_1) \to V$ by $f_1 = f + p_2 \circ s$. Define $f_2 : P_2(V_1) \to C_w$ by $f_2 = (Id - f) \circ P_1 \circ P_2^{-1}$. Then $\psi : V \to V$ is defined by

$$\psi = \begin{cases} f & \text{on} \quad V_4, \\ f_1 & \text{on} \quad P_1(V_1), \\ f_2 & \text{on} \quad P_2(V_1), \\ 0 & \text{on} \quad V_3. \end{cases}$$

Since $P_1 \circ f_1 = f$, it follows that $P_1 \circ \psi \circ j_1 = f$. Consider $P_2 \circ \psi \circ j_2(V_2) = P_2 \circ \psi \circ j_2(P_2(V_1)) = P_2 \circ f_2(P_2(V_1)) = 0$ as f_2 maps into C_w.

It remains to check that V_1 is kept invariant by ψ. Let

$$(v_1 + v_2) \in V_1, v_1 \in P_1(V_1), v_2 \in P_2(V_1).$$

Then

$$\begin{aligned}
\psi(v_1 + v_2) &= f_1(v_1) + f_2(v_2) \\
&= f(v_1) + P_2 \circ s(v_1) + (Id - f)(v_1) \\
&= v_1 + P_2 \circ s(v_1) \\
&= P_2 \circ s(v_1) + P_2 \circ s(v_1) \\
&= s(v_1) \in V_1.
\end{aligned}$$

COROLLARY 3.4: *In the notations of Proposition* 3.2, *let* $V = \Sigma C_w$ *with* C_w *of dimension* 1 *for all* w. *Then* S_k *and* $S'_{k'}$ *intersect transversely. In particular* $S_k \cap S'_{k'}$ *is nonsingular.*

PROOF: We claim that if C_w is of dimension 1, then for a subspace V_1 of V, either $V_1 \to C_w$ is onto or $V_1 \cap C_w = 0$. For, C_w being one dimensional, $V_1 \to C_w$ not onto means $V_1 \to C_w$ is zero i.e. $V_1 \subset \Sigma_{w' \neq w} C_{w'}$, i.e. $V_1 \cap C_w = 0$. The assertions now follow from Propositions 3.2 and the fact that S_k, S_k', are nonsingular being G/P's.

COROLLARY 3.5: *For a subset X of Grass V, let X_{ss} denote the set of semistable points of X, for the action of $\Pi_{w \in W} O(Q_w)$. Then $(S_0 \cap S_k')_{ss}$ and $(S_k \cap S_0')_{ss}$ are nonsingular. In particular, the space of semistable fixed dimensional spaces contained in the intersection of quadrics $Q_1 = \Sigma Q_w$, $Q_2 = \Sigma k_w Q_w$ is nonsingular.*

PROOF: By the proof of Proposition 0.9 [1], it follows that for $V_1 \in S_0$ or S_0', $V_1 \cap C_w = 0$ for all w in W. Proposition 3.2 now implies that $(S_0')_{ss}$ and $(S_k)_{ss}$ (respectively $(S_0)_{ss}$ and $(S_k')_{ss}$ intersect transversely and hence $(S_0' \cap S_k)_{ss}$ and $(S_0 \cap S_k')_{ss}$ are nonsingular.

Henceforth we will work with $\text{Grass}_N(V)$. We have $R' = S_0' \cap S^n$, n integer denoting the rank of the orthogonal bundles. The subset $(R_n')_{ss} = (S_0' \cap S_n)_{ss}$ is nonsingular in itself by Corollary 3.5 and being an open subset of R_{ss}' is nonsingular in R_{ss}' too. In view of proposition 3.1 and transversality, its complement $(S_0' \cap S^{n-1})_{ss}$ has codimension $N - n + 1$ in R_{ss}'. Hence the singular set of R_{ss}' has codimension $\geqslant 2$ if $N \geqslant n + 1$. In any case, $n \leqslant N$ and for $n = N$, $S_0' \cap S^n = S_0'$ and hence nonsingular. So R_{ss}' will be normal if it is Cohen-Macaulay.

PROPOSITION 3.6: *R_{ss}' is Cohen-Macaulay.*

PROOF: Let A be the coordinate ring of an affine open subset U of $(S_0')_{ss}$. Since $(S_0')_{ss}$ is a nonsingular variety, A is a regular ring. The ideal I of A defining $A \cap (S^n)_{ss}$ is the ideal generated by all $(n+1) \times (n+1)$ minors of A. Since A is regular, grade I = height I = codimension of $U \cap (S^n)_{ss}$ in U. By Proposition 3.1,

$$\dim U \cap (S^n)_{ss} \leqslant \dim(S^n \cap S_0')_{ss} = N(n-1) - \frac{n(n-1)}{2},$$

$$\dim(S_0')_{ss} = \frac{N(N-1)}{2}$$

so that

$$gr\, I \geqslant \frac{N(N+1)}{2} - Nn + \frac{n(n-1)}{2}.$$

Applying a result of Kutz (See abstract, [3]) it follows that A/I is Cohen-Macaulay. Thus R_{ss}' is Cohen-Macaulay.

COROLLARY 3.7: *$(S_0' \cap S_n)_{ss}$ is dense in R'_{ss}.*

PROOF: Follows as Cohen-Macaulay implies equidimensionality and $\dim(S_0' \cap S_k)_{ss} < \dim(S_0' \cap S_n)_{ss}$ if $k < n$.

PROPOSITION 3.8: *R is normal.*

PROOF: Since R'_{ss} is normal and R is the quotient of R'_{ss} by $\Pi O(Q_w)$, it follows that R is normal.

§4. Applications to Clifford bundles and rank two vector bundles

Let Γ and Γ^+ denote respectively the Clifford group and even Clifford group. We recall that if a $\Gamma(2n)$ bundle P is a lift of an $O(2n)$-bundle F, then an i-action on F induces an isomorphism $i^*P \approx \alpha \circ P$, α being a line bundle (see § 3, [6] for details).

THEOREM 2: *The moduli space $\overline{M}$ of semistable Γ^+-bundles P of rank $2n$ and of fixed norm, together with isomorphisms $i^*P \approx \alpha \circ P$ of Γ^+-bundles (α a line bundle) is isomorphic to the quotient by $\Pi SO(Q_w)$ (in the sense of geometric invariant theory) of the component R_0 of R' corresponding to a fixed system of maximal isotropic spaces for Q_1.*

PROOF: The composite $R_0 \hookrightarrow R' \to R'/\Pi O(Q_w)$ gives an isomorphism $R_0/S\Pi O(Q_w) \overset{\approx}{\to} R'/\Pi O(Q_w)$, where $S\Pi O(Q_w)$ denotes the subgroup of $\Pi O(Q_w)$ consisting of elements with determinant 1. Hence, in view of Theorem 1, to show that $\overline{M} \to R_0/\Pi SO(Q_w)$ is a bijection, we have only to show that the space $\overline{M}$ of Γ^+-bundles with i-action modulo $S((\mathbb{Z}/2)^W)$ is in bijective correspondence with the space M of orthogonal bundles with i-action. Notice that $S\Pi O(Q_w)/\Pi SO(Q_w) \approx S(\mathbb{Z}/2)^W$. Now, the group J_2 of elements of order 2 of the Jacobian of X acts simply transitively on the set $\bar{s}$ of Γ^+-bundles P of fixed norm with a fixed associated (special) orthogonal bundle, the action of α in J_2 on P given by $P \to \alpha \circ P$. There is a surjection from $S(\mathbb{Z}/2)^W$ onto J_2 (Lemma 2.1, [2]) with kernel $\mathbb{Z}/2$, using which we get an action of $S(\mathbb{Z}/2)^W$ on $\bar{s}$. The assertion now follows. Note that an orthogonal bundle can have only two special orthogonal structures.

Now to check that the morphism $\Psi : \overline{M} \to R_0/\Pi SO(Q_w)$ is an isomorphism, we have only to check that the set theoretic inverse Ψ^{-1} is a morphism. In view of Theorem 1, Ψ^{-1} composed with the canonical morphism $\overline{M} \to M$ is a morphism as can be seen from the diagram

$$\begin{array}{ccc} \overline{M} & \overset{\psi}{\to} & R_0/\Pi SO(Q_w) \\ \downarrow & & \downarrow \\ M & \underset{f}{\overset{\approx}{\to}} & R_0/S\Pi O(Q_w). \end{array}$$

The result now follows from the fact that given an orthogonal bundle on $T \times X$ and t in T, there exists an etale neighbourhood $U \to T$ of t such that the induced bundle on $U \times X$ can be lifted to a Γ^+-bundle (see: Proof of Theorem 2, [6]).

Specialising to the case $\dim C_w = 1$ for all w, we get the following theorem.

THEOREM 3: *The moduli space of semistable orthogonal bundles E of rank* $2n$ *with i-action and with* Γ^+*-structures such that* $\dim(E)_w^- = 1$ *for all w, is isomorphic to the space R of* $(g+1)$*-dimensional subspaces of* k^{2g+2} $(k = \text{base field})$ *which are isotropic for* Q_1*, belong to one component of maximal isotropic spaces for* Q_1 *and have rank* $\leqslant 2n$*, for* Q_2*, where* $Q_1 = \Sigma X_i^2$, $Q_2 = \Sigma k_i X_i^2$, k_i *being mutually distinct scalars corresponding to Weierstrass points.*

From this theorem, we can now deduce some results of [2] viz. Theorem 4 below. This throws more light on the proof of Theorem 4 given in [2].

THEOREM 4: *(I) The quotient by involution i of the moduli space U of semistable vector bundles of rank 2 and fixed determinant of even degree on X is isomorphic to the space R of* $(g+1)$*-dimensional subspaces of* k^{2g+2} *which are maximal isotropic for the quadratic form* $Q_1 = \Sigma_{i=1}^{2g+2} X_i^2$*, belong to the same system of maximal isotropic spaces for* Q_1 *and have rank* $\leqslant 4$ *for the quadratic form* $Q_2 = \Sigma k_i X_i^2$, k_i *being mutually distinct scalars corresponding to Weierstrass points.*

(II) For $m \leqslant 4$*, if* R_m *denotes the subset of R consisting of those spaces which are of rank* $\leqslant m$ *for* Q_2*, then under the above isomorphism, the i-fixed subvariety of U, the Kummer variety and the set of elements of order 2 of the Jacobian correspond respectively to* R_3*,* R_2 *and* R_1*.*

PROOF: We shall deduce Theorem 4 from Theorem 3 by taking $n = 2$. The group $\Gamma^+(4)$ is isomorphic to the subgroup G of $GL_2 \times GL_2$ consisting of elements (A, B) such that $\det A \cdot \det B = 1$. Hence a $\Gamma^+(4)$-bundle P on X is essentially a pair of rank two vector bundles (M, N) with $\det M \otimes \det N =$ trivial bundle. There is an i-action on it given by $(M, N) \mapsto (N, M)$ and the associated orthogonal bundle is given by $F = M \otimes N = M \otimes i^*M$. Thus, on F, there is an i-action given by switching of tensors, so that F_w^- is the space of skew symmetric tensors in $M_w \otimes M_w$ and hence of dimension one. Since i acts trivially on h^g at all Weierstrass points, it follows that $\dim E_w^-$ is also one for all w. Thus Theorem 3 is applicable. To complete the proof of Part I, we have only to check that $\det M$ is of even degree. Since $\dim F_w^- = 1$ for all w, it follows from Proposition 3.3, [6] that F is of even type i.e. the $\Gamma^+(4)$-bundle P has spinor norm of even degree. Hence from the following lemma, we get that M has even degree.

LEMMA 4.9: *Under the identification $\Gamma^+(4) \approx G =$ the subgroup of $GL_2 \times GL_2$ consisting of elements (A, B) such that $\det A \cdot \det B = 1$, we have*

$$\text{spinor norm of } (A, B) = \det A \text{ or } (\det A)^{-1}.$$

PROOF: We first show that any homomorphism $\varphi: G \to G_m$ is of the form $f^r: (A, B) \mapsto \det A^r$, r integer. For $r = 1$, f^1 induces an isomorphism $G/SL_2 \times SL_2 \approx G_m$. Since there exists no nontrivial homomorphism $SL_2 \times SL_2 \to G_m$, any φ goes down to a map $\bar{\varphi}: G/SL_2 \times SL_2 = G_m \to G_m$, hence $\bar{\varphi}$ must be of the form $x \mapsto x^r$ i.e. $\varphi = f^r$, for some integer r.

In particular, the norm map $G \equiv \Gamma^+(4) \to G_m$ also must be equal to f^r for some r. For $r \neq \pm 1$, the kernel of f^r will contain roots of unity of order > 2 and hence will be strictly bigger than $SL_2 \times SL_2$; whereas the kernel of the norm map is $\Gamma_0^+(4) \approx SL_2 \times SL_2$. Hence the norm map must equal f^r, $r = \pm 1$.

For the proof of Theorem 4, (II), we only remark that by arguments after Lemma 6.8 [2], the i-fixed subvariety of U, the Kummer variety and the set of elements of order two of the Jacobian correspond respectively to P_3, P_2 and P_1 in the notations of § 2.

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School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road
Bombay 400 005
India