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GONZALO RIERA

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THE GEOMETRY OF THE PERIOD MAPPING ON CYCLIC COVERS OF \mathbb{P}_1

Gonzalo Riera

Introduction

The tangent space to the period space for Riemann surfaces of genus g at a curve C is naturally isomorphic to the second symmetric product

$$S^{(2)}H^0(C; \Omega_C^1)^* \quad (0)$$

of the dual of the vector space of holomorphic differentials on C . If C is Galois, then its group of automorphisms acts on the vector space (0), and the representation theory of this situation was analyzed classically by Chevalley and Weil. Our aim in this and a future paper is to analyze the relationship between subspaces of (0) described representation-theoretically and the geometric properties of deformations of C in directions lying in these subspaces. The present paper deals with the case in which C is cyclic over \mathbb{P}_1 .

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Algebraic differential on \mathbb{P}_1

A rational differential form ω on $\mathbb{C} - \{0\}$ is the pull-back of a form on \mathbb{P}_1 if is homogeneous of degree 0 and satisfies

$$\langle \omega, \theta \rangle = 0$$

where $\theta = x_0 \partial / \partial x_0 + x_1 \partial / \partial x_1$ (see [6]).

Thus an algebraic one-form on \mathbb{P}_1 can be expressed as

$$p(x_0, x_1)\Omega/q(x_0, x_1)$$

where $\Omega = \langle \theta, dx_0 \wedge dx_1 \rangle = x_0 dx_1 - x_1 dx_0$ and p, q are homogeneous polynomials such that

$$\deg p = \deg q - 2$$

Algebraic differentials on a cyclic cover of \mathbb{P}_1

Let n, m be integers $n \geq 2, m \geq 1$ and consider a divisor

$$a_1 + a_2 + \dots + a_{nm} \tag{1}$$

of distinct non-zero complex numbers in $\mathbb{C} \cup \{\infty\} = P_1$. Also, denote by M the line bundle associated to that divisor. Since $H^1(\mathbb{P}_1, \mathcal{O}^*) = \mathbb{Z}$ and the Chern class $c_1(M) = nm$ there exists a line bundle L such that the following diagram commutes

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ \downarrow \text{id} & & \downarrow \\ \mathbb{P}_1 & \rightarrow & \mathbb{P}_1 \end{array}$$

where λ raises a local section to the power n .

If s is a global section of M given by a homogeneous polynomial

$$F(x_0, x_1) = \prod_{i=1}^{nm} (x_0 - a_i x_1)$$

then $C = \lambda^{-1}(s(\mathbb{P}_1))$ is a cyclic covering whose ramification divisor lies over the given divisor (1).

The function $y = F^{1/n}$ is a homogeneous form of degree m on C so that in affine coordinates for \mathbb{P}_1 , the surface is the Riemann surface of the equation

$$y^n = (x - a_1) \dots (x - a_{nm}).$$

Let $\sigma: C \rightarrow C$ be a generator of the natural cyclic group of analytic automorphisms of the curve. The automorphism σ^* acts on differentials on C and thus any differential can be expressed as a linear combination of

$$\omega_k \quad k = 0, 1, \dots, n-1$$

where

$$\sigma^*(\omega_k) = \zeta^{-k} \omega_k \quad (\zeta = \exp(2\pi i/n)).$$

If we multiply this differential by the function

$$y^k / x_0^{mk}$$

we obtain a differential on \mathbb{P}_1 :

$$y^k \omega_k / x_0^{mk} = p\Omega/q.$$

Thus redefining p and q we have that any algebraic differential on C is a linear sum of

$$p\Omega/qy^k \tag{2}$$

with p, q relatively prime homogeneous polynomials such that

$$\deg p + 2 = \deg q + km.$$

A differential (2) will have no-poles on C if and only if $q = 1$ and we can write

$$H^{1,0}(C, \mathbb{C}) = \bigoplus_{k=1}^{n-1} H_k^{1,0}$$

where

$$H_k^{1,0} = \{ p\Omega/y^k; \deg p = km - 2 \} \tag{3}$$

for all $k = 1, 2, \dots, n-1$. Adding up the dimensions of these eigenspaces we obtain

$$\sum_{k=1}^{n-1} (km - 1) = m(n-1)/2 - (n-1) = g$$

where g is the genus of C .

Algebraic Hodge decomposition

Since we have just obtained an explicit expression for $H^{1,0}$ we have to characterize the quotient

$$H^1(C, \mathbb{C})/H^{1,0}(C, \mathbb{C}) = H^{0,1}(C, \mathbb{C}).$$

The answer is given in terms of meromorphic differentials on the curve that have poles on the branch points.

THEOREM: *The automorphism $\sigma : C \rightarrow C$ induces a decomposition in eigenspaces*

$$H^1/H^{1,0} = \bigoplus_{k=1}^{n-1} (H^1/H^{1,0})_k$$

where the k^{th} summand can be naturally identified with

$$\{\omega^k = p\Omega/y^{n+k}; \deg p = m(n+k) - 2\}/I_k \quad (4)$$

where I_k consists of those forms ω_k that satisfy

$$p \subset (\partial F/\partial x_0, \partial F/\partial x_1)I,$$

the homogeneous Jacobian ideal.

PROOF: Let B be the divisor on C which maps to the divisor (1) on P . The natural inclusion $C-B \rightarrow C$ induces in cohomology the exact sequence

$$0 \rightarrow H^1(C) \rightarrow H^1(C-B) \xrightarrow{R} \bigoplus_{p \in B} \mathbb{C}_p \xrightarrow{\Sigma} \mathbb{C} \rightarrow 0$$

where R applied to a differential form gives the residues at the branch points.

The cyclic group generated by σ acts on each term of this sequence and if we set

$$H_k^1(C) = \{w \in H^1(C); \sigma^*(w) = \zeta^{-k}w\}$$

($k = 0, \dots, n-1$) and similarly for $H_k^1(C-B)$ we obtain the exact sequences:

$$0 \rightarrow H_0^1(C) \rightarrow H_0^1(C-B) \rightarrow \bigoplus_{p \in B} \mathbb{C}_p \rightarrow \mathbb{C} \rightarrow 0$$

$$0 \rightarrow H_k^1(C) \rightarrow H_k^1(C-B) \rightarrow 0 \quad k = 1, \dots, n-1.$$

Since $H_0^1(C)$ is the space of forms invariant under σ , it is the space of forms on \mathbb{P}_1 with no singularities, thus

$$H_0^1(C) = 0$$

and

$$H^1(C) = \bigoplus_{k=1}^{n-1} H_k^1(C) \cong \bigoplus_{k=1}^{n-1} H_k^1(C-B).$$

Moreover since σ is an analytic map, the decomposition into eigenspaces is compatible with the Hodge decomposition, that is

$$H_k^{1,0} \subset H_k \quad \text{and}$$

$$H^1/H^{1,0}(C) \cong \bigoplus_{k=1}^{n-1} H_k^1(C-B)/H_k^{1,0}(C-B).$$

We will compute these last terms using the “algebraic de Rham theorem” by Grothendieck (cf. [5]).

For an affine variety S , $H^1(S) \cong H^1(A^*)$, where A^* is the complex of algebraic differentials.

The decomposition into eigenspaces gives

$$H_k^1(C - B) \cong H^1(A_k^*, d)$$

where the complex $A_k^0 \xrightarrow{d} A_k^1$ has an increasing filtration

$$\begin{aligned} A_k^0(l) &= \{p/y^{nl+k}\} & l = 0, 1, 2, \dots \\ A_k^1(l) &= \{q\Omega/y^{n(l+1)+k}\} & l = 1, 0, 1, \dots \end{aligned} \quad (5)$$

for homogeneous polynomials p and q of appropriate degrees. We can now write a Koszul resolution as in Clemens (cf. [4]).

Recall that

$$\theta = x_0 \partial / \partial x_0 + x_1 \partial / \partial x_1$$

and set

$$v = \mathbb{C} dx_0 \oplus \mathbb{C} dx_1.$$

For $l, r \in \mathbb{Z}$ let $P_{k,l}^r$ be the vector space of homogeneous forms of degree $mn(l+r) + mk - 1$ in x_0 and x_1 . We can then define natural epimorphisms

$$\alpha: P_{k,l}^0 v \rightarrow A_k^0(1) / A_k^0(l-1)$$

$$\omega \rightarrow \langle \theta, \omega \rangle / y^{nl+k}$$

and

$$\beta: P_{k,l}^1 \oplus \Lambda^2 v \rightarrow A_k^1(l) / A_k^1(l-1)$$

$$\omega \rightarrow \langle \theta, \omega \rangle / y^{n(l+1)+k}.$$

Next recall that

$$\langle \theta, dF \wedge \omega \rangle = mnF\omega - dF \wedge \langle \theta, \omega \rangle. \quad (6)$$

Now $\alpha(\omega) = 0$ if and only if F divides $\langle \theta, \omega \rangle$ that is

$$\langle \theta, \omega \rangle = F\gamma.$$

But then

$$\langle \theta, \omega - \gamma dF/nm \rangle = 0$$

so that

$$\omega - \gamma dF/nm = \langle \theta, q dx_0 \wedge dx_1 \rangle.$$

Thus

$$\ker \alpha = \{ p dF + q \Omega \}.$$

Also

$$\ker \beta = \{ p F d x_0 \wedge d x_1 \}.$$

Using (6) to insure commutativity, we can write the following diagram

$$\begin{array}{ccccccc}
 0 & \rightarrow & P_{k,l}^{-1} & \rightarrow & \ker \alpha & \rightarrow & \ker \beta & \rightarrow & P_{k,l}^1/I_{k,l} & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 0 & \rightarrow & P_{k,l}^{-1} & \rightarrow & P_{k,l}^0 \oplus V & \rightarrow & P_{k,l}^1 \oplus \Lambda^2 V & \rightarrow & P_{k,l}^1/I_{k,l} & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 0 & \rightarrow & 0 & \rightarrow & A_k^0(l)/A_k^0(l-1) & \rightarrow & A_k^1(l)/A_k^1(l-1) & \rightarrow & 0 & &
 \end{array} \tag{7}$$

where $I_{k,l} = I \cap P_{k,l}^1$.

The middle row is exact, it is just a Koszul resolution. The top row is exact except at $P_{k,l}^1/I_{k,l}$. To see this, exactness at $\ker \beta$ is just the identity (6) that, for two-forms ω ,

$$mnF\omega = dF \wedge \langle \theta, \omega \rangle.$$

Exactness at $\ker \alpha$ is just the fact that

$$dF \wedge \Omega = mnF d x_0 \wedge d x_1.$$

So (7) is a short exact sequence of complexes. Our remarks on each of the other two complexes then gives, via the long exact sequence in cohomology,

$$H^0(A_k^*(l)/A_k^*(l-1)) = 0$$

$$H^1(A_k^*(l)/A_k^*(l-1)) = P_{k,l}^1/I_{k,l}. \tag{8}$$

The complexes $A_k^*(l)$ filter the complex A_k^* whose cohomology is

$H_k^*(C-B)$. The spectral sequence associated to this filtration has

$$E_1^{l,q} = H^{l+q}(A_k^*(l)/A_k^*(l-1))$$

so that

$$E_1^{l,q} = 0$$

unless $l+q=1$. So this spectral sequence degenerates at E_1 and we can compute the resulting filtration on $H_k^1(C-B)$ via (8), namely,

$$E_\infty^{-1,2} = H^1(A_k^*(-1)) = P_{mk-2}$$

$$E_\infty^{0,1} = H^1(A_k^*(0)/A_k^*(-1))$$

$$= P_{m(n+k)-2}/I_{m(n+k)-2}$$

etc.

To finish the proof of the theorem, notice that, referring to (3),

$$E_\infty^{-1,2} = H_k^{1,0}$$

and, by the exactness of the middle row of (7), we have that if $l \geq 1$,

$$\begin{aligned} \dim E_\infty^{l,1-l} &= [m(l+1)n+k] - 1 - 2[m(l+1)n+k - mn] \\ &\quad + [m(l+1)n+k - 2mn + 1] = 0. \end{aligned}$$

We will compute the dimension of $E_\infty^{0,1}$ in order to motivate our next result. Namely $\dim E_\infty^{0,1} = [m(n+k) - 1] - 2[m(n+k) - mn] = -[m(n+k) - 2mn + 1] = mn - k + 1 = \dim H_{n-k}^{1,0}$.

This is as it should be since, under the cup-product pairing, if $w \in H_r^{1,0}$, $y \in H_s^1$, then

$$w \wedge y \in H_{r+s}^2 = 0 \quad \text{unless } r+s=n.$$

More precisely, we have the following result.

PROPOSITION: *Under the cup product $H^{1,0} \times H^{0,1} \rightarrow \mathbb{C}$ the space $H_k^{1,0}$ is orthogonal to $(H^1/H^{1,0})_i$ for $i \neq n-k$.*

For differential forms

$$\varphi = p\Omega/y^k \in H_k^{1,0}$$

$$\psi = q\Omega/y^{2n-k} \in (H^1/H^{1,0})_{n-k}$$

the following identity holds

$$\begin{aligned} (\varphi, \psi) &= \int \varphi \wedge \psi \\ &= (2\pi i/n - k)n^2 \sum_{j=1}^{nm} p(a_j)q(a_j)/ \\ &\quad \times (a_j - a_1)^2 \dots \wedge \dots (a_j - a_{nm})^2. \end{aligned} \quad (9)$$

PROOF: The first statement follows from the fact that, under the cup-product,

$$H_{k_1}^{1,0}(C) \otimes (H^1(C)/H^{1,0}(C))_{k_2}$$

goes into $H_{k_1+k_2}^{1,1}(C)$, which is zero for $k_1 + k_2 \neq n$.

For the second statement, we proceed as follows. Using affine coordinates for our curve C

$$y^n = F(x) = \prod_{j=1}^{nm} (x - a_j)$$

so that

$$ny^{n-1}dy = F'(x)dx.$$

Near $x = a_j$ we write

$$\begin{aligned} \varphi &= p(x)dx/y^k = p(x)y^{n-k-1}dx/y^{n-1} = np(x)dy/F'(x)y^{n-k-1} \\ &= \beta_j(x)y^{n-k-1}dy. \end{aligned}$$

Similarly $\psi = nq(x)dy/F'(x)y^{n-k+1} = \alpha_j(x)dy/y^{n-k+1}$.

For each $j = 1, \dots, nm$, let ρ_j be a C^∞ function on C supported in a neighborhood $|y| < \varepsilon$ of $x = a_j$ and identically equal to one the smaller neighborhood $|y| \leq \varepsilon/2$.

The form

$$\tilde{\psi} = d\left(\frac{1}{n-k} \sum_{j=1}^{nm} \rho_j \alpha_j / y^{n-k}\right) + \psi$$

is in the same cohomology class as ψ but is C^∞ in C , analytic outside the neighborhoods $|y| < \varepsilon$.

We the have

$$(\varphi, \psi) = (\varphi, \tilde{\psi}) = \sum_{j=1}^{nm} \int_{|y| \leq \varepsilon} \varphi \wedge \tilde{\psi}.$$

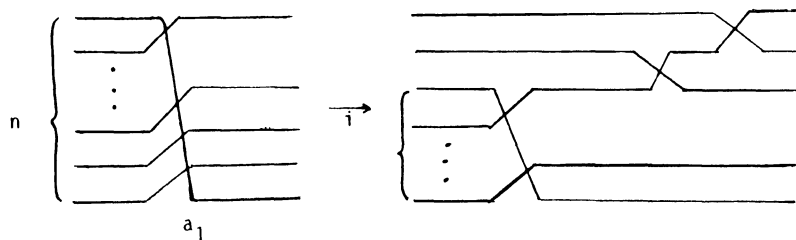


Figure 1

Since $\tilde{\psi}$ is analytic also in $|y| \leq \epsilon/2$ we get

$$(\varphi, \psi) = \sum_{j=1}^{nm} \int_{\epsilon/2 \leq |y| \leq \epsilon} \varphi \wedge \partial/\partial \bar{y} (1/n - k\rho_j \alpha_j / y^{n-k}) d\bar{y}$$

and by Stokes theorem

$$\begin{aligned} (\varphi, \psi) &= \frac{1}{n-k} \sum_{j=1}^{nm} \int_{|y|=\epsilon/2} \beta_j(x) y^{n-k-1} \alpha_j dy / y^{n-k} \\ &= \frac{2\pi i}{n-k} \sum_{j=1}^{nm} \beta_j(a_j) \alpha_j(a_j). \end{aligned}$$

proving the identity.

Deformations of a cyclic cover

Let τ_n^1 be the period space of curves of a given genus that can be expressed as an n to 1 branched cover of \mathbb{P}_1 . We will consider a neighborhood of the cyclic cover C given by an equation

$$y^n = (x - a_1) \dots (x - a_{nm}).$$

A deformation within τ_n^1 will split each branch point of order n into several branch points of lower order; a stratification of these deformations can be given in the following form:

At... a branch point, a_1 say, there is associated a cyclic permutation $(1, 2 \dots n)$ of the sheets of the Riemann surface. For each $i = 2, \dots, n$ we will say that a deformation is of type i if it splits a_1 into a branch point with associated permutation $(1, 2 \dots i)$ and $(n - i)$ simple branch points (see Fig. 1).

A locally irreducible, analytic curve in the period space will determine a non-zero tangent vector (which generates its tangent cone at C) which we consider as an element in

$$Sym^{(2)}(H^{1,0})^*.$$

We denote by v_i the tangent cone to deformations of type greater than or equal to i . Thus $v_{i+1} \subset v_i$, v_n is the set of all tangent vectors to deformations by cyclic covers and v_2 is the tangent space to τ_n^1 .

On the other hand, the cyclic group of order n generated by the automorphism $\sigma : C \rightarrow C$ acts on the space $Sym^{(2)}(H^{1,0})$. We say that a quadratic differential q_j is of type j , $1 \leq j \leq n$, if it is an eigenvector for the eigenvalue ζ^j , i.e.

$$\sigma^*(q_j) = \zeta^j q_j \quad (\zeta = \exp(2\pi i/n)).$$

An element q in $Sym^{(2)}(H^{1,0})$ is a linear sum of these eigenvectors, that is

$$q = \sum_{j=1}^n q_j,$$

and we will call q_j the component of type j of the quadratic differential q .

There is a close relationship between eigenvalues and splitting of branch points given by the following.

THEOREM: *The tangent cone v_i is always a (possibly non reduced) linear space.*

A quadratic differential is orthogonal to v_i if and only if its components of type $j = 2, \dots, n - i + 1, n$ vanish to orders

$$n, \dots, 2n - i - 1, 2n - 2 \text{ at branch points.}$$

In particular the cotangent space of cyclic deformations consists of those quadratic differentials whose invariant component vanishes at branch points to order $2n - 2$.

PROOF: We first fix the notation for a basis of $H^{1,0}(C)$.

$$\text{Let } e_\nu(\lambda) = x^\nu dx / y^\lambda$$

where $0 \leq \nu \leq \lambda m - 2$, $\lambda = 1, \dots, n - 1$. For fixed λ these differentials form a basis of $H_\lambda^{1,0}(C)$; $\sigma^*(e_\nu(\lambda)) = \zeta^{-\lambda} e_\nu(\lambda)$.

In terms of a local parameter centered at a_1 ,

$$(x - a_1) = z^n$$

we can write

$$e_\nu(\lambda) = p(z) dz, p(z) = C_{\lambda,\nu} z^{n-\lambda} (1 + \alpha z^n + \dots)$$

where $C_{\lambda, \nu} = na_1^\nu / [(a_1 - a_2) \dots (a_1 - a_{nm})]^{\lambda/\nu}$ and $\alpha \in C$. (There will of course be analogous expressions for a_2, \dots, a_{nm} .)

The vectors of the dual basis of $(H^{1,0})^*$ will be denoted by $e^\nu(\lambda)$.

To deform the curve C we will take a covering of C by coordinate charts whose changes of parameters depend on t , $|t| < \varepsilon$ (see [1]).

Here we take a local deformation of the parameter z centered at a_1 given by

$$\psi(z, t) = (x - a_1)$$

with $\psi(z, 0) = z^n$; the rest of the local parameters remain fixed. A differential of the first kind on C_t will be written in terms of z as

$$p(z, t)dz = p(z, t)dx/(\partial\psi/\partial z)$$

where $p(z, t)$ is an analytic function. Since

$$\frac{\partial\psi}{\partial z} \frac{dz}{dt} + \frac{\partial\psi}{\partial t} = 0$$

we compute the derivate of the differential at $t = 0$ to be

$$\left(\frac{\partial\psi}{\partial z}\right)^{-2} \left[p(z) \left(-\frac{\partial^2\psi}{\partial z\partial t} \frac{\partial\psi}{\partial z} + \frac{\partial^2\psi}{\partial z^2} \frac{\partial\psi}{\partial t} \right) - \frac{\partial p}{\partial z} \frac{\partial\psi}{\partial z} \frac{\partial\psi}{\partial t} \right] dz + \frac{\partial p}{\partial t} dz. \quad (10)$$

This is in general a differential of the 2nd kind with pole at a_1 , that is, an element in

$$H^1/H^{1,0}.$$

The linear map $\varphi \in \text{Hom}(H^{1,0}, H^1/H^{1,0}) \cong H^{1,0*} \otimes H^{1,0*}$ corresponding to the deformation associates to $p(z)dz$ the differential (10) (where we can disregard the analytic part $\partial p/\partial t dz$). In general, a deformation ψ depends on $n - 1$ parameters t_2, \dots, t_n given as

$$\psi(z, t_2, \dots, t_n) = z^n + t_2 z^{n-2} + \dots + t_n = (x - a_1).$$

(See [2] also.) We compute separately the linear mapping φ_i associated to

$$\psi_i(z, t_i) = z^n + t_i z^{n-i} \quad i = 2, \dots, n.$$

In this case (10) gives

$$(nz^i)^{-1} ((i-1)p(z) - z(p'(z)))dz$$

and replacing $p(z)$ by the corresponding expressions for the differentials $e_\nu(\lambda)$ we obtain

$$n^{-1} C_{\lambda, \nu} z^{n-(\lambda+i)-1} [(i + \lambda - n) + \beta z^n + \dots] dz$$

for some constant $\beta \in \mathbb{C}$. This differential has no poles for $i + \lambda \leq n$ and for $i + \lambda \geq n + 1$ we can compute the cup product

$$\begin{aligned} & (e_\mu(\lambda'), \varphi_i(e_\nu(\lambda))) \\ &= C_{\lambda, \nu} C_{\lambda', \mu} \int_{|z|=e} z^{2n-(\lambda+\lambda'+i)-1} (1 + \gamma z^n + \dots) dz. \end{aligned}$$

This integral vanishes unless $\lambda + \lambda' + i = 2n$ and we obtain

$$\varphi_i = \sum_{\nu, \mu} d_{\nu, \mu} \sum_{\lambda \geq n-i+1} e^\nu(\lambda) \otimes e^\mu(2n - (i + \lambda))$$

where

$$d_{\nu, \mu} = (2\pi\sqrt{-1}) a_1^{\nu+\mu} / [(a_1 - a_2) \dots (a_1 - a_{nm})]^{2-i/n}.$$

Thus $\sigma^*(\varphi_i) = \zeta^{-i} \varphi_i$ and it follows that a quadratic differential will be orthogonal to φ_i if and only if its component of type i vanishes at a_1 at the order $n + i - 2$. More precisely, let

$$q_i = \sum_{\nu, \mu} \alpha^{\nu, \mu} \left(\sum_{\lambda+\lambda'=n-i} e_\nu(\lambda) \otimes e_\mu(\lambda') + \sum_{\lambda+\lambda'=2n-i} e_\nu(\lambda) \otimes e_\mu(\lambda') \right)$$

be a differential of type i ; the first sum in parenthesis vanishes always to order $n + i - 2$. q_i is orthogonal to φ_i if

$$(q_i, \varphi_i) = (cte) \sum_{\nu, \mu} \alpha^{\nu, \mu} a_1^{\nu+\mu} = 0.$$

But this means that in terms of the local parameter z the expansion of the second sum has a first coefficient equal to zero, and thus q_i vanishes to order $n + i - 2$.

It remains to characterize the tangent cones v_i in terms of the parameters t_2, \dots, t_n . We have drawn in Fig. 2 the situation for $n = 4$.

The mapping, from the z -disk to a neighborhood of a_1 given by

$$\psi(z, t_2, \dots, t_n) = x - a_1$$

will be branched at the zeroes of the derivative with respect to z . Two of these branch points will coincide for the values of (t_2, \dots, t_n) that satisfy both equations

$$\partial\psi/\partial z = nz^{n-1} + (n-2)t_2z^{n-3} + \dots + t_{n-1} = 0$$

$$\partial^2\psi/\partial z^2 = n(n-1)z^{n-2} + \dots + 2t_{n-2} = 0.$$

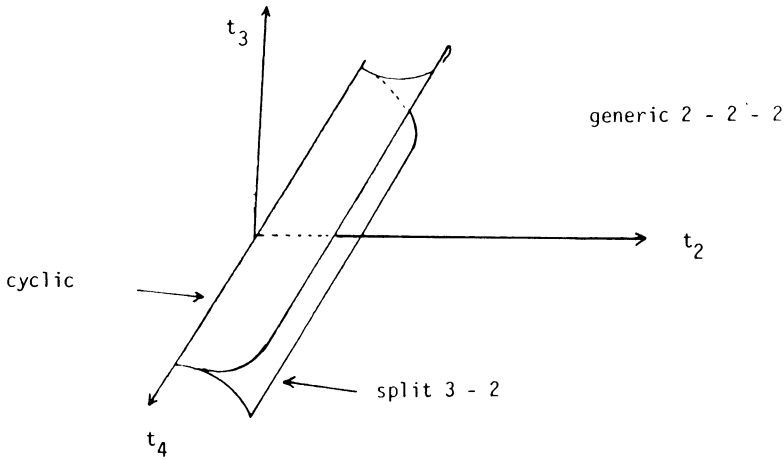


Figure 2

These equations in z will have a common root if their resultant

$$A = \begin{vmatrix} n & 0 & & \dots & & t_{n-1} & 0 & 0 \\ 0 & n & 0 & \dots & & t_{n-1} & 0 & 0 \\ & & & & & & & 0 \\ n & (n-1) & 0 & \dots & & 2t_{n-2} & 0 & 0 \\ 0 & n & (n-1) & \dots & & 2t_{n-2} & 0 & 0 \\ & & & \ddots & & \dots & & \\ & & & & \dots & & & 0 \\ 0 & & & \dots & n(n-1) \dots & & 2t_{n-2} & \end{vmatrix}$$

vanishes. This represents an hypersurface in (t_2, \dots, t_n) space corresponding to a triple branch point or more. More generally the rank of the matrix is smaller or equal to $2n - i - 1$ if and only if $\psi(z, t_2 \dots t_n)$ has a branch point of order i at least.

The hypersurface $A = 0$ is singular at 0 and its tangent cone is given by

$$\sum_{\alpha_2 + \dots + \alpha_n = n-2} \partial^{n-2} A / \partial t_2^{\alpha_2} \dots \partial t_n^{\alpha_n} t_2^{\alpha_2} \dots t_n^{\alpha_n} = 0$$

But from the last $n - 2$ columns in the matrix it is clear that all partial derivatives vanish at zero except

$$\partial^{n-2} A / \partial t_{n-1}^{n-2}(0)$$

and so the tangent cone is given by

$$t_{n-1}^{n-2} = 0, \quad \text{or} \quad t_{n-1} = 0.$$

This is then the tangent cone corresponding to v_3 , the tangent cone to deformations of type greater or equal to 3.

For a deformation to be of type greater than or equal to 4 it is necessary and sufficient that the discriminant of $\partial^2\psi/\partial z^2$, $\partial^3\psi/\partial z^3$ vanishes also. Denoting this discriminant by B , the locus in (t_2, \dots, t_n) space is given by $A = 0$ and $B = 0$. (Furthermore, the singular locus of the surface $A = 0$ is given by this intersection.)

The tangent cone at 0 is then the intersection of the tangent cones to both surfaces, that is

$$\{t_{n-1}^{n-2} = 0, t_{n-2}^{n-3} = 0\} \quad \text{or} \quad \{t_{n-1} = t_{n-2} = 0\},$$

and so on.

The tangent cone at 0 corresponding to v_i will be given by

$$\{t_{n-1} = \dots = t_{n-i+2} = 0\}$$

in general, and then the tangent cone to v_i will be the linear space generated by

$$\{\varphi_n, \varphi_2, \dots, \varphi_{n-i+1}\}.$$

A quadratic differential will be orthogonal to this subspace if and only if its components

$$\{q_n, q_2, \dots, q_{n-i+1}\}$$

vanish at branch points to orders $2n-2$, n , $n+1, \dots, 2n-i-1$ as required.

We observe also that in the generic case, v_2 , we can obtain for these cyclic covers the result proved in general by Donagi-Green concerning the orthogonal space to deformations (see [1]).

Finally, it is interesting to consider the cases $n = 2, 3$ since it is possible to write global variations for the original equation.

For $n = 2$, the cyclic cover is given as

$$C: y^2 = (x - a_1) \dots (x - a_{2m})$$

and a basis for $H^{1,0}(C)$ is

$$e_i = x^i/y dx \quad 0 \leq i \leq m-2.$$

The only possible deformation is given by the families of curves

$$C_t: y^2 = (x - a_1) \dots (x - a_k + t) \dots (x - a_{2m})$$

with a basis for $H^{1,0}(C_t)$

$$e_i(t) = x^i dx / ((x - a_1) \dots (x - a_k + t) \dots (x - a_{2m}))^{1/2}.$$

The derivative at $t = 0$ gives

$$(-1/2)x^i(x - a_1) \dots \hat{k} \dots (x - a_{2m}) dx / y^3$$

and in view of the proposition, its cup product with e_j is

$$\lambda_k^{i,j} = (2\pi\sqrt{-1}) a_k^{i+j} / (a_k - a_1) \dots (a_k - a_{2m}).$$

The corresponding tangent vector is then

$$\sum_{i,j} \lambda_k^{i,j} e^i \otimes e^j$$

and a quadratic differential is orthogonal to all deformations if it vanishes at the branch points a_1, \dots, a_{2m} .

For $n = 3$, the equation of C is

$$y^3 = (x - a_1) \dots (x - a_{3m})$$

with a basis of holomorphic differentials

$$e_i = x^i dx / y, \quad f_j = x^j dx / y^2.$$

A cyclic variation is similar to the preceding case the tangent vector being

$$\sum \lambda_k^{i,j} e^i \otimes f^j.$$

However, to write a global equation in the case of splitting of a branch point the following considerations are necessary:

We have to find an equation

$$f(x, y) = y^3 + p(x)y + q(x) = 0$$

where p, q are polynomials in x whose coefficients depend on a parameter t . The polynomials should be chosen so that the equation represents a

Riemann surface ramified of order 3 over the points $a_2 \dots a_{3m}$ and branched of order 2 over $a_1, a_1 - t$ and such that for $t = 0$ reduces itself to C .

The branch points are the common solutions of $f = 0$ and $\partial f / \partial y = 0$, and they are found among the solutions of the equation obtained by setting the discriminant equal to zero:

$$\Delta = -(4p^3 + 27q^2) = 0.$$

The branch points of order three have to satisfy also

$$\partial f / \partial y, \quad \partial^2 f / \partial y^2 = 0$$

and this implies that they are among the roots of

$$36p = 0.$$

We choose then

$$p(x) = t^{2/3}(x - a_2) \dots (x - a_{3m})$$

where the particular power of t not only appears there for convenience but also reflects the fact that the parameter t is not natural. A natural parameter will be taken to be $s = t^{2/3}$. With this value for p we may compute again the discriminant

$$-\Delta = (x - a_2)^2 \dots (x - a_{3m})^2 \Delta_2$$

for q has to vanish also at these $3m - 1$ points. Here we have written

$$\Delta_2 = 4t^2(x - a_2) \dots (x - a_{3m}) + 27h^2$$

where $q(x) = (x - a_2) \dots (x - a_{3m})h(x)$. The polynomial h must be chosen so as to satisfy the last condition concerning the branch points at $a_1, a_1 - t$. This forces the vanishing of Δ_2 there exactly to the first order, and we are led to an equation of the form

$$4t^2(x - a_2) \dots (x - a_{3m}) + 27h^2(x) = g^2(x)(x - a_1)(x - a_1 + t).$$

This is a sort of Pell's equation for the polynomials $h(x), g(x)$ and can be solved explicitly; to make a long story short, the solution appears as power series in t with polynomial coefficients, that is

$$h(x) = h_0(x - a_1) + h_1(x)t + \dots$$

$$g(x) = g_0 + g_1(x)t + \dots$$

for some constants h_0 and g_0 that can be computed directly from the equation.

Given now the formula for C_t , we have to write a basis for the analytic differentials:

$$\left(x^i y - l(x) dx/f_y, \quad x^i g(x) dx/f_y \right)$$

where l is some polynomial (see [3] for the details). We compute now the derivative with respect to $s = t^{2/3}$ and the answer does not depend on $l(x)$ or $g_1(x)$ but is of the form

$$x^i (x - a_2) \dots (x - a_{3m}) / y^4 dx,$$

in the second case say, and the tangent vector then is obtained as

$$\sum \lambda_i^{i,j} f^i \otimes f^j$$

as in the proof of the theorem.

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Universidad de Santiago de Chile
 Casilla S659 Correo 2
 Santiago
 Chile