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PRINCIPAL SERIES REPRESENTATIONS OF SPECIAL UNITARY GROUPS OVER LOCAL FIELDS

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Intertwining operators are used with a multiplicity-one result to compute certain c-functions on SU(3) over a non-Archimedean local field. Reducibility of the unitary and non-unitary principal series representations is determined. The c-functions are used to determine the Plancherel measure for the unitary principal series of SU$(2n+1)$. The R-groups for SU$(2n+1)$ are explicitly classified.

1. Definitions and preliminaries

Let F be a non-Archimedean local field and E a separable quadratic extension of F with Galois automorphism $x \to \bar{x}$.

Define $G = \mathrm{SU}(2n+1)$ to be the group

$$\{ g \in SL_{2n+1}(E) | {}^t g J \bar{g} = J \} \text{ where } J = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}.$$

G is the algebraic group preserving the hermitian form

$$h(x, y) = x_n \bar{y}_{-n} + \cdots + x_0 \bar{y}_0 + \cdots + x_{-n} \bar{y}_n \quad \text{on} \quad E^{2n+1}.$$

Let A be the torus $\{\mathrm{diag}(d_n, \cdots, 1, \cdots, d_{-n}) | d_i \in F,\ d_i d_{-i} = 1 \text{ for all } i\}$, $M = Z_G(A) \cong (E^*)^n$, and N the group of upper triangular unipotent matrices of G. $P = MN$ is a minimal parabolic subgroup of G.

Extend a character λ of M trivially across N and define the principal series representations $\mathrm{Ind}_P^G \lambda$ using left translation and unitary induction.

Let $K_0 = K = G(\mathcal{O}_E)$ be the standard maximal compact subgroup of G, and set

$$K_m = \{ g \in G | g \equiv I \bmod p_E^m \} \quad \text{for } m \geqslant 1.$$

More generally, if H is any subgroup of G, define

$$H_m = \{ h \in H | h \equiv I \bmod p_E^m \} \quad \text{for } m \geqslant 1.$$

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If $V = {}^tN$ is the lower unipotent subgroup of G, then K_m has Iwahori factorization $V_m M_m N_m$, $m \geqslant 1$.

Let $\mathfrak{a} = \mathrm{Hom}(X(A), \mathbb{Z}) \otimes R$ be the real Lie algebra of A, with dual $\mathfrak{a}^* = X(A) \otimes \mathbb{R}$. Canonically, $\mathfrak{a} = \mathrm{Hom}(X(M), \mathbb{Z}) \otimes \mathbb{R}$ and $\mathfrak{a}^* = X(M) \otimes \mathbb{R}$, as in §5.3.1 of [9]. An element $s \in \mathfrak{a}^*_{\mathbb{C}}$ determines a quasi-character χ_s of M as follows. Let H: $M \to \mathfrak{a}$ be the mapping which satisfies $|\chi(m)| = q^{\langle \chi, H(m) \rangle}$ for all $m \in M$, $\chi \in X(M)$. Then define $\chi_s(m) = q^{i\langle s, H(m) \rangle}$ for $m \in M$. For $\lambda \in M\hat{}$, we set $\lambda_s(m) = \lambda(m)\chi_s(m)$. If $n = 1$, $\lambda \in (E^*)\hat{}$, we set $\lambda_s(x) = \lambda(x)|x|_E^s$ for $s \in \mathbb{C}$. Also define Re λ_s to be Re s.

Each element of $X(A)$ corresponds to a unique element of $\mathfrak{a}^*$, called the associated weight. A non-trivial rational character α of A which occurs in the adjoint representation of A on the Lie algebra $\mathscr{I}$ of G is called a root character. Let $\mathscr{I}_\alpha$ be the eigenspace in $\mathscr{I}$ associated to α. The weights of the root characters form a root system Φ in $\mathfrak{a}^*$. A root α is said to be reduced if $t\alpha \in \Phi$ with $t \in \mathbb{Q}$ implies $t \in \mathbb{Z}$.

In Section 2 we prove a multiplicity one result based on ideas of R. Howe. Intertwining operators are studied and used to define certain meromorphic functions in s in Section 3.

Reducibility, of the representation $\mathrm{Ind}_P^G \lambda_s$ is related to the zeros and poles of these functions in Section 4. The c-functions are computed for the rank 1 groups SL_2 and SU(3) in Section 5. The explicit formulas for the c-functions and Plancherel measure are used to construct complementary series in Section 6 and to determine reducibility in Sections 7 and 8.

We note that Casselman has independently determined the reducible unitary principal series of SU(3) by a different analysis based on Jacquet modules.

2. A multiplicity-one result

We first define a character ψ_m of K_m for each $m \geqslant 0$. Let $\psi_0 \equiv 1$ on K_0.

For α positive, let V_α be the subgroup of V whose Lie algebra is $\mathscr{I}_{-\alpha} + \mathscr{I}_{-2\alpha}$. The group $V_\alpha / V_{2\alpha}$ is isomorphic to the additive group of E. We fix a character ψ of E with conductor the ring of integers $\mathcal{O}_E$, and define a character $\psi_{\alpha,m}$ of $V_\alpha / V_{2\alpha}$ by $\psi_{\alpha,m}(x) = \psi(\pi^{-2m}x)$, $x \in E$, where π is a prime in $\mathcal{O}_E$. We will assume $\psi_{\alpha,m}$ factors through the trace from E to F. Then $\Pi\psi_{\alpha,m}$ (α simple) determines a character of $V/\Pi V_\beta$ (β non-simple), hence of V. Let ψ_m be its restriction to V_m.

Now define ψ_m on $K_m = V_m M_m N_m$ by $\psi_m(vln) = \psi_m(v)$ for $v \in V_m$, $l \in M_m$, $n \in N_m$.

We want to show

THEOREM 1: *Suppose π is an irreducible representation of G. Then the multiplicity of $\bar{\psi}_m$ in $\pi|K_m$ is* 0 *or* 1.

Define an algebra of (K_m, ψ_m)-spherical functions, $m \geqslant 0$, by $\mathscr{S}_m = \{f \in C_c^\infty(G) | f(k_1 g k_2) = \psi_m(k_1 k_2) f(g)$ for $k_1, k_2 \in K_m$, $g \in G\}$.

Theorem 1 will follow from

THEOREM 2: *S_m is a commutative algebra under convolution.*

The case $m = 0$ is well-known, i.e., the theory of K-class-1 representations. The proof for general m is based on arguments due to Roger Howe.

With restrictions on F and G a Chevalley group, the multiplicity one result is equivalent to uniqueness of Whittaker models. See Corollary 2 of the main theorem of [7].

For $G = \mathrm{SU}(2n+1)$, char $F = 0$, and E/F unramified, one may use arguments similar to the proof of Lemma 4 of [7] to prove the commutativity of $\mathscr{S}_m$ and the multiplicity one result. These arguments were originally developed by Howe in the case of GL_n.

Theorem 2 holds for $SL_2(E)$ and SU(3) with no restrictions on E, F. The proof is based on a standard general argument. Let σ be an antiautomorphism of G which preserves K_m and ψ_m. Then σ acts on $\mathscr{S}_m$. Further,

$$\sigma_{(f_1 * f_2)} = \sigma_{f_2} * \sigma_{f_1}. \tag{2.1}$$

A double coset $K_m g K_m$ supports a non-zero spherical function in $\mathscr{S}_m$ if and only if

$$\psi_m(k) = \psi_m(gkg^{-1}) \quad \text{for all} \quad k \in K_m \cap K_m^g. \tag{2.2}$$

Suppose that any double coset $K_m g K_m$ contains an element fixed by σ, up to an element of ker ψ_m. Then σ acts as the identity on $\mathscr{S}_m$, so $\mathscr{S}_m$ is commutative by (2.1).

For $G = \mathrm{SU}(3)$, we set

$$\sigma(g) = w{}^t g w^{-1}, \quad \text{where} \quad w = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}.$$

Then a long calculation shows that if g satisfies (2.2), then the double coset $K_m g K_m$ contains an element fixed by σ, up to an element of ker ψ_m.

In Section 3 we will apply the multiplicity one result for SL_2 and SU(3) to the principal series representations, in which case it is easy to give a direct proof.

3. Intertwining operators and Plancherel measure

Fix a coset representative $\bar{w}$ for $w \in W$ and define the usual intertwining operators between $\mathrm{Ind}_P^G \lambda$ and $\mathrm{Ind}_P^G w\lambda$ (λ a quasi-character) by

$$A(\bar{w}, \lambda) f(g) = \int_{N \cap wVw^{-1}} f(gn\bar{w}) \mathrm{d}n.$$

It is well-known that these operators satisfy the cocycle relation

$$A(\bar{w}_1 \bar{w}_2, \lambda) = A(\bar{w}_1, w_2\lambda) A(\bar{w}_2, \lambda) \tag{3.1}$$

provided

$$l(w_1 w_2) = l(w_1) + l(w_2).$$

Choose a coset representative for each $w \in W$ as in [3] and write $A(w, \lambda)$ instead of $A(\bar{w}, \lambda)$. We will define normalized operators $\mathfrak{A}(w, \lambda)$ which satisfy the cocycle relation

$$\mathfrak{A}(w_1 w_2, \lambda) = \mathfrak{A}(w_1, w_2\lambda)\, \mathfrak{A}(w_2, \lambda) \tag{3.2}$$

with no condition on the lengths of w_1 and w_2.

We define a function on VP, which is open and dense on G, by

$$f_{m,\lambda,s}(vln) = \begin{cases} 0 & \text{if } v \notin V_m \\ \psi_m(v) \lambda_s^{-1} \delta_P^{-1/2}(l) & \text{if } v \in V_m \end{cases} \tag{3.3}$$

where

$$\lambda \in M^{\wedge}, \quad s \in \mathfrak{A}_{\mathbb{C}}^*, \quad h = \text{conductor } \lambda, \quad \text{and } m \geqslant \max\{1, h\}.$$

Then $f_{m,\lambda,s}$ is in $\mathrm{Ind}_P^G \lambda_s$ and transforms as $\bar{\psi}_m$ under K_m. Recall that, by Bruhat Theory, the representations $\mathrm{Ind}_P^G \lambda_s$ are generically irreducible.

Consider first the action of $A(w_\alpha \lambda_s)$ on $f_{m,\lambda,s}$ for $w = w_\alpha$, α a simple root. Suppose Re $s_\alpha > 0$. Then $A(w_\alpha, \lambda_s) f_{m,\lambda,s}$ converges, is in $\mathrm{Ind}_P^G w_\alpha \lambda_s$ and transforms as $\bar{\psi}_m$ under K_m. By the multiplicity one result for $M_\alpha = Z_G(\ker \alpha) = SL_2$ or SU(3), it must be a scalar times $f_{m,w\lambda,ws}$, i.e.,

$$A(w, \lambda_s) f_{m,\lambda,s}(g) = \gamma_{m,\alpha}(\lambda, s) f_{m,w\lambda,ws}(g) \quad \text{for all} \quad g \in G, \tag{3.4}$$

$w = w_\alpha$ a simple reflection.

Now, for any $w \in W$, consider the product decomposition of $A(w, \lambda_s)$ into rank-one operators corresponding to the reduced expression $w = w_{\alpha_1} w_{\alpha_2} \dots w_{\alpha_l}$ as a product of simple reflections. (3.1) and (3.4) give

$$A(w, \lambda_s) f_{m,\lambda,s}(g) = \gamma_{m,w}(\lambda, s) f_{m,w\lambda,ws}(g) \tag{3.4'}$$

for all $g \in G$, where $\gamma_{m,w}(\lambda, s)$ is given by a product formula in terms of the $\gamma_{m,\alpha}(\lambda, s)$. The formula (3.4′) is similar to the case $m = h = 0$; see for example, [2].

Besides the product formula, the $\gamma_{m,w}$ also satisfy

$$\overline{\gamma_{m,w}(\lambda, s)} = \gamma_{m,w^{-1}}(w\lambda, -ws).$$

In Section 5, we explicitly calculate $\gamma_m(\lambda, s)$ for $G = SL_2$ and SU(3) by setting $g = 1$ in (3.4) and evaluating the integrals

$$\gamma_m(\lambda, s) = \int_N f_{m,\lambda,s}(n\bar{w})\,\mathrm{d}n \quad \text{for} \quad \operatorname{Re} s > 0 \quad \text{and} \quad m \geqslant 1. \tag{3.5}$$

The c-functions $\gamma_m(\lambda, s)$ are defined for all s by analytic continuation. Then

$$A(w^{-1}, w\lambda_s)A(w, \lambda_s) = \gamma_m(w\lambda, -s)\gamma_m(\lambda, s)I. \tag{3.6}$$

But, from Harish-Chandra's theory of the intertwining operators,

$$A(w^{-1}, w\lambda_s)A(w, \lambda_s) = \gamma_w(G|P)^2\mu_w(\lambda, s)^{-1}I \tag{3.7}$$

where $\gamma(G/P)$ is a constant and $\mu_w(\lambda, s)$ is Plancheral measure. See [8], or works of Silberger. Set $\mu_\alpha = \mu_w$ if $w = w_\alpha$, with α reduced. If both α and 2α are roots, formally set $\mu_{2\alpha} = \mu_\alpha$.

Thus for the rank 1 group SL_2 and SU(3),

$$\mu_w(\lambda, s) = \gamma_w^2(G|P)[\gamma_m(w\lambda, -s)\gamma_m(\lambda, s)]^{-1}. \tag{3.8}$$

We may determine the Plancherel measure for the unitary principal series in general by a product formula

$$c\mu(\lambda, s) = \prod_{\alpha \in \Phi_r} \mu_\alpha(\lambda, s).$$

This is a product of rank 1 factors μ_α, where α ranges over the set Φ_r of positive reduced roots. Here, c is a positive constant. See Corollary 5.4.3.3 of [9].

REMARK 1: By a general theorem of Silberger [11], for reductive groups the Plancherel measure is a product of Euler factors $U_j(\omega, z)$, $j = 0, 1$. Our results in Section 5 determine, for SU(3), whether either factor is identically one and the values of Silberger's constants λ_j, $j = 0, 1$. We note also that Silberger has shown that one may associate complementary series representations to any zero of the Plancherel measure.

REMARK 2: The functions $\gamma_{m,w}(\lambda, s)$ are related to the local coefficients $C_\chi(\gamma, \pi, \theta, w)$ studies by Shahidi in [8]. Section 3.2 of [8] contains formulas for $C_\chi(\lambda, \pi, \theta, w)$ for the real groups $SL_2(F)$, $F = \mathbb{R}$ or $\mathbb{C}$, and SU(2,1).

4. The reducibility group

We fix an $m \geqslant h$ and define the normalized operators

$$\mathfrak{A}(w, \lambda_s) = (\gamma_{m,w}(\lambda, s))^{-1} A(w, \lambda_s).$$

Then $\mathfrak{A}(w^{-1}, w\lambda)\mathfrak{A}(w, \lambda) = I$, first for $w = w_\alpha$ a simple reflection. Then one may show as in Chapter 1, Section 2 of [3] that the cocycle relation (3.2) holds for normalized operators with no conditions on the lengths of w_1 and w_2.

Thus if $W_\lambda = \{w \in W | w\lambda = \lambda\}$, then $w \to \mathfrak{A}(w, \lambda)$ is a homomorphism on W_λ and not merely a projective homomorphism.

We define the p-adic analogue of the Knapp-Stein R-group.

For λ unitary, set $\Delta' = \{\alpha | \mu_\alpha(\lambda, 0) = 0\}$ and let W' be the subgroup of W generated by the reflections w_α, $\alpha \in \Delta'$. Then $W' = \{w \in W_\lambda | \mathfrak{A}(w, \lambda)$ is scalar$\}$ by results of [10].

Define $R = \{w \in W_\lambda | \alpha \in \Delta'$ and $\alpha > 0$ imply $w\alpha > 0\}$. Then W_λ is a semi-direct product $W_\lambda = R \propto W'$.

THEOREM 1 (Harish-Chandra [9]): $\{\mathfrak{A}(w, \lambda) | w \in W_\lambda\}$ *spans the commuting algebra* $\mathscr{C}(\lambda)$ *of the unitary principal series* $\mathrm{Ind}_P^G \lambda$.

THEOREM 2 (Silberger [10]): *The dimension of the commuting algebra* $\mathscr{C}(\lambda)$ *is* $|W_\lambda / W'|$.

A priori, the map $w \mapsto \mathfrak{A}(w, \lambda)$ from W_λ into the commuting algebra is only a projective homomorphism. Thus R abelian does not immediately imply that multiplicity one holds. But with our normalization and choice of coset representatives for Weyl group elements, the cocycle relation (3.2) holds with no condition on the lengths of the Weyl group elements. Then the above map is actually a homomorphism.

Since the self-intertwining operators corresponding to elements of W' are scalar, Theorem 1, the cocyle relation (3.2), and the decomposition $W_\lambda = R \propto W'$ imply that $\{\mathfrak{A}(w, \lambda) | w \in R\}$ spans the commuting algebra. Then, by Theorem 2, these operators form a linear basis for the commuting algebra. Since $r \mapsto \mathfrak{A}(r, \lambda)$, $r \in R$, is a homomorphism, we get the following.

THEOREM 3: *The commuting algebra* $\mathscr{C}(\lambda)$ *of* $\mathrm{Ind}_P^G \lambda$ *is isomorphic to the group algebra* $\mathbb{C}[R]$.

The R-groups for p-adic Chevalley groups are classified in [3]. We show in Section 8 that $R \cong \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$, with the number of factors of $\mathbb{Z}_2$ bounded by $n = \operatorname{rank} G$, for $G = \mathrm{SU}(2n+1)$.

In fact, we explicitly construct a list of characters λ of M with non-trivial R_λ, such that any character with non-trivial R group is conjugate to one on the list.

Note that the zeros of the $\gamma_m(\lambda, s)$ are related to the reducibility of the non-unitary principal series by Theorem 6.6.2 of [1].

5. Computation of some c-functions

See [4] for the case $m = h = 0$, with applications to reducibility. $\gamma_0(\lambda, \mathrm{s})$ is the Harish-Chandra c-function.

We first indicate the calculation of $\gamma_m(\lambda, s)$ for $SL_2(F)$, $m \geqslant \max\{1, h\}$. Fix an additive character ψ of F with conductor the ring of integers $\mathcal{O}_F$ and set $\psi_m(v) = \psi(\pi^{-2m}v)$.

For $\lambda \in (F^x)^\wedge$, define

$$f_{m,\lambda,s}\left(\begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) =$$

$$\begin{cases} 0 & \text{if } v \notin p^m \\ \psi_m(v)\lambda^{-1}(a)|a|^{-s-1} & \text{if } v \in p^m \end{cases}.$$

Then for Re s > 0,

$$\gamma_m(\lambda, s) = \int f_{m,\lambda,s}\left(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \mathrm{d}n$$

$$= \int f_{m,\lambda,s}\left(\begin{pmatrix} 1 & 0 \\ 1/n & 1 \end{pmatrix}\begin{pmatrix} -n & 1 \\ 0 & -\dfrac{1}{n} \end{pmatrix}\right) \mathrm{d}n$$

$$= \int_{|n| \geqslant q^m} \psi_m\left(\frac{1}{n}\right)\lambda^{-1}(-n)|n|^{-s}\frac{\mathrm{d}n}{|n|}$$

$$= \int_{|n| \leqslant q^{-m}} \psi(\pi^{-2m}n)\lambda(-n)|n|^{s}\frac{\mathrm{d}n}{|n|}$$

$$= \lambda(\pi^{2m})|\pi|^{2ms}\int_{|n| \leqslant q^m} \bar{\psi}(n)\lambda(n)|n|^{s}\frac{\mathrm{d}n}{|n|}$$

$$= \lambda(\pi)^{2m}q^{-2ms}\Gamma(\lambda_s)$$

where $\Gamma(\lambda_s)$ is the gamma function of [6], i.e., one of Tate's local factors $\rho\,(\lambda_s)$.

Thus on $SL_2(F)$, we have the well-known result

$$A(w^{-1}, w\lambda)A(w, \lambda) = \Gamma(w\lambda_s)\Gamma(\lambda_s)I.$$

Now, suppose $G = \mathrm{SU}(3)$, $m \geqslant \max\{1, h\}$. Then for Re $s > 0$,

$$\gamma_m(\lambda, s) = \int_N f_{m,\lambda,s}(n\overline{w})\mathrm{d}n$$

$$= \int f_{m,\lambda,s}\left(\begin{pmatrix} 1 & x & y \\ & 1 & -\overline{x} \\ & & 1 \end{pmatrix}\begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}\right)$$

$$= \int f_{m,\lambda,s}\begin{pmatrix} 1 & & \\ -\overline{x}/y & 1 & \\ 1/y & x/\overline{y} & 1 \end{pmatrix}\lambda_s^{-1}\delta^{-1/2}(y).$$

The difficulty is that the integral is over those x, $y \in E$ satisfying $y + \overline{y} + x\overline{x} = 0$.

We use the formulas

$$\lambda_s(x) = \lambda(x)|x|_E^s, \quad |x|_E = |x\overline{x}|_F, \quad \text{and} \quad \delta(y) = |y|_E|y\overline{y}|_F = |y|_E^2.$$

For E/F unramified, $q_F = q$, $q_E = q^2$. For E/F ramified, $q_E = q_F = q$.

Suppose for now char $F \neq 2$ and write x, y satisfying $y + \overline{y} + x\overline{x} = 0$ as $x = u + v\tau$, $y = -\frac{1}{2}x\overline{x} + z\tau$, where u, v, $z \in F$, $E = F[\tau]$, $\overline{\tau} = -\tau$ and $\mathcal{O}_E = \mathcal{O}_F \oplus \tau\mathcal{O}_F$.

Let $\mathrm{d}_E x = \mathrm{d}x$ be Haar measure on E with $\mathrm{vol}(\mathcal{O}_E) = 1$ and $\mathrm{d}_F z = \mathrm{d}z$ be Haar measure on F with $\mathrm{vol}(\mathcal{O}_F) = 1$. Haar measure on N is given by $\mathrm{d}_E x\ \mathrm{d}_F z$.

It is convenient to assume $m \geqslant \mathrm{ord}_E 2$.

The case char $F = 2$ may be handled similarly.

Making the change of variables $z \mapsto -\frac{1}{2}x\overline{x}z$ in the last integral for $\gamma_m(\lambda, s)$, then letting $x \mapsto 2x(1 + z\tau)^{-1}$ and finally taking $x \mapsto x^{-1}$, we get $\gamma_m(\lambda, s) =$

$$\int f_{m,\lambda,s}\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ -\frac{1}{2}x\overline{x}(1 - z\tau) & -\overline{x} & 1 \end{pmatrix}$$

$$\lambda_s^{-1}\delta^{-1/2}\left(\frac{-2}{x\overline{x}(1 - z\tau)}\right)|2|_E|2|_F|x|_E^{-2}|1 + z\tau|_E^{-2}\frac{\mathrm{d}x}{|x|_E}\mathrm{d}z$$

$$= \lambda_s^{-1}(-2)|2|_F.$$

$$\int_{x\in p_E^m}\psi_m(x)\lambda_s(x\bar{x})\int_{z\in\mathscr{B}(x)}\lambda_s(1+z\tau)|1+z\tau|_E^{-1}\mathrm{d}z\frac{\mathrm{d}x}{|x|_E}$$

$$=\lambda^{-1}(-2)|2|_F^{1-2s}.$$

$$\sum_{n=m}^{\infty}\int_{|x|_E=q_E^{-n}}\psi_m(x)\lambda_s(x\bar{x})\frac{\mathrm{d}x}{|x|_E}\int_{z\in\mathscr{B}(n)}\lambda(1+z\tau)|1+z\tau|_E^{s-1}\mathrm{d}z.$$

where $\mathscr{B}(x)=F\cap(x\bar{x}\tau)^{-1}p_E^{m+\mathrm{ord}2}$ and $\mathscr{B}(n)=F\cap\tau^{-1}p_E^{m-2n+\mathrm{ord}2}$.

Now, if degree $\lambda=h\geqslant 1$, let $\mathscr{B}(n,h)=\mathscr{B}(n)\cap\tau^{-1}p_E^{-h+1}$, and write

$$\int_{\mathscr{B}(n)}\lambda(1+z\tau)|1+z\tau|_E^{S-1}\mathrm{d}z$$

$$=\int_{\mathscr{B}(N)\setminus\mathscr{B}(n,h)}\lambda(z\tau)|z|_F^{2s-1}|\tau|_E^{s-1}|z|_F^{-1}\mathrm{d}z$$

$$+\int_{\mathscr{B}(n,h)}\lambda(1+z\tau)|1+z\tau|_E^{s-1}\mathrm{d}z.$$

The last integral, over $F\cap\tau^{-1}\mathrm{P}_E^{-h+1}$, is similar to a Beta function and is the only tricky part of $\gamma_m(\lambda,s)$ to evaluate. One way is to use Fourier analysis as follows: For notational simplicity, assume E/F is unramified.

Define a function $B(u)$ on E by

$$B(u)=\begin{cases}0 & \text{if } u\notin p_E^{-h+1}\\ \int_{z\in p_F^{-h+1}}\lambda(u+z\tau)|u+z\tau|_E^{s-1}d_Fz & \text{if } u\in p_E^{-h+1}\end{cases}.$$

we want the value $B(1)$. $B(u)\in L^1(E)$ if $\mathrm{Re}\, s>0$. Define the Fourier transform $\hat{B}(w)=\int_E B(u)\bar{\psi}(uw)\mathrm{d}u$

$$=\lambda^{-1}(w)|w|_E^{-s}\int_{wp_E^{-h+1}}\lambda(u)|u|_E^{s-1}\bar{\psi}(u)\mathrm{d}u\int_{p_F^{-h+1}}\psi(z\tau w)\mathrm{d}z$$

where we let $u\mapsto u-z\tau$ and then $u\mapsto w^{-1}u$.

If $w=a+b\tau$ and we choose $\psi(w)=\psi(a)$ to have conductor $\mathscr{O}_E$, then $\hat{B}(w)=\hat{B}(a+b\tau)$

$$=\begin{cases}0 & \text{if } w\in\mathscr{O}_E \text{ or } b\notin p_F^{h-1}\\ \lambda_s^{-1}(a)\Gamma_E(\lambda_s)q^{h-1} & \text{if } a\notin\mathscr{O}_F \text{ and } b\in p_F^{h-1}\end{cases}$$

using results of [6].

If Re $s>1$, then $\hat{B}\in L^1(E)$, and by the Fourier inversion formula, $B(u)=\int_E \hat{B}(w)\psi(uw)\mathrm{d}w$. But it is then easy to calculate $B(1)$ in terms of the gamma functions of [6].

This line of analysis leads to the following result.

THEOREM: *Let $m\geqslant \max\{1,h\}$.*

(1) *For $G=SL_2(F)$, $\gamma_m(\lambda, s)=\lambda(\pi)^{2m}q^{-2ms}\Gamma(\lambda_s)$.*

(2) *for $G=SU(3)$, char $F\neq 2$, the $\gamma_m(\lambda, s)$ are given as follows: It is convenient to change s to assume $\lambda(\pi_E)=1$.*

First assume E/F is unramified.

(a) If λ is unramified, $\lambda_s(x)=|x|_E^s$, then

$$\gamma_m(\lambda, s)=|2|_F^{1-2s}q_F^{-8ms}\frac{\left(1-q^{2s-2}\right)\left(1+q^{2s-1}\right)}{\left(1-q^{-2s}\right)\left(1+q^{-2s}\right)}.$$

(b) If λ is ramified of degree $h\geqslant 1$ and $\lambda|_F x\equiv 1$, then

$$\gamma_m(\lambda, s)=|2|_F^{1-2s}\lambda(\tau)q^{-8ms}q^{h(2s-1)}\frac{1+q^{2s-1}}{1+q^{-2s}}.$$

(c) If λ is ramified of degree $h\geqslant 1$, $\lambda|_F x$ ramified of degree $h'\geqslant 1$, then $w\lambda_s\neq\lambda_s$ for any s and $\gamma_m(\lambda, s)=$

$$|2|_F^{1-2s}\lambda^{-1}(2)q^{-8ms}\lambda(\pi^{4m})\Gamma_E\left(\lambda(x\bar{x})|x|_E^{2s}\right)$$

$$\cdot\Gamma_E(\lambda_s)\Gamma_F\left(\lambda^{-1}|_F x\cdot|\cdot|_F^{-2s+1}\right).$$

Now assume F/F is ramified.

(d) If $\lambda(x)=|x|_E^s$ is unramified, then

$$\gamma_m(\lambda, s)=|2|_F^{1-2s}q^{-4ms}q^s\frac{1-q^{s-1}}{1-q^{-s}}.$$

(e) If λ is ramified of degree h $\geqslant 1$ and $\lambda|_{\mathrm{F}}\mathrm{x}\equiv 1$, then

$$\gamma_m(\lambda, s)=|2|_F^{1-2s}q^{-4ms}\lambda(\tau)q^s q^{(h/2)(2s-1)}.$$

(f) If λ is ramified of degree $h\geqslant 1$, and $\lambda|_{\mathcal{O}_F}x$ has order 2, then

$$\gamma_m(\lambda, s)=|2|_F^{1-2s}\lambda^{-1}(2)q^{-4ms}\Gamma_E(\lambda_s)\Gamma_F\left(\lambda^{-1}|_{Fx}\cdot|\cdot|_F^{-2s+1}\right)$$

$$\frac{1-q^{2s-1}}{1-q^{-2s}}=|2|_F^{1-2s}\lambda^{-1}(2)q^{-4ms}c_1 q^{h(s-1/2)}c_2$$

$$\times q^{1/2-2s}\frac{1-q^{2s-1}}{1-q^{-2s}}.$$

(g) If λ is ramified of degree $h \geqslant 1$ and $x \mapsto \lambda(x\bar{x})$ is ramified on $E^\times$ of degree h', then $\gamma_m(\lambda, s) = |2|_F^{1-2s}\lambda^{-1}(2)q^{-4ms}\lambda(\pi\bar{\pi})^{2m}$.

$$\Gamma_E\big(\lambda(x\bar{x})|\cdot|_E^{2s}\big)\Gamma_E(\lambda_s)\Gamma_F\big(\lambda^{-1}|_F x\cdot|\cdot|_F^{-2s+1}\big).$$

Remark: The case char $F = 2$ may be handled similarly. Write x and y with $y + \bar{y} + x\bar{x} = 0$ as $y = z + xx(\tau + \bar{\tau})^{-1}\tau$, where $z \in F$, $x \in E$, and $\mathcal{O}_E = \mathcal{O}_F \oplus \tau\mathcal{O}_F$. Note that trace $\tau = \tau + \bar{\tau}$ is nonzero since E/F is assumed separable. Then $\gamma_m(\lambda, s) =$

$$\lambda_s^{-1}(\tau + \bar{\tau})|\tau + \bar{\tau}|_F \sum_{n=m}^{\infty} \int_{|x|_E = q_E^{-n}} \psi_m(x)\lambda_s(x\bar{x})$$

$$\times \frac{\mathrm{d}x}{|x|} \int_{\mathscr{B}(n)} \lambda(z + \bar{\tau})|z + \bar{\tau}|_E^{s-1}\mathrm{d}z$$

where $\mathscr{B}(n) = \{z \in F | z + \bar{\tau} \in (\tau + \bar{\tau})p_E^{m-2n}\}$.

6. Complementary series

Complementary series representations for SU(3) may be constructed in the standard manner.

Note that the contragredient of $\mathrm{Ind}_P^G\lambda_s$ is naturally isomorphic to $\mathrm{Ind}_P^G(\lambda_s)^{-1}$ via the non-degenerate G-invariant bilinear form

$$\langle f|g\rangle = \int_{G/P} f(x)g(x)\mathrm{d}\dot{x} = \int_K f(k)g(k)\mathrm{d}k, \quad f \in \mathrm{Ind}_P^G(\lambda_s)^{-1},$$

$$g \in \mathrm{Ind}_P^G\lambda_s.$$

For λ_s with $w\lambda_s = (\bar{\lambda}_s)^{-1}$, define the hermitian form

$$(f|g) = \langle \mathfrak{a}(w, \lambda_s)f|\bar{g}\rangle \quad \text{on} \quad \mathrm{Ind}_P^G\lambda_s.$$

If $\mu(\lambda\colon 0) = 0$, then $\mathfrak{a}(w, \lambda)$ acts as the identity on $\mathrm{Ind}_P^G\lambda$, and $(\cdot|\cdot)$ is the usual inner product. But the restriction of $f \in \mathrm{Ind}_P^G\lambda_s$ to K depends only on λ, not on s. Thus by continuity, $(\cdot|\cdot)$ will be positive definite of s near 0, i.e., until $\mathfrak{a}(w, \lambda_s)$ has a pole, i.e., until $\gamma_m(\lambda, s)$ has a zero.

Thus $\mathrm{Ind}_P^G\lambda_\nu$ is in the complementary series if $\lambda \in (E^\times)\hat{}$, $w\lambda = \lambda$, $\nu \in \mathbb{R}$, and if $\gamma_m(\lambda, s)$ has a pole at $s = 0$ and is nonzero for $0 < s \leqslant \nu$.

7. Reducible principal series of SU(3)

We determine the reducibility of the unitary and non-unitary principal series of SU(3).

Reducibility of the unitary principal series $\mathrm{Ind}_P^G\lambda$ is determined by the theory of the R-group outlined in Section 3 and knowledge of the Plancherel measure; $\mathrm{Ind}_P^G\lambda$ is reducible if and only if $w\lambda = \lambda$ and $\gamma_m(w\lambda, -s)\gamma_m(\lambda, s)$ is holomorphic at $s = 0$. In this case, $\mathrm{Ind}_P^G\lambda$ splits as the sum of two inequivalent irreducible subrepresentations, each occurring with multiplicity 1.

Let λ be a quasicharacter.

By Theorem 6.6.2. of [1], the non-unitary principal series $\mathrm{Ind}_P^G\lambda$ is reducible if and only if $\gamma_m(w\lambda, -s)\gamma_m(\lambda, s)$ is zero at $s = 0$. We may assume $\mathrm{Re}\,\lambda > 0$. Then the kernel of $A(w, \lambda)$ is an irreducible invariant subspace of $\mathrm{Ind}_P^G\lambda$, which transforms as a special representation of G.

THEOREM: *Let $G = SU(3)$, $\lambda \in (E^\times)^\wedge = M^\wedge$. Note the $w\lambda = \lambda$ if and only if $\lambda(x\bar{x}) = 1$.*

(1) The unitary principal series $\mathrm{Ind}_P^G\lambda$ is reducible if and only if $\lambda \not\equiv 1$, $w\lambda = \lambda$, and $\lambda|_F x \equiv 1$.

(2) Suppose $\lambda \in (E^\times)^\wedge$ and $\mathrm{Re}\,s > 0$. The reducible non-unitary principal series $\mathrm{Ind}_P^G\lambda_s$ are the following:
Assume E/F unramified.

(a) The unramified $\lambda_s(x) = |x|_E^s$ for $s = 1$ or $s = \frac{1}{2} + \pi i(2 \ln q)^{-1}$.

(b) λ ramified of degree $h \geqslant 1$, $\lambda|_F x \equiv 1$, and $s = \frac{1}{2} + \pi i(2 \ln q)^{-1}$.
Now assume E/F ramified.

(c) Unramified $\lambda_s(x) = |x|_E$ for $s = 1$.

(d) λ ramified of degree h, $\lambda|_{\mathcal{O}_F} x$ of order 2, and $s = \frac{1}{2}$.

8. R-groups for SU(2n + 1)

We classify the R-groups which occur for the quasi-split special unitary groups in odd dimension associated to separable quadratic extensions E/F.

We realize the non-reduced root system Φ as $\{\pm e_i \pm e_j, \pm e_i, \pm 2e_i | 1 \leqslant i \neq j \leqslant n\}$ with $\{e_i - e_{i+1}, e_n | 1 \leqslant i < n\}$ as a basis for the positive roots. The Weyl group $S_n \propto \mathbb{Z}_2^n$ acts as permutations and sign changes on the e_i. Let c_i denote the sign change on e_i. Then c_n is the simple reflection corresponding to the root e_n.

Extend the dual roots α^v: $F^\times \to A$ to α^v: $E^\times \to M$. For $\lambda \in M^\wedge$, define a character λ_α of $E^\times$ by $\lambda_\alpha(x) = \lambda(\alpha^v(x))$. We will be concerned with the λ_α corresponding to $\alpha \in \{\pm e_i \pm e_j, \pm 2e_j\}$. Let $x \in E^\times$. The elements $\alpha^v(x)$ of M are given by the following diagonal matrices in G.

For $\alpha = 2e_i$, $\alpha^v(x) = \mathrm{diag}(1, \cdots, 1, x, \cdots, 1, \bar{x}x^{-1}, 1, \cdots, \bar{x}^{-1}, 1, \cdots, 1)$, with an x in the i^{th} place.

For $\alpha = e_i - e_j$, $\alpha^v(x) = \mathrm{diag}(1, \cdots, x, \cdots, x^{-1}, \cdots, 1, \cdots, \bar{x}, \cdots, \bar{x}^{-1}, \cdots, 1)$ with x and x^{-1} in the i^{th} and j^{th} places.

For $\alpha = e_i + e_j$, $\alpha^v(x) = \mathrm{diag}(1, \cdots, x, \cdots, \bar{x}, \cdots, 1, \cdots, x^{-1}, \cdots, \bar{x}^{-1}, \cdots, 1)$.

There is a well-defined action of the Weyl group $W \cong N(A)/Z(A)$ on characters of $M = Z(A)$, given by $w^{-1}\lambda(m) = \lambda(wmw^{-1})$. Thus W acts on the characters λ_α of E^x, by $w^{-1}\lambda_\alpha(x) = \lambda(w\alpha^v(x)w^{-1}) = \lambda_{w\alpha}(x)$.

A character λ of M is determined by the n characters λ_α of E^x corresponding to the roots $\alpha = e_1 - e_2,\ e_2 - e_3, \dots, e_{n-1} - e_n, 2e_n$, since $M \cong (E^x)^n$ by the n corresponding one-parameter subgroups α^v. Note that $w\lambda = \lambda$ if and only if $w\lambda_\alpha = \lambda_\alpha$ for each $\alpha = e_1 - e_2, \dots, e_{n-1} - e_n$, $2e_n$.

From the definitions of the α^v, we get the relations

$$\lambda_{e_i - e_j} = \lambda_{2e_i}\lambda_{2e_j}^{-1},$$

$$\lambda_{e_i - e_j}\lambda_{e_j - e_k} = \lambda_{e_i - e_k},$$

and

$$\lambda_{e_i + e_j}(x) = \lambda_{e_i - e_j}(x)\lambda_{2e_j}(x\bar{x}).$$

These relations hold in general. Additional relations will follow from the conditions $w\lambda_\alpha = \lambda_\alpha$.

Recall that $R = \{w \in W_\lambda | \alpha \in \Delta' \text{ and } \alpha > 0 \text{ imply that } w\alpha > 0\}$. It is enough to consider roots $\alpha \in \Delta'$ such that 2α is not a root, since α is positive if and only if 2α is positive. Here, $\Delta' = \{\alpha | \mathfrak{A}(w_\alpha, \lambda) \text{ is scalar}\}$. For $\alpha = e_i \pm e_j$, $\alpha \in \Delta'$ if, and only if $\lambda_\alpha \equiv 1$, a well-known SL_2 result. For $\alpha = 2e_i$, by our Theorems for SU(3), $\alpha \in \Delta'$ if, and only if $\lambda_\alpha(x\bar{x}) = 1$ for all $x \in E^x$, and either $\lambda_\alpha \equiv 1$ or $\lambda_\alpha|_{F^x} \not\equiv 1$.

THEOREM: *For $G = \mathrm{SU}(2n+1)$, $R \cong \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ with the number of factors of the 2-element group $\mathbb{Z}_2$ bounded by n = rank G.*

Explicitly, any λ with non-trivial R-group is conjugate to a character λ defined by specifying distinct characters λ_α of E^x, for $\alpha = 2e_i$, where $k \leqslant i \leqslant n$ for some k, satisfying $\lambda_\alpha(x\bar{x}) = 1$, $\lambda_\alpha \not\equiv 1$ and $\lambda_\alpha|_{F^x} \equiv 1$. Then R_λ is $\langle c_k, c_{k+1}, \cdots, c_n \rangle \cong \mathbb{Z}_2^{n-k+1}$.

The commuting algebra of $\mathrm{Ind}_P^G\lambda$ is given as the group algebra $\mathbb{C}[R]$: thus $\mathrm{Ind}_P^G\lambda$ decomposes into $|R| \leqslant 2^n$ irreducible inequivalent components.

The proof of the Theorem is similar to the one in [3] for groups of type C_n. The difference is that the reflection c_n sends $\lambda_{2e_n}(x)$ to $\lambda_{2e_n}(\bar{x})^{-1}$ instead of $\lambda_{2e_n}(x)^{-1}$. Note also that $\lambda_{e_i + e_j}(x) = \lambda_{e_i - e_j}(x)\lambda_{2e_j}(x\bar{x})$.

We may replace λ by a conjugate $w\lambda$ with no loss of generality. Then W_λ and W' are replaced by conjugates ${}^wW_\lambda$ and ${}^wW'$. But $R_{w\lambda} \neq {}^wR_\lambda$ in general, although these groups are isomorphic. However, we prove that certain w are not in any R_λ in an invariant manner. Consider the sublattice $\mathscr{L}=\mathscr{L}_\lambda=\mathbb{Z}[\alpha^v|\lambda_\alpha\equiv 1]$ in $\mathbb{Z}[\Phi^v]$. Then $w\lambda=\lambda$ if, and only if $w\alpha^v-\alpha^v$ is in $\mathscr{L}$ for all α. Suppose there is a $\beta^v=w\alpha^v-\alpha^v$ in $\mathscr{L}\cap\Phi^v$, so $\lambda_\beta\equiv 1$ and thus $\beta\in\Delta'$. Then, summing over i, $0\leqslant i<\operatorname{ord}(w)$, $\Sigma w^i\beta^v=\Sigma w^i(w\alpha^v-\alpha^v)=0$. This implies there is a $\beta\in\Delta'$, $\beta>0$, with $w^i\beta<0$ for some i. So $w^i\notin R$ and then $w\notin R$. This argument is invariant under conjugation, so no conjugate of w may be in any R-group.

LEMMA 1: *Suppose $w=sc\in R$, with $s\in S_n$ and $c\in\mathbb{Z}_2^n$. Then $s=1$.*

PROOF: Suppose s has a non-trivial cycle. By the above remark, we may conjugate by a permutation to assume that one such cycle is $(k\ k+1\ \cdots\ n)$. Then, conjugating by a sign change, we may assume that c changes the sign of at most one e_i, $k\leqslant i\leqslant n$.

First, suppose $c(e_i)=e_i$ for all $k\leqslant i\leqslant n$. Then $w\lambda=\lambda$ implies $\lambda_{2e_n}=w\lambda_{2e_n}=\lambda_{w-1(2e_n)}=\lambda_{2e_{n-1}}$. This implies $\lambda_{e_{n-1}-e_n}=\lambda_{2e_{n-1}}\lambda_{2e_n}^{-1}\equiv 1$, so $e_{n-1}-e_n\in\Delta'$. But then $e_{n-1}-e_n>0$ and $w(e_{n-1}-e_n)=e_n-e_k<0$, contradicting $w\in R$.

Now suppose $c(e_i)=e_i$ for $k\leqslant i<n$ and $c(e_n)=-e_n$. Then $w\lambda=\lambda$ implies $\lambda_{e_i-e_{i+1}}=w\lambda_{e_i-e_{i+1}}+\lambda_{e_{i-1}-e_i}$ for $k<i<n$. Hence $\lambda_{e_i-e_{i+1}}=\lambda_{e_{n-1}-e_n}$ for $k\leqslant i<n$. Also, $\lambda_{2e_n}=w\lambda_{2e_n}=\lambda_{2e_{n-1}}$ implies $\lambda_{e_{n-1}-e_n}=1$. Multiplying the trivial characters λ_α together, for $\alpha=e_k-e_{k+1},\dots,e_{n-1}-e_n$, we get $\lambda_{e_k-e_n}\equiv 1$. Also, $\lambda_{2e_k}(x)=w\lambda_{2e_k}(x)=c_n^{-1}\lambda_{2e_n}(x)=\lambda_{2e_n}(\bar{x})^{-1}$. So we get $\lambda_{2e_n}(x\bar{x})=\lambda_{e_k-e_n}(x)\lambda_{2e_n}(x\bar{x})=\lambda_{2e_k}(x)\lambda_{2e_n}(x)^{-1}\lambda_{2e_n}(x\bar{x})=\lambda_{2e_k}(x)\lambda_{2e_n}(\bar{x})\equiv 1$. Finally, $\lambda_{e_{n-1}+e_n}(x)=\lambda_{e_{n-1}-e_n}(x)\lambda_{2e_n}(x\bar{x})\equiv 1$ implies $e_{n-1}+e_n\in\Delta'$. However, $e_{n-1}+e_n>0$ and $w(e_{n-1}+e_n)<0$ contradict $w\in R$.

Thus s cannot contain a non-trivial cycle, and R is contained in the group $\mathbb{Z}_2^n$ of sign changes.

LEMMA 2: *If $c_kc_{k+1}\cdots c_n\in R$, then each $c_i\in R$, $k\leqslant i\leqslant n$.*

PROOF: Since $w\lambda=\lambda$ if, and only if $w\lambda_\alpha=\lambda_\alpha$ for all $\alpha=e_1-e_2,\dots,e_{n-1}-e_n, 2e_n$, $w=c_k\cdots c_n$ fixes λ if, and only if $\lambda_{2e_n}(x\bar{x})=\lambda_{e_i-e_{i+1}}(x\bar{x})\equiv 1$ for all $k\leqslant i<n$. But then each product $c_j\cdots c_n$ fixes λ, $k\leqslant j\leqslant n$, so in fact each c_j fixes λ, $k\leqslant j\leqslant n$. Note that this result is true for any product of sign changes. Since for any $\alpha>0$, $c_kc_{k+1}\dots c_n\alpha>0$ implies that $c_j\alpha>0$ for each $k\leqslant j\leqslant n$, we get that $c_kc_{k+1}\cdots c_n\in R$ implies that each $c_j\in R$, $k\leqslant j\leqslant n$.

We may now explicitly classify all $\lambda\in M^\wedge$ with non-trivial R_λ. We know that R-group is contained in the group $\mathbb{Z}_2^n$ of sign changes in W. Consider the longest product of sign changes in R. Each factor will then be in R, and R is generated by these sign changes. We may in fact replace

λ by a conjugate to assume that $c_k c_{k+1} \cdots c_n$ is the longest product of sign changes in R. Then $R \cong \langle c_k, \cdots, c_n \rangle$.

To complete the classification, we show that characters λ exist having these R-groups by listing all of them. A character λ of M will be fixed by each c_j generating R, $k \leqslant j \leqslant n$, if, and only if $\lambda_{2e_n}(x\bar{x}) = \lambda_{e_i - e_{i+1}}(x\bar{x}) \equiv 1$ for all $k \leqslant i < n$. Since $\lambda_{e_i - e_{i+1}} = \lambda_{2e_i}\lambda_{2e_{i+1}}^{-1}$, we may equivalently require $\lambda_{2e_i}(x\bar{x}) = 1$ for $k \leqslant i \leqslant n$. Note that this condition implies $\lambda_{e_i + e_j} = \lambda_{e_i - e_j}$. Then, to have $R_\lambda = \langle c_k, \cdots, c_n \rangle$, we need $\alpha \notin \Delta'$ for each $\alpha = e_i \pm e_j$ and $2e_m$, where $k \leqslant i < j \leqslant n$ and $k \leqslant m \leqslant n$. So we must require $\lambda_{2e_m} \not\equiv 1$ and $\lambda_{2e_m}|_{F^\times} \equiv 1$, $k \leqslant m \leqslant n$, and also $\lambda_{e_i - e_j} \not\equiv 1$ for $k \leqslant i < j \leqslant n$.

Equivalently, we may define the character λ by specifying distinct characters λ_α of $E^\times$, $\alpha = 2e_i$, $k \leqslant i \leqslant n$, satisfying $\lambda_\alpha(x\bar{x}) = 1$, $\lambda_\alpha \not\equiv 1$ and $\lambda_\alpha|_{F^\times} \equiv 1$.

We note that the conditions for reducibility, as well as the explicit classification of the R-groups, are the same as these for the analogous real group, $F = \mathbb{R}$, $E = \mathbb{C}$.

Remark: The above theorem determines the reducibility of the unitary principal series of G. But note that for $R = \langle c_k, \cdots c_n \rangle$, the characters $\lambda_{e_i - e_{i+1}}$, $1 \leqslant i < k$, are arbitrary. If we replace these characters by quasi-characters, the non-unitary principal series $\mathrm{Ind}_P^G \lambda$ will also be reducible. One may use the intertwining operators $\{\mathscr{A}(w, \lambda) | w \in R\}$ to construct $|R|$ nonzero projections onto invariant subspaces, thus showing operator reducibility.

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