M. VAN DE VEL

Dimension of convex hyperspaces : nonmetric case

Compositio Mathematica, tome 50, nº 1 (1983), p. 95-108
<http://www.numdam.org/item?id=CM_1983__50_1_95_0>

Compositio Mathematica 50 (1983) 95–108

DIMENSION OF CONVEX HYPERSPACES: NONMETRIC CASE

M. van de Vel

Abstract

Equality of rank and convex hyperspace dimension is extended from metric to nonmetric spaces. This is done by means of a quotient technique for convex structures. We determine some invariants of convex hyperspaces and we derive two combinatorial results on subcontinua in a tree.

0. Introduction

It was shown in [12, 2.3] that for a separable metric convex structure with connected convex sets, with compact polytopes, and with a certain separation property (S_4), the dimension of the convex hyperspace is equal to the rank of the convex structure. The rank $d = d(X)$ of a convex structure is determined as follows. A set $F \subset X$ is *free* if no point of F belongs to the convex hull of the other points of F. For $n < \infty$, $d \leqslant n$ iff no finite set with $n+1$ or more points is free.

For the obtaining of the above result it was necessary to interpret "dimension" both in a topological and in a convex setting. Each of these interpretations is used to derive one inequality between convex hyperspace dimension and rank. For separable metric spaces, both dimension functions were shown to be equal in [10].

Recently, Jan van Mill and the author developed a quotient technique for convex structures, which was used to obtain equality of "topological" and "convex" dimension for nonmetric compact convex structures [6]. We will now use the same quotient technique to derive the above quoted result without metrizability from the metric case. This is the main result:

0.1. THEOREM: *Let X be a compact space equipped with a uniform and S_4 convexity with connected convex sets. Then the dimension of $\mathcal{C}^*(X)$, the convex hyperspace of X, equals the rank $d(X)$ of X.*

The term "dimension" need not be specified, since by the main result of [6] all topological dimension functions ind, Ind, dim coincide with "convex" dimension on compact spaces with a suitable convexity.

AMS Subject Classification: 54B20, 52A01, 54H99, 54F45.

From 0.1 we also derive the following result for noncompact spaces:

0.2. THEOREM: *Let X be equipped with a uniform and S_4 convexity with connected convex sets and with compact polytopes. Then the convex dimension of $\mathcal{C}^*(X)$ equals the rank $d(X)$ of X.*

According to results in [9], there is a close relationship between the (convex) dimension and certain invariants of a convex structure. For a certain class of convex hyperspaces we determine the exact value of such invariants as the rank and the Radon number, and we apply these results to obtain some combinatorial theorems on subcontinua of trees.

Let us first explain the terminology used above.

1. Preliminaries

1.1. ABSTRACT CONVEXITY: A *convex structure* consists of a set X together with a collection $\mathcal{C}$ of subsets of X, called *convex sets*, such that $\mathcal{C}$ is closed under the formation of intersections and of updirected unions. In particular, $\emptyset$, $X \in \mathcal{C}$. It will also be assumed throughout that singletons are convex. The family $\mathcal{C}$ is called a *convexity* on X.

The *$\mathcal{C}$-hull* of a set $A \subset X$ is the convex set

$$h(A) = \cap\{C \mid A \subset C \in \mathcal{C}\}.$$

For finite A, $h(A)$ is called a *polytope*. A convex set with a convex complement is called a *half-space*. The convex structure $(X, \mathcal{C})$ has the *separation property* S_4 if for each two disjoint convex sets C, C' in X there is a half-space H with

$$C \subset H, \quad C' \cap H = \emptyset.$$

See [2,7].

1.2. UNIFORM CONVEXITY: Let μ be a uniformity on X, defined in terms of diagonal entourages, and let $\mathcal{C}$ be a convexity on X. Then μ is *compatible* with $\mathcal{C}$ provided that for each $U \in \mu$ there is an "*associated*" $V \in \mu$ such that for each $C \in \mathcal{C}$,

$$hV[C] \subset U[C].$$

It is required at the same time that $\mathcal{C}$ be a *topological convexity,* that is: the $\mathcal{C}$-polytopes are closed in the (uniform) topology of X.

If X is a topological space, then a topological convexity $\mathcal{C}$ on X is called *uniformizable* (*metrizable*) if there exists a (metric) uniformity for X compatible with $\mathcal{C}$.

For notational convenience, all additional structures on a set X (topology, uniformity, convexity) will be included in the symbol X whenever possible. If needed, "the" convexity of X will be denoted by $\mathcal{C}(X)$. The collection of all nonempty compact convex sets in X will be denoted by $\mathcal{C}^*(X)$. Note that $\mathcal{C}^*(X)$ is a subset of the *hyperspace*, $H(X)$, of X, which is the collection of all nonempty closed subsets of X equipped with the usual Vietoris topology. $\mathcal{C}^*(X)$, equipped with the relative topology, will be called the *convex hyperspace* of X.

In addition to a convex hull operator, a topological convexity on X also gives rise to a *convex closure operator* h^*: for $A \subset X$,

$$h^*(A) = \cap\{C \mid A \subset C,\ C \subset X \text{ convex closed}\}.$$

We note that a uniformizable convexity is *closure-stable* (that is, the closure of a convex set is convex; see [10, 2.4]), whence $h^*(A) = Clh(A)$ in this case.

1.3. Subbase for a Convexity: A convexity $\mathcal{C}$ on a set X is *generated* by the family $\mathcal{S} \subset \mathcal{C}$ (which is then called a *subbase for* $\mathcal{C}$) if $\mathcal{C}$ is the coarsest among all convexities including $\mathcal{S}$. Every family $\mathcal{S}$ of subsets of X generates a convexity on X: one first constructs the family $\mathcal{B}$ of all intersections of subfamilies of $\mathcal{S}$, and the desired convexity is then obtained by taking all unions of updirected subfamilies of $\mathcal{B}$.

An essential observation in this process is that *all* $\mathcal{C}$*-polytopes* (except, maybe, the empty one) *are obtained already after the first step*: every nonempty polytope is the intersection of subbase sets (see [2,7]). Note that a topological convexity is simply a convexity with a subbase of closed sets.

Subbases are involved in constructing a suitable convexity on a convex hyperspace:

1.4. Induced Convexity on a Convex Hyperspace: Let X be a topological convex structure. If $A_1,\ldots,A_n$ are subsets of X, then we write

$$\langle A_1,\ldots,A_n\rangle = \left\{A \mid A \in H(X), A \subset \bigcup_{i=1}^{n} A_i, A \cap A_i \neq \emptyset \text{ for } i = 1,\ldots,n\right\}.$$

Note that the closed sets of the (Vietoris) topology of the hyperspace $H(X)$ are generated by the sets of type

$$\langle A\rangle \quad \text{or} \quad \langle A, X\rangle, \quad A \subset X \text{ closed}.$$

The *induced convexity* on the convex hyperspace $\mathcal{C}^*(X)$ is defined to be the one generated by the sets of type

$$\langle C\rangle \cap \mathcal{C}^*(X) \quad \text{or} \quad \langle C, X\rangle \cap \mathcal{C}^*(X), \quad C \subset X \text{ convex closed}.$$

Note that this is a topological convexity, since it has a subbase of closed sets.

The fact that polytopes must be subbase intersections was used in [11, 1.1] to obtain the following hull formula: if $C_1, \ldots, C_n \in \mathcal{C}^*(X)$, then $h\{C_1, \ldots, C_n\}$ equals the collection of all $C \in \mathcal{C}^*(X)$ with the following properties:

(1) $C \subset h^*\left(\bigcup_{=1}^{n} C_i\right)$

(2) if $c_1 \in C_1, \ldots, c_n \in C_n$, then $C \cap h\{c_1, \ldots, c_n\} \neq \emptyset$.

REMARK: The consistent use of h (hull operator) in all circumstances seems not to lead to ambiguity: the argument of h will make clear which space is involved.

With most of the terminology available now, let us quote some results for later use.

1.5. THEOREM [11, §1]: *Let X be a uniformizable and S_4 convex structure such that the convex closure of the union of two compact convex sets is compact.*† *Then the following are true*:

(1) *the convex structure $\mathcal{C}^*(X)$ is uniformizable, S_4, and has compact polytopes*;

(2) *if all convex sets in X are connected, then the same is true in $\mathcal{C}^*(X)$.*

1.6. FACTORIZATION THEOREM [6, 3.1]: *Let X be a compact space equipped with a uniform convexity. Let Y be a topological space, and let $f\colon X \to Y$ be a map. Then there exists a compact space $\tilde{X}$, equipped with a uniform convexity, together with a factorization*

$$X \xrightarrow{q} \tilde{X} \xrightarrow{\tilde{f}} Y$$

of f, such that the following are true:

(1) *q is onto and $\tilde{C} \subset \tilde{X}$ is convex iff $q^{-1}(\tilde{C}) \subset X$ is convex*;

(2) *$\tilde{X}$ is S_4 if X is*;

(3) *the weight of the space $\tilde{X}$ is at most the weight of Y.*

A *map* is a continuous function. A function g from a convex structure X to a convex structure X' is *convexity preserving* (CP) if the inverse images of convex sets in X' are convex in X. Note that there is no condition on *direct* images, but an equivalent condition is as follows:

$$gh(F) \subset hg(F), \qquad F \subset X \text{ finite.}$$

Note that the above q is a CP map. An additional fact, not needed below,

† In particular, polytopes are compact.

is that a direct q-image of a convex set is also convex. The convex structure $\tilde{X}$ is called the *quotient of X under q.*

One other fact, which is not needed below, is that $\tilde{X}$ is also subject to a dimension restriction. This condition played a mayor role in [6] in obtaining an equality of convex and topological dimension (see 1.8 below).

1.7. CONVEX DIMENSION: The *trace* of a convex structure X on a subset Y is obtained by taking on Y the convexity

$$\{C \cap Y \mid C \in \mathcal{C}(X)\}.$$

Let X now be a topological convex structure. Its *convex small inductive dimension* is defined to be the number cind $(X) \in \{-1, 0, 1, \ldots, \infty\}$ satisfying the following conditions:

(1) cind $X = -1$ iff $X = \emptyset$;

(2) cind $X \leqslant n + 1$ (where $n < \infty$) iff for each convex closed set $C \subset X$ and for each $x \in X \setminus C$ there exist convex closed sets D, D' in X with

$$(*) C \subset D \setminus D', \quad x \in D' \setminus D, \quad D \cup D' = X$$

and cind $(D \cap D') \leqslant n$

(the pair of sets D, D' with properties $(*)$ above is usually called a *screening* of C and x). The set $D \cap D'$ is equipped with the trace convexity. See [8] for motivation and for detailed results on cind.

1.8. EQUALITY THEOREM [6, 3.2]: *Let X be a compact space equipped with a uniform and S_4 convexity with connected convex sets. Then*

$$\text{ind } X = \text{Ind } X = \dim X = \text{cind } X.$$

Let us note that a compact space X has only one uniformity. This simplifies the problem in which circumstances a convexity on X is uniformizable or metrizable: by [10, 3.3] the following are equivalent for a topological convexity $\mathcal{C}(X)$:

(1) $\mathcal{C}(X)$ is compatible with the unique uniformity on X;

(2) the convex closure operator $h^* : H(X) \to \mathcal{C}^*(X)$ is continuous;

(3) $\mathcal{C}^*(X)$ is closed in $H(X)$ and for each $C \in \mathcal{C}^*(X)$ and each open set $O \supset C$ there is a $D \in \mathcal{C}^*(X)$ with

$$C \subset \text{int } D \subset D \subset O.$$

With the notion of rank available from the introduction, let us quote the main theorem of [12]:

1.9. THEOREM: *Let X be a separable space equipped with a metrizable S_4 convexity with connected convex sets and with compact polytopes. If X has more than one point, then*

$$\dim \mathcal{C}^*(X) = d(X).$$

Note that for a one-point space X, $\dim \mathcal{C}^*(X) = 0$, whereas $d(X) = 1$.

2. Proof of the main theorem

We begin with a result on induced maps in convex hyperspaces:

2.1. THEOREM: *Let X be a topological convex structure, and let X' be a completely uniformizable S_4 convex structure with compact polytopes. If $f\colon X \to X'$ is a CP map, then the function*

$$f^* : \mathcal{C}^*(X) \to C^*(X'),$$

defined by

$$f^*(C) = h^* f(C),$$

is well-defined, continuous, and CP.

PROOF: If $C \subset X$ is compact, then $f(C)$ is compact. By [10, 2.6] the convex closure of a compact set is compact in a completely uniformizable convexity with compact polytopes. Hence f^* is well-defined. To see that f^* is continuous, it suffices to show that the convex closure operator of X' is continuous on compact sets. It is known [10, 2.4] that the hull operator of X' is continuous on *finite* sets. Let $A \subset X'$ be compact, and let $O \subset X'$ be open.

Case 1. O meets $h^*(A)$. As X' is closure-stable, we find that O meets $h(A)$, and hence there exist $a_1, \dots, a_n \in a$ with $h\{a_1, \dots, a_n\} \cap O \neq \emptyset$. As h is continuous on finite sets, there exist neighborhoods O_i of a_i such that $h\{a'_1, \dots, a'_n\}$ meets O whenever $a'_i \in O_i$, $i = 1, \dots, n$. Hence for each $A' \in \langle O_1, \dots, O_n, X \rangle$, $h^*(A)$ meets O.

Case 2. $h^*(A) \subset O$. As $h^*(A)$ is compact and as $X' \setminus O$ is closed, there is a uniform diagonal neighborhood U_0 of X' with $U_0[h^*(A)] \subset O$. Let U_1 be a uniform diagonal neighborhood with $U_1 \ U_1 \subset U_0$, and let the uniform diagonal neighborhood V be associated to U_1. For

$$C = h(V[h^*(A)])$$

we find that

$$A \subset h^*(A) \subset \operatorname{int} C,$$

and that

$$\begin{aligned}\bar{C} \subset U_1[C] = U_1[hV[h^*(A)]] &\subset U_1[U_1[h^*(A)]]\\ &\subset U_0[h^*(A)] \subset O.\end{aligned}$$

Hence if $A' \subset \operatorname{int} C$, then $h(A') \subset C$ and consequently

$$h^*(A') \subset \bar{C} \subset O,$$

establishing continuity of h^* on compact sets.

We finally show that f^* is CP. It suffices to prove that for each finite collection $\mathfrak{F} \subset \mathcal{C}^*(X)$,

$$f^*h(\mathfrak{F}) \subset h(f^*(\mathfrak{F})).$$

Let $\mathfrak{F} = \{C_1, \ldots, C_n\}$, and let

$$C \in h\{C_1, \ldots, C_n\}.$$

Then

$$C \subset h^*\left(\bigcup_{i=1}^{n} C_i\right),$$

and it directly follows that

$$\begin{aligned}f^*(C) = h^*(f(C)) \subset h^*fh^*\left(\bigcup_{i=1}^{n} C_i\right) &\subset h^*\left(\bigcup_{i=1}^{n} f(C_i)\right)\\ &\subset h^*\left(\bigcup_{i=1}^{n} f^*C_i\right).\end{aligned}$$

Next, let $d_i \in f^*(C_i)$ for $i = 1, \ldots, n$, and suppose that

$$f^*(C) \cap h\{d_1, \ldots, d_n\} = \emptyset.$$

By $[V_4, 2.5]$ there exists a CP map $g\colon X' \to [0,1]$ ([0,1] with "linear" convexity) such that

$$gf^*(C) = \{0\};\ gh\{d_1, \ldots, d_n\} = \{1\}.$$

Note that $g^{-1}(1)$ is a closed half-space of X' meeting $h^*f(C_i)$. Hence $g^{-1}(1)$ meets $f(C_i)$, for otherwise there is a $t < 1$ with

$$gf(C_i) \subset [0, t],$$

and then

$$h^*f(C_t) \subset g^{-1}[0, t].$$

Fix $c_t \in C_t$ with $gf(c_t) = 1$. Then

$$gfh\{c_1, \dots, c_n\} \subset gh\{fc_1, \dots, fc_n\} = \{1\}.$$

However, as C meets $h\{c_1, \dots, c_n\}$, we obtain that $gf(C)$ includes 1, contradicting that

$$gf(C) \subset gf^*(C) = \{0\}. \qquad \square$$

2.2. PROOF OF THE MAIN THEOREM: Let X be a compact space with a uniform S_4 convexity, the convex sets of which are connected. By [12, 2.1] the "convex" dimension of $\mathcal{C}^*(X)$ does not exceed the rank of X. As noted in 1.5 above, $\mathcal{C}^*(X)$ carries a uniform S_4 convexity with connected convex sets, and $\mathcal{C}^*(X)$ is compact, being a retract of $H(X)$ under convex closure. Hence by 1.8, the "convex" dimension of $\mathcal{C}^*(X)$ is simply dim $\mathcal{C}^*(X)$, showing that

$$\dim \mathcal{C}^*(X) \leqslant d(X).$$

In order to see that

$$d(X) \leqslant \dim \mathcal{C}^*(X),$$

we show that for all $n \geqslant 1$,

$$n \leqslant d(X) \Rightarrow n \leqslant \dim \mathcal{C}^*(X).$$

Note that $n = 1$ is a trivial case, since X is connected and has more than one point, whence dim $X \geqslant 1$, and consequently dim $\mathcal{C}^*(X) \geqslant 1$ since X is a closed subspace of $\mathcal{C}^*(X)$.

So assume $n > 1$, and let

$$F = \{x_1, \dots, x_n\}$$

be a free set in X with exactly n points. For each $k = 1, \dots, n$ we put

$$F_k = F \setminus \{x_k\}$$

for convenience. Then there exist CP maps ([0,1] with "linear" convexity)

$$f_k \colon X \to [0,1]$$

with $f_k(F_k) = \{0\}$ and $f_k(x_k) = 1$, $k = 1, \dots, n$. This leads to a CP map

$$f = (f_1, \dots, f_n) \colon X \to [0,1]^n$$

(the n-cube carries the "subcube" convexity).

By 1.6, there is a factorization

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "q"'] & {[0,1]^n} \\ & \tilde{X} \arrow[u, "\tilde{f}"'] \end{tikzcd}$$

of f, where $q: X \to \tilde{X}$ is a CP quotient, $\tilde{X}$ is a uniform S_4 convex space, and the weight of the topological space $\tilde{X}$ is at most the weight of $[0,1]^n$. In particular, $\tilde{X}$ is metrizable. Note that $\tilde{f}$ is CP since for each convex set $C \subset [0,1]^n$.

$$q^{-1}(\tilde{f}^{-1}(C)) = f^{-1}(C),$$

which is convex in X. Let $\tilde{f}_k$ be the k^{th} component function of $\tilde{f}$. Note that

$$f_k = \tilde{f}_k \circ q.$$

Hence,

$$\tilde{f}_k(qx_k) = 1,$$

whereas

$$\tilde{f}_k(qF_k) = \{0\} \qquad (k = 1, \ldots, n).$$

It follows that $q(F)$ is a free collection in $\tilde{X}$ with n points, that $d(\tilde{X}) \geqslant n$, and by 1.9 (the main theorem for the metric case), that $\dim \mathcal{C}^*(\tilde{X}) \geqslant n$. By Theorem 2.1, q induces a CP map

$$q^* : \mathcal{C}^*(X) \to \mathcal{C}^*(\tilde{X}),$$

which is onto because q is onto. A CP map does not raise (convex) dimension by [8, 4.8], from which it finally follows that

$$\dim \mathcal{C}^*(X) \geqslant n. \qquad \square$$

The above result enables us to derive a similar theorem, valid for *noncompact* spaces as well:

2.3. Proof of Theorem 0.2: Let X be equipped with a uniform S_4 convexity having connected convex sets and compact polytopes. To see that

(1) $\operatorname{cind} \mathcal{C}^*(X) \leqslant d(X)$,

we may assume that $d(X) < \infty$. Then the induced hyperspace convexity is "nice enough" to allow application of most results of convex dimension theory, [12, 1.7]. By [8, 4.5], there is a polytope

$$h\{C_1, \ldots, C_n\}$$

in $\mathcal{C}^*(X)$ with

$$\operatorname{cind} \mathcal{C}^*(X) = \operatorname{cind} h\{C_1, \ldots, C_n\}.$$

Note that $C_1, \ldots, C_n$ are compact convex sets in X, and hence that

$$C = h^*\left(\bigcup_{i=1}^{n} C_i\right)$$

is compact convex too. Then $\langle C \rangle \cap \mathcal{C}^*(X)$ equals the convex hyperspace of C, and it is also a convex subset of $\mathcal{C}^*(X)$, including the polytope $h\{C_1, \ldots, C_n\}$. Hence

$$\operatorname{cind} \mathcal{C}^*(X) = \operatorname{cind} \mathcal{C}^*(C).$$

C being compact, it follows from 1.8 and 0.1 that

$$\operatorname{cind} \mathcal{C}^*(C) = \dim \mathcal{C}^*(C) = d(C),$$

where, obviously, $d(C) \leqslant d(X)$. This establishes (1).

On the other hand, if $n \leqslant d(X)$, then there is a free collection F in X with exactly n points, whence by 0.1 and by [8, 2.5]

$$n \leqslant d(h(F)) = \operatorname{cind} \mathcal{C}^*(h(F)) \leqslant \operatorname{cind} \mathcal{C}^*(X). \qquad \square$$

It appears from [9] that (convex) dimension is rather closely related to certain "classical" invariants of convex structures. As a matter of observed fact, convex dimension is usually much easier to compute than the rather combinatorially behaved invariants. In this way, it is a considerable step forwards to possess information on convex dimension, like in the case of convex hyperspaces. By way of example, let us derive some "dependency" results for subcontinua of a tree, which seem difficult to obtain "by hand". The first result, 2.4, is more or less auxilliary to 2.5.

2.4. THEOREM: *Let T be a compact tree with n endpoints* ($2 \leqslant n < \infty$), *let $r > 2n$, and let $C_1, \ldots, C_r$ be nonempty subcontinua. Then there is an $i \in \{1, \ldots, r\}$ such that*

(1) *every connected set including all C_j, $j \neq i$ also includes C_i;*

(2) *every connected set meeting all C_j, $j \neq i$ also meets C_i.*

Moreover, the lower bound for r is sharp.

2.5. THEOREM: *Let T be a compact tree with n endpoints ($2 \leqslant n < \infty$), and let r be such that $C(r,[r/2]) > 2n$. If $C_1, \ldots, C_r$ are nonempty subcontinua, then there is a partition $I \cup II$ of $\{1, \ldots, r\}$ such that*

(1) *every connected set including all C_i, $i \in I$, intersects every connected set meeting all C_j, $j \in II$;*

(2) *idem, with I and II interchanged.*

Moreover, the lower bound for r is sharp.

$[r/2]$ designates the lower integer approximation to $r/2$, and $C(q, p)$ denotes the number of combinations of p elements in a q-point set. Here are some examples of lower bounds for r in 2.5:

$$n = 2: r \geqslant 4;\ n = 3, 4: r \geqslant 5;\ n = 5, \ldots, 9: r \geqslant 6;$$

$$n = 10, \ldots, 17: r \geqslant 7.$$

It is somewhat surprising (especially in 2.5) that the number of endpoints is the *only* information needed.

Let us first discuss some concepts needed for our proofs. A convex structure is *binary* (or: has Helly number $\leqslant 2$) if every finite collection of pairwise intersecting convex sets has a common point.

The *generating degree*, gen (X), of a convex structure X is defined in [11, 2.1] as follows. For $n < \infty$, gen $(X) \leqslant n$ iff there is a subbase for the convexity $\mathcal{C}(X)$ which can be decomposed into (at most) n totally ordered subfamilies. This invariant was introduced to obtain an upper estimate for the rank $d(X)$ of X: if X has more than one point, then $d(X) \leqslant$ gen (X), as was observed in [11, 2.2].

The *Radon number, $r(X)$*, of a convex structure X can be defined as follows: For $n < \infty$, $r(X) \leqslant n$ iff for each finite set $F \subset X$ with $n + 1$ or more points there is a partition $F_1 \cup F_2$ of F with $h(F_1) \cap h(F_2) \neq \emptyset$ (the set F is then called "*dependent*", and F_1, F_2 is a *Radon partition* of F). See [9,1.3].

The following auxilliary result almost directly leads to 2.4, and will be used in the proof of 2.5 as well:

2.6. LEMMA: *Let X be a uniformizable, S_4, and binary convex structure such that the convex closure of the union of two compact convex sets is compact again.* † *Then*

$$d(\mathcal{C}^*(X)) \leqslant d(X) + \text{gen}\,(X).$$

PROOF. Let $d = d(X)$ and $g =$ gen (X). We may assume that $d < \infty$ and

† In view of a result in [7, 2.9] on binary convexities, this condition is actually equivalent with compactness of polytopes.

$g < \infty$. Let $D_1, \ldots, D_{d+g+1}$ be members of $\mathcal{C}^*(X)$ which form a free collection. Then, for each i, there is a closed half-space $\mathcal{H}_i$ of $\mathcal{C}^*(X)$ with

$$D_i \notin \mathcal{H}_i,\, h\{D_j \mid j \neq i\} \subset \mathcal{H}_i.$$

By [12, 1.2] each $\mathcal{H}_i$ is of type

$$(^3)\ \langle H_i \rangle \cap \mathcal{C}^*(X), \quad \text{or } (^4)\ \langle H_i, X \rangle \cap \mathcal{C}^*(X),$$

where H_i is a closed half-space of X. Let us assume that (^{3}) is in order for $i \in I$ only, where I has p members ($p \geqslant 0$). For each $i \in I$ we have

$$D_i \not\subset H_i;\ D_j \subset H_i \quad \text{for } j \neq i.$$

Choose a point $x_i \in D_i \setminus H_i$. Then for $i \neq j$ in I, we have $x_i \notin H_i$, $x_j \in D_j \subset H_i$, and hence that $\{x_i \mid i \in I\}$ is a free collection in X. Consequently, $p \leqslant d$, and there are at least $g+1$ indices left, for which $\mathcal{H}_i = \langle H_i, X \rangle \cap \mathcal{C}^*(X)$. For every two of such remaining indices $i \neq j$, we have $H_i \not\subset H_j$ since H_i meets D_j and D_j is disjoint with H_j. It was shown in [11, 2.3] that for a binary S_4 convexity the generating degree is "realized" by the subbase of all (closed) half-spaces. Hence this subbase can be decomposed into g chains, contradicting with the obtaining of at least $g+1$ mutually incomparable closed half-spaces of X. $\square$

Proof of 2.4 and 2.5: The collection $\mathcal{C}(T)$ of all connected subsets of T constitutes a uniform, binary and S_4 convexity on T by [7, 2.10] and [10, 3.9]. It was shown in [11, 3.1] that

$$d(T) = \text{gen}(T) = \text{number of endpoints of } T.$$

whence by lemma 2.6, the rank of $\mathcal{C}^*(T)$ is at most $2n$. By [11, 4.2], the rank of an n-dimensional "sufficiently nice" convex structure (like $\mathcal{C}^*(T)$) is *at least* equal to twice its dimension. Hence,

$$(^5)\ d(\mathcal{C}^*(T)) = 2n.$$

Taking $2n+1$ or more subcontinua of T, one of them must be in the hull of the other ones, which is expressed more explicitly in (1) and (2) of 2.4 (the hull of a compact set in T is closed by [9, 2.14]).

It was shown in [9, 2.11] that the Radon number of an n-dimensional uniformizable, S_4, binary convexity with connected convex sets and with compact polytopes must be equal to r_n or $r_n + 1$, where r_n is the Radon number of the n-cube, equipped with its "subcube" convexity. Also, equality with $r_n + 1$ can occur only for a restricted number of dimensions. This result was slightly improved in [12, 3.3] with the aid of convex hyperspace techniques: if the rank of X (as above) is minimal (twice the dimension n of X) then $r(X)$ equals r_n.

Note that $\mathcal{C}^*(T)$ is n-dimensional since $n = d(T)$, and that the convexity on $\mathcal{C}^*(T)$ is also binary by [5, 5.4]. Hence by ([5]) and by the above quoted result, the Radon number of $\mathcal{C}^*(T)$ equals r_n. By a result of Eckhoff, r_n is the largest among all r with $C(r, [r/2]) \leqslant 2n$ ([1, Satz 3]). Hence, taking r subcontinua $C_1, \ldots, C_r$ of T with r *not* satisfying this inequality, there must be a partition $I \cup II$ of $\{1, \ldots, r\}$ with

$$h\{C_i \mid i \in I\} \cap h\{C_j \mid j \in II\}$$

nonempty. Using the fact that the convex hull of a compact set in T is closed, it is not difficult to see that the above property is equivalent to (1) and (2) in 2.5. □

We note that binarity is involved in lemma 2.6 only to ensure that the generating degree is "realized" by the subbase of all half-spaces. For S_3-convexities it is known, [11, 2.3], that gen is realized by a subbase consisting of *some* (not all) half-spaces, but it is not known whether the subbase of *all* half-spaces behaves the same way.

2.7. Remark: Let X be a compact uniformizable convex structure. Then the convex hyperspace $\mathcal{C}^*(X)$ can be looked upon as a *topological semilattice,* where the "infimum" of $C, D \in \mathcal{C}^*(X)$ is taken as $h^*(C \cup D)$. Certain results of Lawson, [3, 3.4] and [4, 2.2], assert that for a compact chain-wise connected semilattice S, breadth equals the "cohomological" dimension, $cd\ S$, or equals $cd\ S + 1$. This suggests a close relationship between the rank of X and the breadth of $\mathcal{C}^*(X)$ (which is chain-wise connected iff all convex sets in X are connected). And indeed, it is easy to show that both invariants are equal. With some more efforts, it can be shown that if $d(X)$ is strictly larger than the Helly number of X (which is the case if X is as in 0.1 and has more than one point) then the breadth of $\mathcal{C}^*(X)$ equals $cd\ \mathcal{C}^*(X)$, not $cd\ \mathcal{C}^*(X) + 1$. Recently, we have been able to show that "cohomological" dimension equals convex dimension for compact, uniformizable S_4-convexities with connected convex sets. A paper on this topic is in preparation.

Joining these efforts together leads to an independent proof of theorem 0.1.

References

[1] J. Eckhoff: Der Satz von Radon in konvexen Produktstrukturen II. *Monatsh. für Math. 73* (1969) 7–30.

[2] R.E. Jamison: A General Theory of Convexity. Dissertation, University of Washington, Seattle, Washington, 1974.

[3] J.D. Lawson: The relation of breadth and co-dimension in topological semilattices. *Duke Math. J. 37* (2) (1970) 207–212.

[4] J.D. LAWSON: The relation of breadth and co-dimension in topological semilattices II. *Duke Math. J. 38* (3) (1971) 555–559.

[5] J. VAN MILL and M. VAN DE VEL: Subbases, convex sets, and hyperspaces. *Pacific J. Math. 92* (2) (1981) 385–402.

[6] J. VAN MILL and M. VAN DE VEL: Equality of the Lebesgue and the inductive dimension functions for compact spaces with a uniform convexity. *Coll. Math.*, to appear.

[7] M. VAN DE VEL: Pseudo-boundaries and pseudo-interiors for topological convexities. *Diss. Math. 210* (1983) 1–72.

[8] M. VAN DE VEL: Finite dimensional convex structures I: general results. *Top Appl. 14* (1982) 201–225.

[9] M. VAN DE VEL: Finite dimensional convex structures II: the invariants. *Top. Appl. 16* (1983) 81–105.

[10] M. VAN DE VEL: A selection theorem for topological convex structures, to appear.

[11] M. VAN DE VEL: On the rank of a topological convexity. *Fund. Math. 119,* to appear.

[12] M. VAN DE VEL: Dimension of convex hyperspaces. *Fund. Math.*, to appear.

(Oblatum 29-VI-1981)

Subfakulteit Wiskunde
Vrije Universiteit Amsterdam
De Boelelaan 1081
1081 HV Amsterdam
The Netherlands