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SELF-DUAL KÄHLER MANIFOLDS AND EINSTEIN MANIFOLDS OF DIMENSION FOUR

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1. Introduction

The aim of the present paper is to study some local and global questions concerning four-dimensional Einstein and Kähler manifolds satisfying various conformally invariant conditions.

For an oriented Riemannian four-manifold (M, g), the bundle of 2-forms over M splits as the Whitney sum $\Lambda^2 M = \Lambda^+ + \Lambda^-$, $\Lambda^\pm$ being the eigenspace bundles of the Hodge star operator $* \in \operatorname{End} \Lambda^2 M$. The Weyl conformal tensor $W \in \operatorname{End} \Lambda^2 M$ leaves $\Lambda^\pm$ invariant, and the restriction $W^\pm$ of W to $\Lambda^\pm$ may be viewed as a (0,4) tensor, operating trivially on $\Lambda^\mp$. The oriented Riemannian four-manifold (M, g) is said to be *self-dual* (resp., *anti-self-dual*) ([2]) if $W^- = 0$ (resp., $W^+ = 0$). Considering Kähler metrics, we shall endow the underlying manifold with the natural orientation and speak, e.g., of self-dual Kähler manifolds (of real dimension four).

Section 3 of this paper deals with conformal changes of Kähler metrics in dimension four. We start by observing (Proposition 3) that, for such a Kähler metric g with scalar curvature u, the conformally related metric $\bar{g} = g/u^2$ (defined wherever $W^+ \neq 0$) satisfies the conditions $\bar{\delta}\bar{W}^+ = 0$ ($\bar{\delta}$ being the formal divergence operator associated to $\bar{g}$) and

$$\#\operatorname{spec}(\bar{W}^+) \leq 2, \tag{1}$$

i.e., $\bar{W}^+ \in \operatorname{End} \Lambda^+$ has, at each point, less than three distinct eigenvalues. Conversely, for any metric $\bar{g}$ satisfying these two conditions, a natural

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conformal change, defined wherever $\bar{W}^+ \neq 0$, leads to a Kähler metric (Proposition 5). Next we characterize, at points where $W^+ \neq 0$, those four-dimensional Kähler metrics which are locally conformally Einsteinian (Proposition 4), as well as the Einstein metrics which are locally conformally Kählerian (Proposition 5 and Remark 4). The latter result, local in nature, has an additional consequence ((iv) of Proposition 5): *A four-dimensional Einstein manifold* (M, g) *satisfying condition* (1) (*which is necessary and, generically, sufficient for* g *to be locally conformally Kählerian*), *has either* $W^+ = 0$ *identically, or* $W^+ \neq 0$ *everywhere.*

In Section 4 we study four-dimensional Kähler manifolds which are self-dual for the natural orientation. We prove there (Corollary 3 and Proposition 6) that, except for some trivial cases, self-dual Kähler metrics coincide, locally, with (four-dimensional) metrics *of recurrent conformal curvature.* The main result of Section 4 is Theorem 1, stating that *any compact self-dual Kähler manifold is locally symmetric.* This assertion fails, in general, for non-compact manifolds (Remark 6). We also prove (Proposition 7) that *any compact analytic self-dual manifold, such that* $\delta W = 0$, *must be conformally flat or Einsteinian.*

Section 5 of this paper is concerned with four-dimensional compact oriented Einstein manifolds satisfying condition (1) (cf. assertion (iv) of Proposition 5, mentioned above). All known examples of compact orientable Einstein four-manifolds satisfy (1) for a suitable orientation. Our main result (Theorem 2) says that *up to a two-fold Riemannian covering, every compact oriented Einstein four-manifold* (M, g), *having property* (1), *belongs to one of the following three classes*: (i) *anti-self-dual Einstein manifolds*; (ii) *non-Ricci-flat Kähler–Einstein manifolds*; (iii) $M = S^2 \times S^2$ *or* $M = CP^2 \#(-kCP^2)$, $0 \leq k \leq 8$, *and* g *is Hermitian* (*but not Kählerian*) *for some complex structure, globally conformal to a Kähler metric and admits a non-trivial holomorphic Killing vector field.* Known examples show that each of these three cases may really occur (Remark 7). As a consequence, we obtain *a finiteness theorem for compact four-dimensional Einstein manifolds admitting sufficiently many local isometries with fixed points* (Theorem 3).

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2. Notations and preliminaries

Throughout this paper we shall use the symbols ∇, R, r, u and W for the Riemannian connection, curvature tensor, Ricci tensor, scalar curvature and Weyl conformal tensor, respectively, of a given Riemannian metric g on a manifold M. For a metric denoted by $\bar{g}$, the corresponding symbols (including δ and Δ) will be barred; however, for the pull-back metric on any covering space of (M, g), we shall use the symbol g again. Instead of $g(Y, Z)$ we shall sometimes write $\langle Y, Z \rangle$. Our conventions are such that, in local coordinates, $r_{ij} = R_{ikj}^{k} = g^{pq}R_{ipjq}$, $u = g^{ij}r_{ij}$ and

$$W_{hijk} = R_{hijk} - (n-2)^{-1}(g_{hj}r_{ik} + g_{ik}r_{hj} - \\ - g_{ij}r_{hk} - g_{hk}r_{ij}) + (n-1)^{-1}(n-2)^{-1}u(g_{hj}g_{ik} - g_{ij}g_{hk}), \tag{2}$$

where $n = \dim M$. Using a fixed metric g, we shall identify covariant vectors with contravariant ones, tensors of type $(0, 2)$ with those of type $(1, 1)$ etc., without any special notation. In particular, a $(0, 2)$ tensor T operates on any tangent vector Y according to the formula

$$(TY)^i = T_k^i Y^k = g^{ij}T_{jk}Y^k. \tag{3}$$

An arbitrary vector field Y satisfies the Ricci identity $\nabla_i\nabla_j Y_k - \nabla_j\nabla_i Y_k = R_{ijkp}Y^p$, which implies the Weitzenböck formula

$$r_{ik}Y^k = \nabla_k\nabla_i Y^k - \nabla_i\nabla_k Y^k. \tag{4}$$

The Ricci identity for any $(0, 2)$ tensor field T can be written as

$$\nabla_h\nabla_j T_{ki} - \nabla_j\nabla_h T_{ki} = R_{hjk}^{p} T_{pi} + R_{hji}^{p} T_{kp}, \tag{5}$$

so that, in view of the first Bianchi identity,

$$\nabla^k\nabla_j T_{ki} - \nabla_j\nabla^k T_{ki} + \nabla_i\nabla^k T_{kj} - \nabla^k\nabla_i T_{kj} = \\ = R_{ijpq}T^{pq} + r_j^p T_{pi} - r_i^p T_{pj}. \tag{6}$$

Denoting by $\Lambda^m = \Lambda^m M$ the bundle of exterior m-forms over (M, g), we use the standard conventions for the inner product and exterior product of forms:

$$\langle \zeta, \eta \rangle = g(\zeta, \eta) = (m!)^{-1}\zeta_{i_1\ldots i_m}\eta^{i_1\ldots i_m},\ (\zeta \wedge \theta)_{i_1\ldots i_{m+l}} = \\ = \sum \operatorname{sign}(j_1, \ldots, j_m, k_1, \ldots, k_l)\zeta_{i_{j_1}\ldots i_{j_m}}\theta_{i_{k_1}\ldots i_{k_l}}$$

for ζ, $\eta \in \Lambda^m$, $\theta \in \Lambda^l$, the latter summation being taken over all permutations $j_1, \ldots, j_m, k_1, \ldots, k_l$ of $\{1, \ldots, m+l\}$ with $j_1 < \ldots < j_m$ and $k_1 < \ldots < k_l$. Thus, for X, $Y \in \Lambda^1$, $\zeta \in \Lambda^2$, we have

$$\langle \zeta, X \wedge Y \rangle = \zeta(X, Y) = \langle X, \zeta Y \rangle = -\langle \zeta X, Y \rangle, \tag{7}$$

where ζX is defined according to (3), i.e., $\zeta X = -\imath_X \zeta$. If the n-dimensional Riemannian manifold (M, g) is oriented, we shall denote by $V \in \Lambda^n$ the corresponding volume form and by dV the measure on M determined by V. The Hodge star operator $* : \Lambda^m \to \Lambda^{n-m}$ is given by $(*\zeta)_{i_{m+1} \ldots i_n} = (m!)^{-1} V_{i_1 \ldots i_n} \zeta^{i_1 \ldots i_m}$, i.e., $*(e_1 \wedge \ldots \wedge e_m) = e_{m+1} \wedge \ldots \wedge e_n$ for any oriented (i.e., compatible with the orientation) orthonormal basis $e_1, \ldots, e_n$ of a tangent space. The composite $** : \Lambda^m \to \Lambda^m$ equals $(-1)^{m(n-m)} id$. The Hodge star is also characterized by $\zeta \wedge \eta = \langle \zeta, *\eta \rangle V$ for $\zeta \in \Lambda^m$, $\eta \in \Lambda^{n-m}$, so that $\langle \zeta, *\eta \rangle = (-1)^{m(n-m)} \langle *\zeta, \eta \rangle$. Thus, in the case where n is a multiple of four, $* : \Lambda^{n/2} \to \Lambda^{n/2}$ is a self-adjoint involution and hence $\Lambda^{n/2}$ splits orthogonally into the Whitney sum $\Lambda^{n/2} = \Lambda^+ + \Lambda^-$, $\Lambda^\pm = \Lambda^\pm(M, g)$ being the (± 1)-eigenspace bundle of $* \in \operatorname{End} \Lambda^{n/2}$. For $x \in M$, any oriented orthonormal basis $e_1, \ldots, e_n$ of $T_x M$ gives rise to a basis $e_1 \wedge e_{i_2} \wedge \ldots \wedge e_{i_{n/2}} \pm *(e_1 \wedge e_{i_2} \wedge \ldots \wedge e_{i_{n/2}})$ of $\Lambda^\pm$, labelled by all sequences of integers $2 \le i_2 < \ldots < i_{n/2} \le n$. Consequently, $\dim \Lambda^\pm = \frac{1}{2}\binom{n}{n/2}$.

The exterior derivative $d\zeta$ of any smooth m-form ζ on a Riemannian manifold (M, g) is given by $(d\zeta)_{i_o \ldots i_m} = \sum_{0 \le s \le m} (-1)^s \nabla_{i_s} \zeta_{i_o \ldots \hat{i_s} \ldots i_m}$. On the other hand, for an arbitrary smooth tensor field T of type $(0, m)$ on M, we define its divergence δT to be the $(0, m-1)$ tensor field with $(\delta T)_{i_2 \ldots i_m} = -\nabla^p T_{p i_2 \ldots i_m}$, ($\delta T = 0$ if $m = 0$), which, in the case where T is an exterior m-form, gives $(-1)^{mn} \delta T = *d*T$. The Laplace operator $\Delta = d\delta + \delta d$ acts on any function f by $\Delta f = -g^{ij} \nabla_i \nabla_j f$.

From Green's formula together with (4) one obtains immediately, for any vector field Y on a *compact* oriented Riemannian manifold (M, g), the well-known equality

$$\int_M r(Y, Y)\, dV = \int_M (\delta Y)^2\, dV - \int_M \operatorname{Trace}(\nabla Y \cdot \nabla Y)\, dV \tag{8}$$

(cf. [5], [21] and [20], pp. 249–251).

For any Riemannian manifold (M, g), the second Bianchi identity easily implies the following divergence formulae: $-\nabla^p R_{pijk} = \nabla_k r_{ij} - \nabla_j r_{ik}$,

$$\delta r = -\tfrac{1}{2} du, \qquad -\nabla^p W_{pkij} = \nabla_j P_{ki} - \nabla_i P_{kj}, \tag{9}$$

where

$$P = (n-3)(n-2)^{-1}(r - (2n-2)^{-1}ug), \qquad n = \dim M. \tag{10}$$

Any $(0,4)$ tensor A on a Riemannian manifold (M,g), having the properties $A_{hijk} = -A_{ihjk} = A_{jkhi}$ can be considered as a self-adjoint endomorphism of $\Lambda^2 M$ by the formula $A(\zeta)_{ij} = \frac{1}{2}A_{ijpq}\zeta^{pq}$. For two such tensors A, C, we set $\langle A, C\rangle = g(A,C) = \frac{1}{4}A_{hijk}C^{hijk}$, so that, if A and C commute (i.e., $AC \in \operatorname{End}\Lambda^2 M$ is self-adjoint), we have $\langle A, C\rangle =$ $=\operatorname{Trace}(AC)$. Examples of $(0,4)$ tensors with these properties are R, W and, if $n = \dim M = 4$ and M is oriented, the volume element V (the corresponding endomorphism being $* \in \operatorname{End}\Lambda^2$). As a trivial consequence of (6) and (2) we obtain, for any 2-form ζ, the formula

$$W(\zeta)_{ij} = \tfrac{1}{2}(\nabla^k\nabla_j\zeta_{ki} - \nabla_j\nabla^k\zeta_{ki} + \nabla_i\nabla^k\zeta_{kj} - \nabla^k\nabla_i\zeta_{kj}) +$$
$$+ (n-1)^{-1}(n-2)^{-1}u\zeta_{ij} + (n-4)(2n-4)^{-1}(r_i^p\zeta_{pj} - r_j^p\zeta_{pi}). \tag{11}$$

Formula (11) is particularly interesting for $n = 4$, since it does not then involve the Ricci tensor explicitly.

In an oriented four-dimensional Riemannian manifold (M,g), the endomorphisms W and $*$ of $\Lambda^2 M$ commute, which follows from the algebraic properties of W ([29], Theorem 1.3). Consequently, W leaves the subbundles $\Lambda^\pm$ invariant. The restrictions $W^\pm$ of W to $\Lambda^\pm$ satisfy the relation

$$\operatorname{Trace} W^\pm = 0 \tag{12}$$

(cf. [29], l.cit.). Whenever convenient, we shall consider $W^\pm$ as $(0,4)$ tensors, i.e., as endomorphisms of $\Lambda^2 M$ with $W^\pm|_{\Lambda^\mp} = 0$. In this sense, $W = W^+ + W^-$. Since $\Lambda^\pm$ are invariant under parallel displacements, we have

$$\delta W^\pm(Y,\cdot,\cdot) \in \Lambda^\pm \tag{13}$$

for any tangent vector Y. Consequently,

$$|\delta W|^2 = |\delta W^+|^2 + |\delta W^-|^2. \tag{14}$$

In the case where the oriented Riemannian four-manifold (M,g) is compact, one can use the Chern–Weil description of characteristic classes ([25], pp. 308, 311) to obtain the relations $8\pi^2\chi(M) =$

$= \int_M \mathrm{Trace}(R * R *) dV$ (cf. [29], p. 359 and [16], p. 48) and $12\pi^2\tau(M) = = \int_M \mathrm{Trace}(RR*) dV = \int_M \langle W, W*\rangle dV$ (cf. [16], p. 48) for the Euler characteristic $\chi(M)$ and the signature $\tau(M)$ of M. Transforming the integrands, one easily obtains

$$12\pi^2\tau(M) = \int_M (|W^+|^2 - |W^-|^2) dV \tag{15}$$

and

$$192\pi^2\chi(M) = 24\int_M (|W^+|^2 + |W^-|^2) dV - \\ - 12\int_M |r - ug/4|^2 dV + \int_M u^2 dV. \tag{16}$$

As an immediate consequence of (15) and (16), we have the Thorpe inequality $3|\tau(M)| \le 2\chi(M)$, valid for any compact orientable Einstein manifold of dimension four. It is also immediate that, for such a manifold, the strict inequality

$$3|\tau(M)| < 2\chi(M) \tag{17}$$

holds unless $r = 0$ and $W^+ = 0$ for some orientation (cf. [31], [17] and [16]).

Consider now a conformal change $\bar{g} = e^\sigma g$ of the Riemannian metric g on an n-dimensional manifold M, σ being a smooth function on M. The Weyl tensor W of type $(1,3)$ is conformally invariant, so that, for the Weyl tensor of type $(2,2)$, $\bar{W} = e^{-\sigma} W \in \mathrm{End}\,\Lambda^2 M$ and

$$\bar{g}(\bar{W}, \bar{W}) = e^{-2\sigma} g(W, W), \tag{18}$$

while the volume element V (defined if M is oriented) and the divergence δW of W, viewed as tensors of types $(0,n)$ and $(0,3)$, respectively, transform like $\bar{\delta}\bar{W} = \delta W - \frac{1}{2}(n-3)W(\nabla\sigma, \cdot, \cdot, \cdot)$ and $\bar{V} = e^{n\sigma/2} V$, which implies that, for $n \equiv 0 \pmod 4$, $\bar{*} = * \in \mathrm{End}\,\Lambda^{n/2} M$ and $\Lambda^\pm(M, \bar{g}) = \Lambda^\pm(M, g)$. Therefore, if $n = 4$ and M is oriented, (13) yields

$$\bar{\delta}\bar{W}^\pm = \delta W^\pm - \tfrac{1}{2} W^\pm(\nabla\sigma, \cdot, \cdot, \cdot). \tag{19}$$

For two conformally related metrics g and $\bar{g} = e^\sigma g$, a tangent vector Y and a smooth 2-form ω, we have

$$\bar{\nabla}_Y \omega = \nabla_Y \omega - d\sigma(Y)\omega + \tfrac{1}{2}\imath_Y\omega \wedge d\sigma + \tfrac{1}{2} Y \wedge \imath_{\nabla\sigma}\omega, \tag{20}$$

the vector Y in the last term being viewed as a 1-form with the aid of g. On the other hand, for the conformal change written as $\bar{g} = h^{-2}g$, h a non-zero function on the n-dimensional manifold M, the transformation rules for the Ricci tensor and for the scalar curvature are

$$\begin{aligned} \bar{r} &= r + (n-2)h^{-1}\nabla\, dh - h^{-2}(h\cdot \Delta h + (n-1)|\nabla h|^2)g, \\ \bar{u} &= uh^2 - 2(n-1)h\Delta h - n(n-1)|\nabla h|^2. \end{aligned} \tag{21}$$

Suppose now that M is a compact oriented four-dimensional manifold. For any Riemannian metric g on M, the 4-form $g(W, W)V$ is invariant under conformal changes of g. Therefore, the formula

$$g \to \int_M |W|^2\, dV \tag{22}$$

defines a conformally invariant functional in the space of all Riemannian metrics on M. It is easy to verify that the critical points of (22) are characterized as follows.

LEMMA 1 (R. Bach, [3]): *A metric g on a compact oriented four-manifold M is a critical point of* (22) *if and only if its Bach tensor B, given by the local coordinate formula*

$$B_{ij} = \nabla^p\nabla^q W_{pijq} + \tfrac{1}{2}r^{pq}W_{pijq} \tag{23}$$

vanishes identically.

REMARK 1: From the conformal invariance of (22) it is clear that condition $B = 0$ should be preserved by conformal changes of metrics (in dimension four); in fact, $\bar{B} = e^{-\sigma}B$ when $\bar{g} = e^{\sigma}g$. On the other hand, (9) implies $B = 0$ for any Einstein metric; thus, $B = 0$ whenever the metric is locally conformally Einsteinian.

The Bach tensor of a metric g ($n = 4$) can also be written as

$$\begin{aligned} B_{kj} = {} & \nabla^p\nabla_j r_{kp} - \tfrac{1}{2}\nabla^p\nabla_p r_{kj} - \tfrac{1}{3}\nabla_k\nabla_j u - \tfrac{1}{12}\Delta u\cdot g_{kj} + \\ & + \tfrac{1}{3}ur_{kj} - r_k^p r_{pj} + \tfrac{1}{12}(3|r|^2 - u^2)g_{kj}, \end{aligned} \tag{24}$$

where $|r|^2 = r_{ij}r^{ij} = \mathrm{Trace}(rr)$. In fact, (9) yields

$$12\nabla^p\nabla^q W_{pkjq} = 6\nabla^p\nabla_k r_{jp} - 6\nabla^p\nabla_p r_{kj} - \nabla_k\nabla_j u - \Delta u\cdot g_{kj}. \tag{25}$$

On the other hand, using (9) and the Ricci identity (5) with $T = r$, we obtain $\nabla^p\nabla_k r_{jp} - \frac{1}{2}\nabla_j\nabla_k u = \nabla^p\nabla_k r_{jp} - \nabla_k\nabla^p r_{jp} = r^{pq}R_{pkjq} + r_j^p r_{pk}$. Consequently,

$$\nabla^p\nabla_k r_{jp} = \nabla^p\nabla_j r_{kp} \tag{26}$$

and, by (2), $6r^{pq}W_{pkjq} = 6\nabla^p\nabla_k r_{jp} - 3\nabla_k\nabla_j u + 4ur_{kj} - 12r_k^p r_{pj} + (3|r|^2 - u^2)g_{kj}$. Formula (24) is now immediate from (25) and (26).

Let (M, g) be an oriented Riemannian four-manifold. For $x \in M$, denote by P_x the set of all oriented orthonormal bases of T_xM. To each $e = (e_1, e_2, e_3, e_4) \in P_x$, we can assign the orthogonal basis $F_x^{\pm}(e)$ of $\Lambda_x^{\pm}$, formed by elements

$$\mp e_1 \wedge e_2 - e_3 \wedge e_4, \quad \mp e_1 \wedge e_3 - e_4 \wedge e_2, \quad \mp e_1 \wedge e_4 - e_2 \wedge e_3 \tag{27}$$

of length $\sqrt{2}$. Since the assignment $F_x^{\pm}$ is continuous, each of $\Lambda^{\pm}$ has a preferred orientation. Let Q_x be the set of all orthogonal bases of $\Lambda^2 T_xM$, consisting of an oriented basis of Λ_x^+ and of an oriented basis of Λ_x^-, with all vectors of length $\sqrt{2}$. The group $SO(3) \times SO(3)$ acts on Q_x freely and transitively and it is easy to verify that the map $F_x = (F_x^+, F_x^-): P_x \to Q_x$ is equivariant with respect to the two-fold covering homomorphism $SO(4) \to SO(3) \times SO(3)$. Thus, F_x is a two-fold covering. Consequently, we obtain

LEMMA 2 (cf. [2]): *Suppose that (M, g) is an oriented Riemannian four-manifold and $x \in M$. Then, every pair of oriented orthogonal bases, one of Λ_x^+ and one of Λ_x^-, consisting of vectors of length $\sqrt{2}$, is of the form* (27) *for precisely two oriented orthonormal bases of T_xM, and*

(i) *Λ_x^+ and Λ_x^- are mutually commuting ideals, both isomorphic to* $\mathfrak{so}(3)$, *in the Lie algebra $\Lambda^2 T_xM \simeq \mathfrak{so}(4)$ of skew-adjoint endomorphisms of T_xM.*

(ii) *Elements of length $\sqrt{2}$ in Λ_x^+ or in Λ_x^- coincide with the almost complex structures in T_xM, compatible with the metric.*

(iii) *Every oriented orthogonal basis ω, η, θ of $\Lambda_x^{\pm}$ such that $|\omega| = |\eta| = |\theta| = \sqrt{2}$, satisfies the conditions $\omega^2 = \eta^2 = \theta^2 = -id$, $\omega\eta = \theta = -\eta\omega$, so that it forms a quaternionic structure in T_xM.*

(iv) *Given elements $\omega \in \Lambda_x^+$, $\omega^- \in \Lambda_x^-$ of length $\sqrt{2}$, their composite $\omega\omega^- = \omega^-\omega$ is a self-adjoint, orientation preserving involution of T_xM, distinct from $\pm id$, whence its (± 1)-eigenspaces form an orthogonal decomposition of T_xM into a direct sum of two planes.*

Let us again consider an oriented Riemannian four-manifold (M, g). For $x \in M$, we can choose an oriented orthogonal basis ω, η, θ (resp., ω^-, η^-, θ^-) of Λ_x^+ (resp., Λ_x^-), consisting of eigenvectors of W such that

$$|\omega| = |\eta| = |\theta| = \sqrt{2}, \ |\omega^-| = |\eta^-| = |\theta^-| = \sqrt{2}. \tag{28}$$

Consequently, we have, at x, a relation of the form

$$\begin{aligned} W &= \tfrac{1}{2}(\lambda\omega\otimes\omega + \mu\eta\otimes\eta + \nu\theta\otimes\theta) + \\ &+ \tfrac{1}{2}(\lambda^-\omega^-\otimes\omega^- + \mu^-\eta^-\otimes\eta^- + \nu^-\theta^-\otimes\theta^-). \end{aligned} \tag{29}$$

λ, μ, ν (resp., λ^-, μ^-, ν^-) being the eigenvalues of W_x^+ (resp., W_x^-). Thus, $|W^+|^2 = \lambda^2 + \mu^2 + \nu^2$ and, in view of (12), we have

$$\lambda + \mu + \nu = 0, \ \lambda^- + \mu^- + \nu^- = 0. \tag{30}$$

An immediate consequence of (29) together with Lemma 2 is the well-known formula $W_i^{kpq} W_{jkpq} = |W|^2 g_{ij}$, valid for any Riemannian four-manifold; similarly, (29) yields

$$(W^\pm)^{ikpq}(W^\pm)_{jkpq} = |W^\pm|^2 \delta_j^i. \tag{31}$$

Let M_W be the open dense subset of M, consisting of points at which the number of distinct eigenvalues of W is locally constant. In M_W, the pointwise formula (29) is valid *locally* in the sense that the mutually orthogonal sections ω, η, θ of Λ^+, ω^-, η^-, θ^- of Λ^- and the functions $\lambda, \mu, \ldots, \nu^-$, satisfying (28)–(30), may be assumed differentiable in a neighbourhood of any point of M_W. Since $\Lambda^\pm$ are invariant under parallel displacements, in a neighbourhood of any $x \in M_W$ we have (29) and

$$\nabla\omega = c\otimes\eta - b\otimes\theta, \ \nabla\eta = -c\otimes\omega + a\otimes\theta, \ \nabla\theta = b\otimes\omega - a\otimes\eta \tag{32}$$

for some 1-forms a, b, c defined near x. (Clearly, similar formulae hold for $\nabla\omega^-, \ldots, \nabla\theta^-$). Note that (32) and (29) remain valid after any simultaneous cyclic permutation of the three ordered triples (ω, η, θ), (λ, μ, ν), (a, b, c), and, therefore, this invariance will hold for all consequences of (32). The Ricci identity (5), applied to $T = \omega$, yields, in view of (32), (2), (29) and Lemma 2,

$$\begin{aligned} &2(\nabla_j c_i - \nabla_i c_j + b_i a_j - b_j a_i)\eta_{kl} + 2(\nabla_i b_j - \nabla_j b_i + \\ &+ c_i a_j - c_j a_i)\theta_{kl} = -2\mu\eta_{ji}\theta_{kl} + 2\nu\theta_{ji}\eta_{kl} + g_{il} r_j^p \omega_{pk} - \end{aligned}$$

$$- g_{ik}r_j^p\omega_{pl} + r_{il}\omega_{jk} - r_{ik}\omega_{jl} + g_{jk}r_i^p\omega_{pl} - g_{jl}r_i^p\omega_{pk} +$$
$$+ r_{jk}\omega_{il} - r_{jl}\omega_{ik} + \tfrac{1}{3}u(g_{ik}\omega_{jl} - g_{il}\omega_{jk} + g_{jl}\omega_{ik} - g_{jk}\omega_{il}).$$

Transvecting this equality with θ^{kl} and applying the cyclic permutations mentioned above, we obtain

$$\begin{aligned} da + b \wedge c &= (\lambda - u/6)\omega + \tfrac{1}{2}(r\omega + \omega r), \\ db + c \wedge a &= (\mu - u/6)\eta + \tfrac{1}{2}(r\eta + \eta r), \\ dc + a \wedge b &= (\nu - u/6)\theta + \tfrac{1}{2}(r\theta + \theta r). \end{aligned} \tag{33}$$

Using now (29), (32) and (iii) of Lemma 2, we obtain

$$\begin{aligned} 2\nabla W^+ &= (d\lambda\otimes\omega + (\lambda-\mu)c\otimes\eta + (\nu-\lambda)b\otimes\theta)\otimes\omega + \\ &+ (d\mu\otimes\eta + (\lambda-\mu)c\otimes\omega + (\mu-\nu)a\otimes\theta)\otimes\eta + \\ &+ (d\nu\otimes\theta + (\nu-\lambda)b\otimes\omega + (\mu-\nu)a\otimes\eta)\otimes\theta \text{ and} \\ 2\delta W^+ &= \omega(d\lambda - (\lambda-\mu)\theta c - (\lambda-\nu)\eta b)\otimes\omega + \\ &+ \eta(d\mu - (\mu-\lambda)\theta c - (\mu-\nu)\omega a)\otimes\eta + \\ &+ \theta(d\nu - (\nu-\lambda)\eta b - (\nu-\mu)\omega a)\otimes\theta. \end{aligned}$$

Consequently, the oriented four-manifold (M, g) satisfies the condition $\delta W^+ = 0$ if and only if, in the above notations, relations

$$\begin{aligned} d\lambda &= (\lambda - \mu)\theta c + (\lambda - \nu)\eta b, \\ d\mu &= (\mu - \lambda)\theta c + (\mu - \nu)\omega a, \\ d\nu &= (\nu - \lambda)\eta b + (\nu - \mu)\omega a \end{aligned} \tag{34}$$

hold (locally) in M_W. For any 2-form ζ and a 1-form Y, we have $\delta(\zeta Y) = \langle\delta\zeta, Y\rangle - \langle\zeta, dY\rangle$. On the other hand, (7) and (33) imply $\langle\eta, db\rangle = \langle a, \eta c\rangle + 2\mu + u/6$ and $\langle\theta, dc\rangle = -\langle a, \theta b\rangle + 2\nu + u/6$, while (32) yields $\delta\eta = -\omega c + \theta a$ and $\delta\theta = \omega b - \eta a$. Therefore, $\delta(\theta c) = -\langle b, \omega c\rangle + \langle a, \eta c\rangle + \langle a, \theta b\rangle - 2\nu - u/6$ and $\delta(\eta b) = -\langle b, \omega c\rangle - \langle a, \eta c\rangle - \langle a, \theta b\rangle - 2\mu - u/6$. Consequently, for any oriented Riemannian four-manifold such that $\delta W^+ = 0$, we have, from (34), the following expression for $\Delta\lambda = \delta\, d\lambda$:

$$\Delta\lambda = 2\lambda^2 + 4\mu\nu - \lambda u/2 + 2(\nu - \lambda)|b|^2 + 2(\mu - \lambda)|c|^2. \tag{35}$$

3. Kähler manifolds of real dimension four and conformal deformations

By a *Kähler form* in a Riemannian manifold (M, g) we shall mean a parallel 2-form ω on M such that the corresponding section of End TM is an almost complex structure on M. The triple (M, g, ω) will then be called a *Kähler manifold*. For Kähler manifolds of any dimension $n \geq 4$, we have the following result, proved by Y. Matsushima [24] in the compact case. The local argument given below is due to S. Tanno [30].

PROPOSITION 1 (Matsushima, Tanno): *Let (M, g, ω) be a Kähler manifold of dimension $n \geq 4$. If the divergence δW of the Weyl tensor vanishes identically, then the Ricci tensor r is parallel.*

PROOF: Formula (9) implies the Codazzi equation $\nabla_j P_{ki} = \nabla_i P_{kj}$, P being given by (10). Since P commutes with the Kähler form ω, the local coordinate formula $T_{ij} = P_{iq}\omega_j^q$ defines an exterior 2-form T. The expression $\nabla_k T_{ij} = (\nabla_k P_{iq})\omega_j^q$ is now symmetric in k, i and skew-symmetric in i, j, and so it must be zero. Thus, $\nabla P = 0$ and, by (9), $0 = 2(n-2)(n-3)^{-1}\delta P = (2-n)(n-1)^{-1}\nabla u$, so that $\nabla r = 0$, which completes the proof.

Suppose now that (M, g, ω) is a Kähler manifold of real dimension four. For the natural orientation, ω is a section of Λ^+. At any point $x \in M$, ω_x can be completed to an orthogonal basis ω_x, η, θ of Λ_x^+ with $|\eta| = |\theta| = \sqrt{2}$. It is clear that the tensor $C_x = \eta \otimes \eta + \theta \otimes \theta$ does not depend on the choice of η and θ, and that the tensor field C on M obtained in this way is parallel. We can now define the non-trivial parallel tensor field

$$A = \tfrac{1}{12}\omega \otimes \omega - \tfrac{1}{24}C. \tag{36}$$

Viewed as an endomorphism of $\Lambda^2 M$, A is given by $\zeta \to \langle \zeta^+, \omega \rangle \omega/8 - \zeta^+/12$, ζ^+ being the Λ^+ component of $\zeta \in \Lambda^2 M$.

The following proposition is well-known (cf. [16], [27]).

PROPOSITION 2: *Let (M, g, ω) be a Kähler manifold of real dimension four, oriented in the natural way. Considering W^+ as an endomorphism of $\Lambda^2 M$, trivial on $\Lambda^- M$, we have*

$$W^+ = u \cdot A,$$

where u is the scalar curvature and A denotes the non-trivial parallel tensor field given by (36). *Moreover,*

(i) $\#\mathrm{spec}_{\Lambda^+}(W^+) \leq 2$, *i.e., the endomorphism* W^+ *of* Λ^+ *has, at each point, less than three distinct eigenvalues*;

(ii) $u^2 = 24|W^+|^2$;

(iii) $u\delta W^+ + W^+(\nabla u, \cdot, \cdot, \cdot) = 0$.

PROOF: Since $W(\omega) = u\omega/6$ by (11), we can find, locally in M_W (notation of Section 2), 2-forms $\eta, \theta, \omega^-, \eta^-, \theta^-$ such that ω, η, θ (resp., $\omega^-, \eta^-, \theta^-$) form an oriented orthogonal frame field for Λ^+ (resp., for Λ^-) and formulae (28)–(30) hold for certain functions $\lambda, \mu, \ldots, \nu^-$. Thus, $\lambda = u/6$. Now, as $\nabla\omega = 0$, we have $b = c = 0$ in (32), so that (33) yields

$$r\eta + \eta r = (u/3 - 2\mu)\eta,\ r\theta + \theta r = (u/3 - 2\nu)\theta. \tag{37}$$

Fix $x \in M_W$. Relation $r\omega = \omega r$, valid for any Kähler manifold, implies that r_x has two double eigenvalues α, β (which may coincide). Given $Y \in T_xM$ with $Y \neq 0$ and $r_x(Y) = \alpha Y$, both Y and ωY lie in the α-eigenspace of r_x and so $\eta Y, \theta Y$ lie in the β-eigenspace. Evaluating (37) on Y, we obtain $u/3 - 2\mu = u/3 - 2\nu = \alpha + \beta = u/2$, so that $\mu = \nu = -u/12$. Thus, $2W^+ = \lambda\omega\otimes\omega + \mu\eta\otimes\eta + \nu\theta\otimes\theta = 2uA$. Assertions (i)–(iii) are now immediate, which completes the proof.

The following lemma is a direct consequence of (15) and (16) together with (ii) of Proposition 2. It implies that CP^2 does not admit a non-standard self-dual Kähler metric, which is crucial for the proof of Theorem 1 in Section 4. The author is obliged to Jean Pierre Bourguignon for bringing this lemma to his attention.

LEMMA 3: *Every compact Kähler manifold of real dimension four, endowed with the natural orientation, satisfies the equality* $16\pi^2(3\tau(M) - \chi(M)) = \int_M |r - ug/4|^2\, dV - 6\int_M |W^-|^2\, dV$.

REMARK: Lemma 3 implies the inequality $3\tau(M) \leq \chi(M)$ for any compact Kähler–Einstein four-manifold, due to H. Guggenheimer (cf. [14], [33]).

From Proposition 2 it follows that, for any four-dimensional Kähler metric g, the conformally related metric $u^{-2}g$ has some interesting properties. Namely, we have

PROPOSITION 3: *Suppose that* (M, g, ω) *is a Kähler manifold of real dimension four, oriented in the natural way, and let* M_{W^+} *be the open set of points at which* $W^+ \neq 0$ (*i.e.,* $u \neq 0$).

(i) *If a metric of the form $\bar{g} = e^{\sigma}g$, defined in an open connected subset of M_{W^+}, satisfies $\bar{\delta}\bar{W}^+ = 0$, then the function e^{σ} is a constant multiple of u^{-2}.*

(ii) *The metric $\bar{g} = u^{-2}g$, defined in M_{W^+}, has the following properties: $\bar{\delta}\bar{W}^+ = 0$, $\#\mathrm{spec}(\bar{W}^+) \leq 2$ (i.e., $\bar{W}^+ \in \mathrm{End}\,\Lambda^+$ has at most two distinct eigenvalues at each point) and $\bar{g}(\bar{W}^+, \bar{W}^+) = u^6/24$, so that, in terms of $\bar{g}$, g is given by $g = (24\bar{g}(\bar{W}^+, \bar{W}^+))^{1/3}\bar{g}$.*

PROOF: Assertion (ii) is immediate from Proposition 2 together with (19) and (18). On the other hand, if two metrics, $\tilde{g}$ and $\bar{g} = e^{\sigma}\tilde{g}$, satisfy $\tilde{\delta}\tilde{W}^+ = \bar{\delta}\bar{W}^+ = 0$, then, by (19) and (31), $\nabla\sigma = 0$ wherever $\bar{W}^+ \neq 0$, which completes the proof.

REMARK 2: Every conformally flat Kähler manifold of dimension greater than four is flat ([32]; this follows also easily from Proposition 1). On the other hand, the Riemannian product of two surfaces with mutually opposite constant curvatures is conformally flat and Kählerian. It is well-known that, up to local isometries, these products are the only conformally flat Kähler four-manifolds, which also follows immediately from Propositions 1 and 2.

REMARK 3: Given a Kähler form ω in a Riemannian four-manifold (M, g), $-\omega$ is another Kähler form, corresponding to the complex conjugate of the original complex structure of M. However, there are no more Kähler structures in (M, g), unless $r = 0$ and $W^+ = 0$, or (M, g) is locally a product of surfaces. This (well-known) statement can be verified as follows. Let η be a parallel 2-form, not collinear to ω. We may assume $\eta \in \Lambda^+$ or $\eta \in \Lambda^-$ and $|\eta| = \sqrt{2}$, $\langle \omega, \eta \rangle = 0$. If $\eta \in \Lambda^+$, then Λ^+ admits, locally, orthogonal parallel sections ω, η, θ with $|\theta| = \sqrt{2}$, which implies $r = 0$ and $W^+ = 0$ (for instance, (11) yields $6W^+ = u \cdot id_{\Lambda^+}$, so that $u = 0$ and $W^+ = 0$ by (12), while $r\omega = \omega r$ and (33) with $\lambda = \mu = \nu = 0$ and $a = b = c = 0$ give $r = 0$). In the case where $\eta \in \Lambda^-$, (iv) of Lemma 2 implies the existence of a parallel plane field in M, so that g is locally reducible.

According to Proposition 3, every Kähler metric (in dimension four) with $W^+ \neq 0$ is conformal, in an essentially unique way, to a metric such that $\delta W^+ = 0$. The latter relation is satisfied, e.g., by every Einstein metric (in view of (9) and (14)). We are now going to characterize, in the generic (four-dimensional) case, those Kähler metrics which are locally conformally Einsteinian. By Remark 1, a necessary condition is that the Bach tensor of such a metric is zero; therefore, we start from studying Kähler metrics with $B = 0$.

LEMMA 4: *Let (M, g, ω) be a Kähler manifold.*
(i) *If* $\dim M = 4$, *then*

$$\tfrac{1}{3}u\cdot r - rr + \tfrac{1}{12}(3|r|^2 - u^2)g = -\tfrac{1}{6}u\left(r - \frac{u}{4}g\right).$$

(ii) *Denoting by $\rho = r\omega = \omega r$ the Ricci form, we have*

$$\nabla^p\nabla_j\rho_{ip} - \tfrac{1}{2}\nabla^p\nabla_p\rho_{ij} = \tfrac{1}{4}(\omega_{ip}\nabla^p\nabla_j u + \omega_{jp}\nabla^p\nabla_i u).$$

PROOF: To prove (i) one can proceed by a direct calculation, using a diagonal matrix representation for r and the fact that r has two double eigenvalues. As for (ii), relation $d\rho = 0$, satisfied by every Kähler metric, yields $\nabla_p\rho_{ij} = \nabla_j\rho_{ip} - \nabla_i\rho_{jp}$, so that $\nabla^p\nabla_p\rho_{ij} = \nabla^p\nabla_j\rho_{ip} - \nabla^p\nabla_i\rho_{jp}$ and $\nabla^p\nabla_j\rho_{ip} - \frac{1}{2}\nabla^p\nabla_p\rho_{ij} = \frac{1}{2}(\nabla^p\nabla_j\rho_{ip} + \nabla^p\nabla_i\rho_{jp})$, the last term being just $\nabla^p\nabla_i\rho_{jp}$ symmetrized with respect to i, j. The Ricci identity (5) with $T = \rho$ implies $\nabla^p\nabla_i\rho_{jp} = \nabla_i\nabla^p\rho_{jp} + R_{pijq}\rho^{qp} + r_i^q\rho_{jq}$. The last two terms of this expression being skew-symmetric in i, j (as $r\rho = \rho r$), we obtain, by its symmetrization, $\nabla^p\nabla_j\rho_{ip} + \nabla^p\nabla_i\rho_{jp} = \nabla_i\nabla^p\rho_{jp} + \nabla_j\nabla^p\rho_{ip}$. From (9), we have $2\delta\rho = \omega(\nabla u)$, so that $2\nabla_i\nabla^p\rho_{jp} = \omega_{jp}\nabla^p\nabla_i u$, which completes the proof.

LEMMA 5: *For any Kähler four-manifold (M, g, ω), the following two conditions are equivalent*:
(i) *The Bach tensor B is zero*;
(ii) $2\nabla^2 u + u\cdot r = (u^2/4 - \Delta u/2)g$.

PROOF: Formula (24) yields $12\omega_{ik}B_j^k = 12\nabla^p\nabla_j\rho_{ip} - 6\nabla^p\nabla_p\rho_{ij} - 4\omega_{ip}\nabla^p\nabla_j u - \Delta u\cdot\omega_{ij} + \omega_{ik}(4ur_j^k - 12r^{kp}r_{pj} + (3|r|^2 - u^2)\delta_j^k)$, so that, by Lemma 4 together with the relation $\rho = r\omega = \omega r$, we have

$$-12\omega B = \omega\nabla^2 u + 3(\nabla^2 u)\omega + \Delta u\cdot\omega + 2u(\rho - u\omega/4), \tag{38}$$

i.e.,

$$-12\omega B = 2\omega(2\nabla^2 u + ur - (u^2/4 - \Delta u/2)g) + 3((\nabla^2 u)\omega - \omega\nabla^2 u). \tag{39}$$

Now, if (ii) holds, then $\nabla^2 u$ commutes with ω, since so does r, and $B = 0$ in view of (39). Conversely, if $B = 0$, then, by (38), ω commutes with $\omega\nabla^2 u + 3(\nabla^2 u)\omega$, i.e., $0 = \omega(\omega(\omega\nabla^2 u \to 3(\nabla^2 u)\omega) - (\omega\nabla^2 u \to 3(\nabla^2 u)\omega)\omega) = 2(\omega\nabla^2 u - (\nabla^2 u)\omega)$. Equality (39) implies now (ii), which completes the proof.

We can now obtain the following characterization of Kähler four-manifolds which are locally conformally Einsteinian, valid wherever $W^+ \neq 0$ (i.e., $u \neq 0$).

PROPOSITION 4: *Let (M, g, ω) be a Kähler manifold of real dimension four, oriented in the natural way. The following four conditions are equivalent*:

(i) *The Riemannian metric $u^{-2}g$, defined wherever $W^+ \neq 0$, is Einsteinian*;

(ii) *In the open set where $W^+ \neq 0$, g is locally conformal to an Einstein metric*;

(iii) *The Bach tensor B vanishes identically*;

(iv) $2\nabla^2 u + u\cdot r = (u^2/4 - \Delta u/2)g$.

Moreover, either of (i)–(iv) *implies the condition*

(v) *$\omega(\nabla u)$ is a holomorphic Killing vector field and*

$$u^3 - 6u\cdot\Delta u - 12|\nabla u|^2 = \kappa \tag{40}$$

for some real constant κ.

On the other hand, if u is not constant, then (v) *implies* (i)–(iv).

PROOF: Implications (i)$\to$(ii)$\to$(iii)$\to$(iv) are immediate from Remark 1 and Lemma 5. On the other hand, in view of (21), (iv) implies (i). Assuming (iv) again, we see (cf. (21)) that the left-hand side of (40) is nothing but the constant scalar curvature of the Einstein metric $u^{-2}g$. Relation $\omega\nabla^2 u = (\nabla^2 u)\omega$, immediate from (iv), means just that the tensor field $\nabla(\omega(\nabla u))$ is skew-adjoint, i.e., $X = \omega(\nabla u)$ is a Killing field. It is holomorphic since $L_X\omega = \nabla_X\omega + \omega(\nabla X)^* + (\nabla X)\omega$ (by our convention (3), $\nabla_Y X = (\nabla X)^* Y$ for any X, Y) and the tensor field $\nabla X = -(\nabla X)^* = -\omega\nabla^2 u$ commutes with ω, which proves (v). On the other hand, assuming (v), we have, by (4),

$$r(\nabla u) = \nabla\Delta u - \delta\nabla^2 u \tag{41}$$

and, for the Killing field $X = \omega(\nabla u)$, $rX = \delta\nabla X$. Therefore $r(\nabla u) = -$ $-r\omega(X) = -\omega r X = -\omega(\delta\nabla X) = -\delta\nabla\omega(X) = \delta\nabla^2 u$, which, together with (41), yields $2\delta\nabla^2 u = \nabla\Delta u$ and $r(\nabla u) = \nabla\Delta u/2$. The symmetric tensor field $T = 2\nabla^2 u + u\cdot r - (u^2/4 - \Delta u/2)g$ must, therefore, satisfy the relation $-12T(\nabla u) = -12\nabla(|\nabla u|^2) - 6u\nabla\Delta u + 3u^2\cdot\nabla u - 6\Delta u\cdot\nabla u =$ $= \nabla(u^3 - 6u\Delta u - 12|\nabla u|^2) = 0$, i.e., $T(\nabla u) = 0$. If u is not constant, then the holomorphic field ∇u is non-zero on a dense subset of M, so that T is singular everywhere. However, Trace $T = 0$ and T commutes with ω, since so does $\nabla^2 u$ by our hypothesis. Thus, the singular tensor T has, at any point, two mutually opposite double eigenvalues, which implies $T = 0$. This completes the proof.

The following local result is a converse of Proposition 3.

PROPOSITION 5: *Let (M, g) be an oriented Riemannian four-manifold such that $\delta W^+ = 0$ and $\#\mathrm{spec}(W^+) \le 2$, i.e., W^+, operating on Λ^+, has less than three distinct eigenvalues at each point. We have*

(i) *Denote by M_{W^+} the open set of points with $W^+ \neq 0$. The metric $\bar{g} = (24g(W^+, W^+))^{1/3} g$, defined in M_{W^+}, is locally Kählerian in a manner compatible with the orientation. The (local) Kähler form $\bar{\omega}$ for $\bar{g}$ can be defined by*

$$\bar{\omega} = (24g(W^+, W^+))^{1/3}\omega, \tag{42}$$

where, for $x \in M_{W^+}$, $\omega_x \in \Lambda_x^+$ is an eigenvector of W_x^+ corresponding to the (unique) simple eigenvalue and normed by $g(\omega_x, \omega_x) = 2$. This way, $\bar{\omega}$ is defined, up to a sign, at each point of M_{W^+}. The scalar curvature $\bar{u}$ is characterized by

$$|\bar{u}| = (24g(W^+, W^+))^{1/6} \tag{43}$$

and $\mathrm{sign}(\bar{u}) = \mathrm{sign}(\det W^+)$ *(i.e., the sign of the simple eigenvalue of W^+ in Λ^+); in other words, $\bar{u} = 2^{5/9} \cdot 3^{1/3} (\det W^+)^{1/9}$, and $g = \bar{g}/\bar{u}^2$.*

(ii) *The conformal deformation of g into a Kähler metric is essentially unique, i.e., if $\bar{g} = e^\sigma g$ is a metric on an open connected subset of M_{W^+}, admitting a Kähler form in Λ^+, then e^σ is a constant multiple of $(g(W^+, W^+))^{1/3}$.*

(iii) *$g(W^+, W^+)$ is constant if and only if $\nabla W^+ = 0$.*

(iv) *If, moreover, g is an Einstein metric, then either $W^+ = 0$ identically, or $W^+ \neq 0$ at every point of M.*

PROOF: Locally, in M_{W^+}, we have relations of the form (28)–(30), (32), where $\omega, \lambda, \ldots$ etc. are differentiable and ω, η, θ (resp., $\omega^-, \eta^-, \theta^-$) is an oriented orthogonal frame field for Λ^+ (resp., for Λ^-). Since $\#\mathrm{spec}(W^+) \le 2$, we may set, without loss of generality, $6\lambda = -12\mu = -12\nu = \xi$ for some function ξ. Note that $\xi^3 = 2^5 \cdot 3^3 \det W^+$, so that ξ^3 is well-defined and smooth everywhere in M. We have, clearly, $24g(W^+, W^+) = \xi^2$ and $\bar{g} = \xi^{2/3} g$. Since $\delta W^+ = 0$, formula (34) yields

$$\nabla \xi = 3\xi\theta c = 3\xi\eta b. \tag{44}$$

We assert that, for any tangent vector Y,

$$3\xi\nabla_Y\omega + \nabla\xi \wedge \omega Y + \omega(\nabla\xi) \wedge Y = 0. \tag{45}$$

In fact, let us denote by α the 2-form occurring on the left-hand side of (45). For any $\zeta \in \Lambda^-$, we have $\langle\zeta, \nabla_Y\omega\rangle = 0$ and, by (i) of Lemma 2, $\zeta\omega = \omega\zeta$. Hence, by (7), $\langle\zeta,\alpha\rangle = \langle\nabla\xi, \zeta\omega Y - \omega\zeta Y\rangle = 0$. Consequently, $\alpha \in \Lambda^+$. On the other hand, relations $\eta\omega = -\theta$, $\theta\omega = \eta$ and $\omega^2 = -id$ together with (32) and (44) imply $\langle\eta,\alpha\rangle = \langle Y, 6\xi c + 2\theta(\nabla\xi)\rangle = 0$ and, similarly, $\langle\theta,\alpha\rangle = \langle\omega,\alpha\rangle = 0$, which proves (45). However (cf. (20)), (45) means just that $\bar\nabla\bar\omega = 0$, where $\bar\omega$ is given by (42) (i.e., $\bar\omega = \xi^{2/3}\omega$) and $\bar\nabla$ is the Riemannian connection of $\bar g = \xi^{2/3}g$. Thus, since $\bar\omega \in \Lambda^+$ and $\bar g(\bar\omega,\bar\omega) = 2$, $\bar\omega$ is a Kähler form for $\bar g$, compatible with the prescribed orientation. By Proposition 2, the simple eigenvalue of $\bar W^+$ in Λ^+ is $\bar u/6$, while that of W^+ is $\xi/6$. The relation $\bar W^+ = \xi^{-2/3}W^+ \in \operatorname{End}\Lambda^+$ yields now $\bar u = \xi^{1/3}$, which implies (i). The uniqueness assertion of (ii) follows easily from (i) of Proposition 3 together with Proposition 2. To prove (iii) note that, if $g(W^+, W^+)$ is constant, then, by (i), g is a Kähler metric with constant scalar curvature and $\nabla W^+ = 0$ in view of Proposition 2. Suppose now that g is an Einstein metric. We may assume that W^+ is not parallel. By (iii) and (i), $\bar g$ is a locally conformally Einsteinian Kähler metric with non-constant scalar curvature $\bar u$. In view of Proposition 4, $X = \bar\omega(\bar\nabla\bar u)$ is a non-trivial Killing vector field for $\bar g$ and hence also for $g = \bar g/\bar u^2$. In terms of g, X is given by

$$X = -\omega(\nabla(\xi^{-1/3})). \tag{46}$$

As shown by DeTurck and Kazdan [13], every Einstein metric is analytic in suitable coordinate systems. Formula (46) defines, up to a sign, a Killing field X in an open subset of M, which gives rise to a Killing field defined in an open connected subset of the Riemannian universal covering space $(\tilde M, g)$ of (M, g). Using the standard procedure of extending germs of local isometries along geodesics in analytic manifolds, we see that the latter field must have an extension to a Killing field $\tilde X$ on the whole of $\tilde M$. Since the function ξ^3, pulled back to $\tilde M$, is well-defined and analytic everywhere, relation $|\tilde X| = |\nabla\xi^{-1/3}|$, valid in some open subset by (46), must hold at all points of $\tilde M$ where it makes sense. Consequently, the function $\phi = \xi^{-1/3}$ is defined and analytic almost everywhere in $\tilde M$, with a possible exception of certain points at which $|\phi| \to \infty$, and its gradient $\nabla\phi$ is bounded on compact subsets of $\tilde M$. It follows now easily from a mean value argument that $|\phi|$ must be finite everywhere in $\tilde M$. Thus, $\xi \neq 0$ and $W^+ \neq 0$ at every point of M, which completes the proof.

REMARK 4: Propositions 3, 4 and 5 give the following local conformal relations between metrics on oriented four-manifolds, valid wherever $W^+ \neq 0$:

a) The conformal structures containing Kähler metrics, compatible with the orientation, are precisely those containing metrics with $\delta W^+ = 0$ and $\#\mathrm{spec}(W^+) \leq 2$;

b) A Kähler metric is locally conformally Einsteinian if and only if $B = 0$, i.e., $2\nabla^2 u + ur = (u^2/4 - \Delta u/2)g$ (examples of such metrics on open manifolds are described in [10]).

c) An Einstein metric is locally conformally Kählerian if and only if $\#\mathrm{spec}(W^+) \leq 2$.

In all these cases, the conformal change in question is given by a natural formula and essentially unique. Moreover, an Einstein metric with $\#\mathrm{spec}(W^+) \leq 2$ is *globally* conformally Kählerian unless $W^+ = 0$ identically. Particular consequences of this fact, for compact manifolds, are studied in Section 5.

4. Self-duality and recurrent conformal curvature

Let us recall that an oriented Riemannian manifold (M, g) is called *self-dual* (resp., *anti-self-dual*) ([2]) if $\dim M = 4$ and $W^- = 0$ (resp., $W^+ = 0$). We may also speak of *self-dual* or *anti-self-dual Kähler manifolds* (M, g, ω), $\dim M = 4$, choosing the orientation in M so that $\omega \in \Lambda^+$.

REMARK 5: It is easy to see that a locally reducible Riemannian four-manifold is self-dual if and only if it is conformally flat.

For any compact oriented Riemannian four-manifold (M, g), formula (15) yields

$$12\pi^2 \tau(M) = \int_M (|W^+|^2 - |W^-|^2)\,dV \leq \int_M |W|^2\,dV \tag{47}$$

with equality if and only if (M, g) is self-dual ([2]). Thus, every self-dual metric on a compact oriented four-manifold M provides an absolute minimum for the functional (22). Consequently, Lemma 1 implies

LEMMA 6: *The Bach tensor of any compact self-dual Riemannian four-manifold vanishes identically.*

Another consequence of (47) is the inequality

$$\tau(M) > 0 \tag{48}$$

valid for any compact, oriented, self-dual Riemannian four-manifold which is not conformally flat ([2]).

The following local results follow immediately from Proposition 2 together with the equality $|W|^2 = |W^+|^2 + |W^-|^2$.

COROLLARY 1: *Let (M, g, ω) be a four-dimensional Kähler manifold. The following conditions are equivalent:*

(i) *(M, g, ω) is anti-self-dual;*

(ii) *The scalar curvature of (M, g) is zero.*

COROLLARY 2: *For any four-dimensional Kähler manifold (M, g, ω), we have $u^2 \leq 24|W|^2$, with equality if and only if (M, g, ω) is self-dual.*

We shall need later the following lemma, obtained independently by J.P. Bourguignon ([7], Proposition 9.3).

LEMMA 7: *Let (M, g, ω) be a self-dual Kähler manifold with constant scalar curvature. Then (M, g) is locally symmetric.*

PROOF: Proposition 2 together with $W = W^+$ yield $\nabla W = 0$. Hence, in view of Proposition 1, $\nabla r = 0$ and, consequently, $\nabla R = 0$. Note that, as shown by W. Roter (see [12]), a Riemannian manifold with $\nabla W = 0$ must satisfy $W = 0$ or $\nabla R = 0$.

A Riemannian manifold is said to have *recurrent conformal curvature* ([1]) if, for any point x and any tangent vector Y at x, the Weyl tensor W_x and its covariant derivative $\nabla_Y W$ are collinear. Clearly, this happens if and only if the restriction of W to the open subset where $W \neq 0$ is a functional multiple of a parallel tensor. Therefore, Proposition 2 implies

COROLLARY 3: *Every self-dual Kähler manifold has recurrent conformal curvature.*

The converse of Corollary 3 fails to hold in general, since the class of four-manifolds with recurrent conformal curvature contains all conformally flat manifolds as well as all products of surfaces (cf. Remarks 5 and 2). However, these are, essentially, the only exceptions. Namely, we have

PROPOSITION 6: *Let (M, g) be a four-dimensional Riemannian manifold with recurrent conformal curvature. If $x \in M$ and $W_x \neq 0$, then either*

(i) *x has a neighbourhood isometric to a product of surfaces, or*

(ii) *A neighbourhood U of x admits a Kähler form ω such that (U, g, ω) is a self-dual Kähler manifold.*

PROOF: In a sufficiently small oriented contractible neighbourhood U of x, $\nabla(FW) = 0$ for some non-zero function F and, consequently, the vector bundles $\Lambda^{\pm}$ split over U into direct sums of eigenspace bundles of $W^{\pm}$, each of which is invariant under parallel displacements. Since $W_x \neq 0$, (12) implies that one of $\Lambda^{\pm}$ must split essentially, which gives rise to a one dimensional eigenspace subbundle of Λ^{+} or Λ^{-}, i.e., to a Kähler form ω in U. Let, e.g., $\omega \in \Lambda^{+}$. If Λ^{-} is an eigenspace of W^{-}, then, by (12), $W^{-} = 0$ and (ii) follows. On the other hand, if Λ^{-} does split, then it contains a Kähler form and (i) is immediate from Remark 3. This completes the proof.

As a consequence, we obtain a direct proof for the following well-known

COROLLARY 4: *Let (M, g) be an orientable, locally irreducible, locally symmetric Riemannian manifold of dimension four. Then (M, g) is self-dual for some orientation. If, moreover, (M, g) is not a space of constant curvature, then either (M, g) itself, or a two-fold Riemannian covering space thereof is a self-dual Kähler manifold.*

PROOF: We may assume $W \neq 0$, which, since g is Einsteinian, is nothing but excluding the case of constant curvature. Analyticity of (M, g) together with Proposition 6 implies now $W^{-} = 0$ for some orientation. Moreover, there exist local Kähler forms in Λ^{+}. At any point, such a form is unique up to a sign, for otherwise W^{+} would be zero (Remark 3). This completes the proof.

Because of strong geometrical consequences of self-duality ([2]), the natural question arises, which compact four-manifolds carry self-dual metrics (or conformal structures). However, the list of known examples is rather scarce:

EXAMPLE 1: The only compact oriented four-manifolds which are known to admit self-dual metrics are the following:

(i) compact conformally flat manifolds;

(ii) compact Ricci-flat Kähler manifolds with the opposite of the standard orientation ([2]; cf. also Corollary 1). They are either flat, or diffeomorphic to quotients of K3 surfaces ([33], [17]);

(iii) compact, locally irreducible, locally symmetric Riemannian four-manifolds (Corollary 4). As shown by A. Borel [6], every simply connected Riemannian symmetric space possesses compact isometric quotients.

Thus, the known examples of non-conformally flat compact self-dual manifolds are Kählerian (up to conformal deformations), but the orientation for self-duality need not coincide with the natural one. In the case where it does, we have the following result (global according to Remark 6).

THEOREM 1: *Every compact Kähler manifold (M, g, ω) of real dimension four self-dual with respect to the natural orientation is locally symmetric.*

PROOF: In view of Lemma 7 we may assume that the scalar curvature u is not constant, so that $W \neq 0$ somewhere (Proposition 2), and the signature $\tau(M)$ is positive by (48). By Lemma 6 and Proposition 4, ∇u is a non-trivial holomorphic vector field with zeros on M. As shown by Carrell, Howard and Kosniowski [9], a compact complex surface which carries such a field and a Kähler metric can be obtained by blowing up finitely many points in CP^2 or in the total space of a CP^1 bundle over a Riemann surface. Condition $\tau(M) > 0$ implies now that M is biholomorphic to CP^2, so that $3\tau(M) = \chi(M)$. In view of Lemma 3, g is Einsteinian and u is constant. This contradiction completes the proof.

REMARK: Another proof of Theorem 1 can be obtained from the Bogomolov–Miyaoka inequality $3c_2 \geq c_1^2$, valid for algebraic surfaces of general type ([33]), together with Lemmas 3, 7, (48) and Kodaira's classification theory for compact complex surfaces.

REMARK 6: The compactness hypothesis is essential in Theorem 1: There exist examples (see [10]) of open self-dual Kähler manifolds which satisfy $B = 0$ and are not locally conformal to a symmetric space. Thus, there exist analytic Riemannian four-manifolds which have recurrent conformal curvature and are neither conformally flat, nor locally symmetric, nor locally reducible (cf. Corollary 3 and Remark 5). As shown recently by W. Roter [28], examples of this sort do not exist in dimensions greater than four.

In [7] J.P. Bourguignon proved (Proposition 9.1) that every compact self-dual manifold satisfying the condition $\delta R = 0$ must be conformally flat or Einsteinian. It appears that, in the analytic case, this assertion follows from the apparently weaker hypothesis $\delta W = 0$. Our argument is based on the following local lemma.

LEMMA 8: *Let (M, g) be an oriented Riemannian four-manifold such that $\delta W = 0$, $W^- = 0$ and the Bach tensor $B = 0$. Then $W \otimes (r - ug/4) = 0$.*

PROOF: Condition $\delta W = 0$ together with (23) yields $r^{pq} W_{pijq} = 0$, i.e., in the notation of (28)–(32), valid in an open dense subset M_W of M, we have $\lambda^- = \mu^- = \nu^- = 0$ and the symmetric tensor $T = \lambda\omega r\omega + \mu\eta r\eta + \nu\theta r\theta$ vanishes identically. If $r - ug/4$ does not vanish identically, then it is non-zero at all points of some connected open subset U of M_W. We shall prove that $W = 0$ in any set U with this property. Fix $x \in U$ and find $Y \in T_x M$ which is not an eigenvector of r_x. Since Y is orthogonal to ωY, ηY and θY, rY is *not* orthogonal to one of them; for instance,

$$r(Y, \omega Y) \neq 0. \tag{49}$$

Relation $T = 0$ gives, for any tangent vector Z, $0 = \langle Z, T\omega Z\rangle = \lambda r(Z, \omega Z) + (\mu - \nu) r(\eta Z, \theta Z)$. Substituting here $Z = Y$ and $Z = \eta Y$, we obtain, respectively, $\lambda r(Y, \omega Y) + (\mu - \nu) r(\eta Y, \theta Y) = 0$ and $(\mu - \nu) r(Y, \omega Y) + \lambda r(\eta Y, \theta Y) = 0$. In view of (49), this linear system must satisfy the determinant relation $0 = \lambda^2 - (\mu - \nu)^2 = (\lambda - \mu + \nu)(\lambda + \mu - \nu)$, i.e., by (30), $\mu\nu = 0$. Therefore $\det W^+ = \lambda\mu\nu = 0$ everywhere in U. Suppose now that $W = W^+ \neq 0$ at all points of some connected open subset U_1 of U. Taking U_1 sufficiently small, we may assume, without loss of generality, that $\lambda = 0$ and, by (30), $\nu = -\mu \neq 0$ everywhere in U_1. In U_1, relations (34) and $d\lambda = 0$ yield $\theta c = \eta b$, so that $|b| = |c|$, and (35) implies $0 = \Delta\lambda = 4\mu\nu$. This contradiction shows that $W = 0$ in U, which completes the proof.

The following global result is now immediate from Lemma 8 and Lemma 6.

PROPOSITION 7: *Let (M, g) be a compact, oriented, analytic Riemannian manifold of dimension four. If $W^- = 0$ and $\delta W = 0$, then (M, g) is conformally flat or Einsteinian.*

5. Four-dimensional Einstein manifolds

Let (M, g) be an oriented Riemannian four-manifold. According to Proposition 2, if g is locally conformal to a Kähler metric compatible with the orientation, then it satisfies the condition $\#\mathrm{spec}(W^+) \leq 2$, i.e., $W^+ \in \mathrm{End}\,\Lambda^+$ has, at each point, less than three distinct eigenvalues. In this section we study the consequences of this condition for Einstein metrics on compact manifolds.

LEMMA 9: *Let (M, g) be an orientable Riemannian four-manifold such that* (1) *does not hold for some orientation. Then M possesses a non-empty open subset U_0 (dense if (M, g) is analytic) with the following property: Every non-trivial Killing field, defined in any open connected subset U of U_0, is non-zero at each point of U.*

PROOF: Let U_0' be the open set of all points $x \in M$ such that $W_x^+ \in \operatorname{End} \Lambda_x^+$ has three distinct eigenvalues. In the analytic case, U_0' is dense in M. At any $x \in U_0'$, we may choose functions λ, μ, ν with $\lambda < \mu < \nu$ and mutually orthogonal sections ω, η, θ of Λ^+, differentiable in a neighbourhood of x and satisfying relations of the form (28)–(30) and (32) with some 1-forms a, b, c. By our hypothesis, these conditions determine ω, η, θ (and hence a, b, c) *uniquely* up to changes of signs. Consequently, $\phi = |a|^2 + |b|^2 + |c|^2$ is a well-defined smooth function in U_0'. If we had $\phi = 0$ in some open subset U_1 of U_0', (32), (11) and (12) would yield $\lambda = \mu = \nu = u/6 = 0$ in U_1, contradicting our choice of U_0'. Thus, the open subset U_0 of U_0', defined by $\phi > 0$, is dense in U_0'. At any point $y \in U_0$ we have, for instance, $a \neq 0$ (notation as before), so that a, ωa, ηa, θa is an orthogonal frame field in a neighbourhood of y, invariant under any local one-parameter group of isometries, defined near y. If the local one-parameter group keeps y fixed, it must, therefore, consist of the identity transformation only. In other words, a Killing field, defined in a connected neighbourhood of y and vanishing at y, must vanish identically. This completes the proof.

COROLLARY 5: *Let (M, g) be an orientable Einstein four-manifold such that* (1) *is not satisfied by some orientation. Denoting by G the group of all isometries of (M, g), we have*

(i) $\dim G \leq 3$;

(ii) *For almost all $x \in M$, the orbit of G through x is of dimension* $\dim G$.

PROOF: Choose U_0 as in Lemma 9. By a recent result of DeTurck and Kazdan [13], every Einstein metric is analytic in a suitable coordinate atlas. Therefore U_0 may be assumed dense. An isometry of (M, g) keeping a point $x \in U_0$ fixed and sufficiently close to the identity must then equal the identity, which yields (ii). Thus, $\dim G \leq 4$. If we had $\dim G = 4$, a theorem of G.R. Jensen [19] would imply that the homogeneous Einstein four-manifold (M, g) is locally symmetric. If the locally symmetric Einstein metric g is locally reducible, it is easy to verify that (1) holds for both orientations. In the irreducible case, g is conformally flat or locally Kählerian and self-dual (Corollary 4), which, again, gives (1) for both orientations, in view of Proposition 2. This contradiction completes the proof.

The manifolds listed below exhaust all known examples of compact, orientable, four-dimensional Einstein manifolds.

EXAMPLE 2: (i) Compact Riemannian four-manifolds, locally isometric to a product of two surfaces with equal constant curvatures, are Einsteinian.

(ii) Locally irreducible, locally symmetric four-manifolds are Einsteinian (for the existence of compact quotients, see [6]).

(iii) Compact Kähler–Einstein four-manifolds and their quotients. Every compact Kähler manifold whose first Chern class is zero or negative, admits a Kähler–Einstein metric ([33]).

(iv) The compact complex surface $F_1 = CP^2 \# (-CP^2)$, obtained by blowing up a point in CP^2, admits a Hermitian Einstein metric, whose group of isometries is four-dimensional (with principal orbits of dimension three). This metric was found by D. Page [26] (for a clear exposition, see [4]).

Note that, except for (iv), the above examples are either conformally flat or locally Kählerian, and so they satisfy (1) for an appropriate orientation (Proposition 2). On the other hand, the metric of (iv) satisfies (1) in view of Corollary 5. Consequently, *the condition* $\#\mathrm{spec}(W^+) \leq 2$, *for some orientation, holds for all known examples of compact, orientable, four-dimensional Einstein manifolds.*

Using the results of Section 3, we shall now prove the following structure theorem for compact Einstein four-manifolds satisfying (1):

THEOREM 2: *Let (M, g) be a compact, oriented, four-dimensional Einstein manifold such that the endomorphism W^+ of $\Lambda^+ M$ has, at each point, less than three distinct eigenvalues. Then there exists a Riemannian covering space $(\tilde{M}, g)$ of (M, g), of multiplicity one or two, for which only the following three cases are possible*:

(i) $W^+ = 0$, *i.e.*, *(M, g) is an anti-self-dual Einstein manifold*; $\tilde{M} = M$.

(ii) *W^+ is parallel and non-zero, and $(\tilde{M}, g)$ is a (non-Ricci-flat) Kähler–Einstein manifold*; *M is either $\tilde{M}$ itself, or a quotient thereof by a free, isometric, antiholomorphic involution.*

(iii) *W^+ is not parallel. In this case, $W^+ \neq 0$ everywhere and g is Hermitian (but not Kählerian) with respect to some complex structure J on $\tilde{M}$, compatible with the original orientation. Either $M = \tilde{M}$, or $M = \tilde{M}/\Phi$ for some free antiholomorphic involutive isometry Φ of $(\tilde{M}, g, J)$. The complex surface $(\tilde{M}, J)$ is biholomorphic to a surface obtained from CP^2 by blowing up k points $(0 \leq k \leq 8)$ or from a holomorphic CP^1*

bundle over CP^1 *by blowing up k points* ($0 \leq k \leq 7$), *so that, topologically,* $\tilde{M}$ *is* $S^2 \times S^2$ *or a connected sum* $CP^2 \# (-kCP^2)$, $0 \leq k \leq 8$. *The metric* $\bar{g} = (24g(W^+, W^+))^{1/3} g$ *on* $\tilde{M}$ *is Kählerian with respect to* J, *its scalar curvature* $\bar{u}$ *is non-constant and positive everywhere and satisfies the relations*

$$2\bar{\nabla}^2\bar{u} + \bar{u}\cdot\bar{r} = (\bar{u}^2/4 - \bar{\Delta}\bar{u}/2)\bar{g},$$
$$\bar{u}^3 - 6\bar{u}\bar{\Delta}\bar{u} - 12\bar{g}(\bar{\nabla}\bar{u}, \bar{\nabla}\bar{u}) = u > 0, \tag{50}$$

u *being the (constant) scalar curvature of* g. *The non-trivial vector field* $X = J(\bar{\nabla}\bar{u})$ *is a Killing field for both* g *and* $\bar{g}$, *holomorphic with respect to* J.

PROOF: Most of our assertion follows trivially from Propositions 5 and 4. In particular, if W^+ is not parallel, then, by (iii) of Proposition 5 and (43), $\bar{u}$ is not constant and $\bar{u} \neq 0$ everywhere. The non-trivial vector field $X = J(\bar{\nabla}\bar{u})$ is a holomorphic Killing field (with zeros) for $\bar{g}$ and for $g = \bar{g}/\bar{u}^2$ (Proposition 4), so that, in view of (8), $u > 0$. Choosing a point x with $\bar{\Delta}\bar{u}(x) = 0$, we obtain from (50) $\bar{u}^3(x) = u + 12\bar{g}(\bar{\nabla}\bar{u}(x), \bar{\nabla}\bar{u}(x)) > 0$ and hence $\bar{u} > 0$ at each point. By a theorem of Carrell, Howard and Kosniowski [9], a compact complex surface admitting a Kähler metric and a non-trivial holomorphic vector field with zeros must be obtained from CP^2 or from a CP^1 bundle over CP^1 by successively blowing up finitely many points. The estimate on the number of these points follows immediately from Thorpe's inequality (17). Finally, it is well-known (cf. [23], p. 277) that $(S^2 \times S^2)\#(-CP^2)$ is diffeomorphic to $CP^2\#(-2CP^2)$. This completes the proof.

REMARK 7: Known examples show that each of the three cases of Theorem 2 may really occur. Assertion (i) is satisfied, e.g., by flat manifolds and by K3 surfaces with Ricci-flat Kähler metrics ([33]); on the other hand, as shown by N. Hitchin ([17], [18]; cf. also [15]), a non-flat (anti-)self-dual compact Einstein four-manifold with non-negative scalar curvature must be isometric to S^4 or to CP^2 with a standard metric, or to K3 surface with a Ricci-flat Kähler metric. As for case (ii), beside many concrete examples, there is the existence theorem of Aubin and Yau ([33]). Finally, the Page metric on F_1 ((iv) of Example 2) satisfies (iii) of Theorem 2. In fact, this metric is neither Kählerian (by a theorem of A. Lichnerowicz [22], F_1 admits no Kähler metric with constant scalar curvature), nor self-dual for any orientation (in view of (48), since $\tau(F_1) = 0$ and the Einstein metric on F_1 is not conformally flat). It follows now from Theorem 2 that the Page metric on F_1 is globally conformal to a Kähler metric. The latter was found by E. Calabi [8], who studied compact Kähler manifolds for which the vector field ∇u is holomorphic.

As a consequence of Theorem 2 together with a result of L. Bérard Bergery [4], we obtain

THEOREM 3: *Let (M, g) be a compact Einstein four-manifold. Denote by G the group of isometries of the Riemannian universal covering space $(\tilde{M}, g)$ of (M, g). If all orbits of G are of dimensions strictly less than $\dim G$, then either (M, g) is locally symmetric, or $(\tilde{M}, g)$ is isometric, up to a scaling factor, to $CP^2 \#(-CP^2)$ endowed with the Page metric.*

PROOF: We may assume that M is orientable and g is not locally symmetric. Corollary 5, applied to $(\tilde{M}, g)$, yields $\#\mathrm{spec}(W^+) \leq 2$ and $\#\mathrm{spec}(W^-) \leq 2$. Since g is not conformally flat, we may choose the orientation so that $W^+ \neq 0$ (everywhere, by (iv) of Proposition 5). If we had $W^- = 0$, (ii) and (iii) of Theorem 2 together with Proposition 5 would imply that (a two-fold cover of) M carries a self-dual Kähler metric $\bar{g}$, with scalar curvature $\bar{u}$ ($\bar{u} \neq 0$ everywhere) and g is a constant multiple of $\bar{g}/\bar{u}^2$; by Theorem 1, $\bar{g}$ would be locally symmetric and hence so would be g. Therefore, by (iv) of Proposition 5, $W^- \neq 0$ everywhere. If both W^+ and W^- were parallel, there would locally exist two Kähler forms $\omega^{\pm} \in \Lambda^{\pm}$ for (M, g) (cf. Proposition 5); by Remark 3, the Einstein metric g would be locally reducible and hence locally symmetric. Thus, reversing the orientation if necessary, we obtain $\nabla W^+ \neq 0$ somewhere. From (iii) of Theorem 2 it follows now (cf. Proposition 5) that some finite covering space of (M, g) admits a non-trivial Killing field X with zeros, defined (up to a sign) by an explicit formula and hence invariant, up to a sign, under all local isometries. Consequently, by (8) and the Myers theorem, u is positive and $\tilde{M}$ is compact, while the pull-back $\tilde{X}$ of X to $\tilde{M}$ is invariant under the connected component G^0 of G. Consequently, G^0 has an infinite center, and our hypothesis implies that $\dim G^0 \geq 2$. We assert that $\dim G^0 \geq 4$. In fact, if we had $\dim G^0 \leq 3$, then the compact connected Lie group G^0 with infinite center would be isomorphic to a torus of dimension 2 or 3. Fix a point $x \in \tilde{M}$ lying in a G^0 orbit of maximal dimension. For all points y sufficiently close to x, the isotropy subgroups $H_y \subset G^0$ are subtori of the same positive codimension and the corresponding subalgebras of the Lie algebra of G^0 vary continuously with y. Consequently, $H_y = H$ for some subtorus H of G^0 and all y close to x. The non-trivial group H operates trivially on a neighbourhood of x and, consequently, on $\tilde{M}$. This contradicts the effectiveness of the action of G and hence proves that $\dim G^0 \geq 4$. Our assertion is now immediate from L. Bérard Bergery's Théorème 1.8 of [4].

Using Lie algebra techniques, G.R. Jensen proved in [19] that every locally homogeneous four-dimensional Einstein manifold is locally symmetric. A direct proof of this result in the *compact* case can be obtained as follows.

For an oriented Riemannian four-manifold (M, g), let us define the *discriminant* $D(W^+)$ (resp., $D(W^-)$) of W^+ (resp., of W^-) to be the smooth function on M given by $D(W^\pm)(x) = (\lambda - \mu)^2(\lambda - \nu)^2(\mu - \nu)^2$, λ, μ, ν being the eigenvalues of $W_x^\pm \in \operatorname{End} \Lambda_x^\pm$.

LEMMA 10: *Let (M, g) be a compact, oriented, Einstein four-manifold. Then both discriminants $D(W^+)$, $D(W^-)$ vanish at certain points of M.*

PROOF: Assume the contrary and choose the orientation so that $D(W^+) > 0$ everywhere. Thus, $W^+ \in \operatorname{End} \Lambda^+$ has three distinct eigenvalues at each point, which implies that $\Lambda^+ M$ is a Whitney sum of line bundles. Consequently, the real Pontryagin class $p_1(\Lambda^+ M)$ is zero. On the other hand, since $p_1(\Lambda^+ M) = p_1(TM) + 2e(TM)$ (see, e.g., [11], p. 490), we have $3\tau(M) + 2\chi(M) = 0$. In view of (15) and (16), this means that $48 \int_M |W^+|^2 dV + \int_M u^2 dV = 0$. Consequently, $W^+ = 0$ and $D(W^+) = 0$. This contradiction completes the proof.

The following result implies immediately that a locally homogeneous compact Einstein four-manifold must be locally symmetric.

PROPOSITION 9: *Let (M, g) be a compact, oriented, Einstein four-manifold. Consider $W^\pm \in \operatorname{End} \Lambda^\pm$. If the functions $|W^+|^2$, $|W^-|^2$,* $\det W^+$ *and* $\det W^-$ *are all constant (i.e., the eigenvalues of W are constant), then (M, g) is locally symmetric.*

PROOF: Since the discriminants $D(W^\pm)$ are symmetric functions of the eigenvalues of $W^\pm$, our hypothesis together with (12) implies that $D(W^\pm)$ are constant. By Lemma 10, $D(W^\pm) = 0$. We may assume that g is not conformally flat, so that, e.g., the constant $|W^+|^2$ is positive. By (i) of Proposition 5, (M, g) admits, locally, a Kähler form $\omega^+ \in \Lambda^+$. If $|W^-|^2 > 0$, the same argument gives a local Kähler form $\omega^- \in \Lambda^-$ and, by Remark 3, g is a locally reducible Einstein metric, so that $\nabla R = 0$. On the other hand, if $W^- = 0$, our assertion is immediate from Lemma 7.

Note. Theorem 1 of this paper was first proved by Bang-yen Chen (Some topological obstructions to Bochner–Kaehler metrics and their applications. *J. Differential Geometry 13* (1978) 547–558), who classified the compact *Bochner–Kähler manifolds* of real dimension four, using Kodaira's classification theory for complex surfaces (cf. Remark following Theorem 1; these manifolds turn out to be just the self-dual Kähler manifolds).

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