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CONVERTING COMPACT ANR FIBRATIONS INTO LOCALLY TRIVIAL BUNDLES WITH COMPACT MANIFOLDS AS FIBRES

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Introduction

In the early 50's, K. Borsuk and J.H. Whitehead raised the following question: "Is any compact ANR homotopy equivalent to a compact manifold (possibly with boundary)?" This problem was solved in the mid 70's by J. West: his solution was the second striking application of the theory of hilbert cube manifolds. In this paper we will be concerned with the parametrized version of the West theorem, precisely with the conversion of a compact ANR fibration $E \xrightarrow{\pi} B$ (i.e., an Hurewicz fibration with E and B ANR's and π-proper) up to concordance into a bundle with compact manifolds (with boundary) as fibres.

The theory of compact ANR fibrations was resurrected by Hatcher [7], Chapman and Ferry [6]; it represents the right context for simple homotopy theory as well as an important link between algebraic K-theory and the geometric topology of manifolds. There are two basic sources of compact ANR fibrations: maps $p: E \to K, K$ polyhedron which over each simplex of K are iterated mapping cylinders and bundles $p: E \to K$ with fibres compact manifolds. The first class of examples realize the link with the algebraic K-theory, the second with the topology of the group of homeomorphisms of compact manifolds. Although not very surprising (but very interesting), each of these classes exhaust the compact ANR fibrations up to concordance.** This paper proves this fact for the second source.

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** Two compact fibrations $E_i \to B$, $i = 1,2$ are concordant if there exists the compact ANR fibration $\tilde{E} \to B \times [1,2]$ whose restriction to $B \times [1]$ is $E_i \to B$.

Before we state the results we introduce the following definition. A compact topological manifold M, possible with boundary *realizes the fibre of the compact ANR fibration* $E \xrightarrow{\pi} B$ if there exists a simple homotopy equivalence $\theta: M^n \to \pi^{-1}(b)$ for some b (and then for any b if B is connected) and a topological euclidean bundle ξ over E so that $\theta_*(\xi_{|\pi^{-1}(b)})$ is stably isomorphic to $\tau(M)$.

THEOREM A: (1) *Given a compact ANR fibration $E \to B$ with B compact, there exists a bundle $L \to B$ with fibres compact manifolds which is concordant (as a compact ANR fibration) to $E \to B$.*

(2) *One can choose $L \to B$ with fibres homeomorphic to $M^n \times D^N$ for some N iff M^n realises the fibre of $E \xrightarrow{\pi} B$.*

(3) *Iff $L_i \xrightarrow{\pi_i} B$ are two bundles with compact manifolds as fibres which are concordant as compact ANR fibrations, then there exists the disc bundles $E_i \to L_i$, $i = 1,2$ so that $E_1 \to L_1 \to B$ and $E_2 \to L_2 \to B$ are isomorphic.*

Part 1) of Theorem A is a consequence of the following result which was first proved by Chapman and Ferry [10].

THEOREM B: (1) *If M^n is a compact topological manifold, let $\mathbb{H}$omeo(M) denote $\varinjlim \mathbb{H}\text{omeo}(M \times I^n)$ (see section I for exact definitions) and Q the hilbert cube. Then the homotopy theoretic fibre of B $\mathbb{H}$omeo$(M) \to B$ Homeo$(M \times Q)$ is homotopy equivalent to* Maps $(M, B\text{Top})_0$ *(the connected component of the constant map).*

(2) *If M^n is stably parallelizable then the fibration* Maps $(M, B\text{Top})_0 \to B$ $\mathbb{H}$omeo$(M) \to B$ Homeo$(M \times Q)$ *has a cross section.*

Prior to this paper Theorem B and hence Theorem A(1) have been proved by Chapman and Ferry in a handwritten manuscript in 1978, and as I understand from the introduction of [9], Theorem A(1) was announced by R. Edwards (unpublished 1978). I heard about the Chapman and Ferry manuscript in the winter of 1980 and noticed that Theorem B is implicit in some previous work of Chapman plus [2], as can be seen in Section I. I was aware of Theorem A long before I wrote this manuscript (winter 1980) but I have not regarded the problem as interesting enough to deserve the appropriate elaborations. My efforts towards the proof of the stability of topological concordances made Theorem A relevant as I will explain in a forthcoming paper [1]. I should also add that the important results of [9] implies Theorem A if one restricts the attention to compact ANR-fibrations with fibres of bounded dimension.

In section I we prove Theorem B and in section II Theorem A.

Section I

If X and Y are two topological spaces, $A \subset X$ is a closed subset of X and $f: A \to Y$ a continuous map, one denotes

$$\text{by} \begin{cases} \operatorname{Homeo}(X; A) \\ \\ \operatorname{Maps}(X, Y|f) \end{cases} \qquad \text{respectively} \begin{cases} \widetilde{\operatorname{Homeo}}(X; A) \\ \\ \widetilde{\operatorname{Maps}}(X, Y|f) \end{cases}$$

the simplicial set resp. Δ-sets whose n-simplexes

$$\sigma \text{ are} \begin{cases} \text{homeomorphisms } h: \Delta[n] \times X \to \Delta[n] \times X, h|\Delta[n] \times A = id \\ \\ \text{maps } l: \Delta[n] \times X \to \Delta[n] \times X \text{ with } l|\Delta[n] \times A = id \times f \end{cases}$$

which commute with the projection on $\Delta[n]$ resp. are faces preserving: this means $h(d_i\Delta[n] \times X) \subset d_i\Delta[n] \times X$, $l(d_i\Delta[n] \times X) \subset d_i\Delta[n] \times Y$. Clearly $\operatorname{Homeo}(X; A)$ resp. $\widetilde{\operatorname{Homeo}}(X; A)$ is a simplicial group resp. Δ-group with s_i's, d_i's resp. d_i's defined in the obvious way. We will denote by $C(X)$ resp. $\tilde{C}(X)$ the simplicial groups $\operatorname{Homeo}(X \times I; X \times 0)$ resp. $\widetilde{\operatorname{Homeo}}(X \times I; X \times 0)$, by $\mathscr{K}(X)$ resp. $\tilde{\mathscr{K}}(X)$ the simplicial resp. Δ submonoid of $\operatorname{Maps}(X, X)$ resp. $\widetilde{\operatorname{Maps}}(X, X)$ whose simplexes are maps $l: \Delta[n] \times X \to \Delta[n] \times X$ which are homotopy equivalences and when X is a compact ANR, simple homotopy equivalences. Both are (associative) simplicial resp. Δ monoids with $\pi_0(\ldots)$ groups. We also consider $\mathscr{E}\mathrm{mb}(X, Y{:}f)$ resp. $\widetilde{\mathscr{E}\mathrm{mb}}(X, Y; f)$ the simplicial resp. Δ subset whose simplexes are maps l which are closed embeddings (of course in this case f is assumed to be a closed embedding). Let $\mathrm{Top}_n = \operatorname{Homeo}(R^n; 0)$, $\widetilde{\mathrm{Top}}_n = \widetilde{\operatorname{Homeo}}(R^n; 0)$, $\mathscr{D}_n = \operatorname{Homeo}(D^n; 0)$, $\tilde{\mathscr{D}}_n = \widetilde{\operatorname{Homeo}}(D^n; 0)$; the product with id_I resp. id_R define the homeomorphisms $\operatorname{Homeo}(X) \to \operatorname{Homeo}(X \times I)$, $\widetilde{\operatorname{Homeo}}(X) \to \widetilde{\operatorname{Homeo}}(X \times I)$, $C(X) \to C(X \times I)$, $\tilde{C}(X) \to \tilde{C}(X \times I)$, $\mathscr{K}(X) \to \mathscr{K}(X \times I)$, $\tilde{\mathscr{K}}(X) \to \tilde{\mathscr{K}}(X \times I)$, $\mathrm{Top}_n \to \mathrm{Top}_{n+1}$, $\widetilde{\mathrm{Top}}_n \to \widetilde{\mathrm{Top}}_{n+1}$, $\mathscr{D}_n \to \mathscr{D}_{n+1}$, $\tilde{\mathscr{D}}_n \to \tilde{\mathscr{D}}_{n+1}$ and therefore $\mathbb{H}\mathrm{omeo}(X) = \varinjlim \operatorname{Homeo}(X \times I^n)$, $\widetilde{\mathbb{H}\mathrm{omeo}}(X) = \varinjlim \widetilde{\operatorname{Homeo}}(X \times I^n)$, $\mathbb{C}(X) = \varinjlim C(X \times I^n)$, $\tilde{\mathbb{C}}(X) = \varinjlim \tilde{C}(X \times I^n)$, $\tilde{\mathscr{K}}(X) = \varinjlim \tilde{\mathscr{K}}(X \times I^n)$, $\mathbb{T}\mathrm{op} = \varinjlim \mathrm{Top}_n$, $\widetilde{\mathbb{T}\mathrm{op}} = \varinjlim \widetilde{\mathrm{Top}}_n$, $\mathscr{D} = \varinjlim \mathscr{D}_n$, $\tilde{\mathscr{D}} = \lim \tilde{\mathscr{D}}_n$. $\tilde{C}(X)$ and $\tilde{\mathbb{C}}(X)$ are contractible and in the diagram

$$\begin{array}{ccc} \mathscr{K}(X) & \subsetneq & \tilde{\mathscr{K}}(X) \\ \downarrow\cap & & \downarrow\cap \\ \boldsymbol{\mathscr{K}}(X) & \subsetneq & \tilde{\boldsymbol{\mathscr{K}}}(X) \end{array} \qquad \begin{array}{ccc} \widetilde{\mathbb{T}\mathrm{op}} & \rightleftarrows & \mathrm{Top} \\ \uparrow\cup & & \uparrow\cup \\ \tilde{\mathscr{D}} & \rightleftarrows & \mathscr{D} \end{array}$$

all inclusions are homotopy equivalences. These are folklor results, easy to be checked for the first diagram, known for the second diagram in the p.l. category and then by Kirby Siebenman in the topological category. The inclusion $\mathrm{Maps}(X, Y|f) \subseteq \widetilde{\mathrm{Maps}}(X, Y|f)$ is also a homotopy equivalence.

Let Q be the hilbert cube, i.e. $A = \prod_{r=1}^{\infty} I_r$ where I_r is a copy of $[0, 1]$. We continue to denote by $0 \in Q$ the point with coordinates zero. The homomorphisms $\mathrm{Homeo}(X) \to \mathrm{Homeo}(X \times Q)$ and $\widetilde{\mathrm{Homeo}}(X) \to \widetilde{\mathrm{Homeo}}(X \times Q)$ induce the commutative diagram

$$\begin{array}{ccc} \mathbb{H}\mathrm{omeo}(X) & \to & \mathrm{Homeo}(X \times Q) \\ \downarrow & & \downarrow \\ \widetilde{\mathbb{H}\mathrm{omeo}}(X) & \to & \widetilde{\mathrm{Homeo}}(X \times Q) \end{array}$$

and if BG denotes the classifying space of the simplicial resp. Δ-group G, the commutative diagram

$$\begin{array}{ccc} B\,\mathbb{H}\mathrm{omeo}(X) & \to & B\,\mathrm{Homeo}(X \times Q) \\ \downarrow & & \downarrow \\ B\,\widetilde{\mathbb{H}\mathrm{omeo}}(X) & \to & B\,\widetilde{\mathrm{Homeo}}(X \times Q). \end{array}$$

The proof of Theorem B is a consequence of Theorem B′ and Theorem C below.

THEOREM B′: (1) *If M^n is a compact topological manifold then the homotopy theoretic fibre of* $B\,\widetilde{\mathbb{H}\mathrm{omeo}}(M) \to B\,\widetilde{\mathrm{Homeo}}(M \times Q)$ *is homotopy equivalent to* $\mathrm{Maps}(M, B\,\mathrm{Top})_0$.

(2) *If M^n is stable parallelizable then the fibration* $\mathrm{Maps}\ (M, B\,\mathrm{Top})_0 \to B\,\widetilde{\mathbb{H}\mathrm{omeo}}(M) \to B\,\widetilde{\mathrm{Homeo}}(M \times Q)$ *has a cross section.*

THEOREM C: *If X is a compact topological manifold, the diagram*

$$\begin{tikzcd} B\,\mathbb{H}\mathrm{omeo}(X) \arrow[r] \arrow[d] & B\,\mathrm{Homeo}(X \times Q) \arrow[d] \\ B\,\widetilde{\mathbb{H}\mathrm{omeo}}(X) \arrow[r] & B\,\widetilde{\mathrm{Homeo}}(X \times Q) \end{tikzcd}$$

is homotopy cartesian.

To prove Theorem C beside a few nontrivial results of geometric topology we will use some elementary properties of Q-manifolds, namely:

PROPOSITION I.1: (general position) *If A and B are two Q-manifolds with A compact, $\mathscr{E}\mathrm{mb}^{\mathrm{v}}(A, B)$ resp. $\widetilde{\mathscr{E}\mathrm{mb}}^{\mathrm{v}}(A, B)$ are the subcomplex resp. Δ-subset of $\mathscr{E}\mathrm{mb}(A, B)$ resp. $\widetilde{\mathscr{E}\mathrm{mb}}(A, B)$ whose simplexes are embeddings $f: \Delta[k] \times A \to \Delta[k] \times B$ so that $f(\Delta[k] \times A)$ is a Z-set in $\Delta[k] \times B$ then in the following commutative diagram*

$$\begin{tikzcd} \mathscr{E}\mathrm{mb}^{\mathrm{v}}(A, B) \arrow[r, hook] \arrow[d, hook] & \mathrm{Maps}(A, B) \arrow[d, hook] \\ \widetilde{\mathscr{E}\mathrm{mb}}^{\mathrm{v}}(A, B) \arrow[r, hook] & \widetilde{\mathrm{Maps}}(A, B) \end{tikzcd}$$

all inclusions are homotopy equivalences.

PROPOSITION I.2: (ambient isotopy extension theorem) *If P is a Q-manifold, V a compact submanifold which is a Z-set then in* (*) *and* (**) $\pi(\mathrm{Homeo}(P))$ *and* $\tilde{\pi}(\widetilde{\mathrm{Homeo}}(P))$ *are unions of connected components*

$$(*) \quad \mathrm{Homeo}(P; V) \to \mathrm{Homeo}(P) \xrightarrow{\pi} \mathscr{E}\mathrm{mb}^{\mathrm{v}}(V, P)$$

$$(**) \quad \widetilde{\mathrm{Homeo}}(P; V) \to \widetilde{\mathrm{Homeo}}(P) \xrightarrow{\tilde{\pi}} \widetilde{\mathscr{E}\mathrm{mb}}^{\mathrm{v}}(V, P).$$

Moreover $\mathrm{Homeo}(P; V) \to \mathrm{Homeo}(P) \to \pi(\mathrm{Homeo}(P))$ *and* $\widetilde{\mathrm{Homeo}}(P; V) \to \widetilde{\mathrm{Homeo}}(P) \to \tilde{\pi}(\widetilde{\mathrm{Homeo}}(P))$ *are fibrations.*

PROOF OF THEOREM C: If suffices to show that

$$\widetilde{\mathbb{H}\mathrm{omeo}}(X) \Big/ \mathbb{H}\mathrm{omeo}(X) \to \widetilde{\mathrm{Homeo}}(X \times Q) \Big/ \mathrm{Homeo}(X \times Q)$$

is a homotopy equivalence which reduces to check that λ_1, λ_2, λ_3 in the diagram below are homotopy equivalences.

$$
\begin{tikzcd}
\widetilde{\mathbb{H}\mathrm{omeo}}(X \times I)\Big/\mathbb{H}\mathrm{omeo}(X \times I) \arrow[r, "\subset"] & \widetilde{\mathrm{Homeo}}(X \times I \times Q)\Big/\mathrm{Homeo}(X \times I\,Q) \\
\tilde{\mathbb{C}}(X)\Big/\mathbb{C}(X) \arrow[u, "\cup"', "\lambda_1"] \arrow[r, "\lambda_2"] & \tilde{C}(X \times Q)\Big/C(X \times Q). \arrow[u, "\lambda_3", "\cup"']
\end{tikzcd}
$$

λ_1 is a homotopy equivalence since as shown in [3] Lemma C.14, the Morlet's disjunction lemma as presented in [2] implies

$$
\tilde{C}(M \times I^r)\Big/C(M \times I^r) \longrightarrow \widetilde{\mathrm{Homeo}}(M \times I^{r+1})\Big/\mathrm{Homeo}(M \times I^{r+1})
$$

is $(r - n - 2)$ connected. Because $\tilde{C}(X)$ and $\tilde{C}(X \times Q)$ are contractible, and $\mathbb{C}(X) \to C(X \times Q)$ is a homotopy equivalence, λ_2 is a homotopy equivalence. To check that λ_3 is a homotopy equivalence we notice that in the diagram below, where "$\circ$" indicates that we restrict only to simple homotopy equivalences, the lines $^+)$ and $^{++})$ are fibrations. This follows from Proposition I.2 which applies because $X \times Q \times 0$ is a Z-set in $X \times Q \times [0,1]$.

$$
\begin{tikzcd}
(^+)\ \tilde{C}(X \times Q) \arrow[r] & \widetilde{\mathrm{Homeo}}(X \times Q \times I) \arrow[r] & \tilde{\mathring{\mathscr{E}\mathrm{mb}}}{}^{\mathrm{v}}(X \times Q \times 0, X \times Q \times I) \arrow[ddr, "\subset"] & \\
(^{++})\ C(X \times Q) \arrow[r] \arrow[u] \arrow[ur, phantom, "\mathrm{II}"] & \mathrm{Homeo}(X \times Q \times I) \arrow[r] \arrow[u] & \mathring{\mathscr{E}\mathrm{mb}}{}^{\mathrm{v}}(X \times Q \times 0, X \times Q \times I) \arrow[u, "\cup"'] \arrow[ddr, "\cap"'] \arrow[dr, phantom, "\mathrm{I}"] & \\
& & & \tilde{\mathring{\mathrm{Maps}}}(X \times Q \times 0, X \times Q \times I) \\
& & & \mathring{\mathrm{Maps}}(X \times Q \times 0, X \times Q \times I) \arrow[u, "\cup"']
\end{tikzcd}
$$

Since in Square I all inclusions are homotopy equivalences we conclude that Square II is homotopy cartesian, hence λ_3 is a homotopy equivalence. Q.e.d.

Proof of Theorem B′: Let us denote by $\mathbb{H}\mathrm{omeo}(M \times D^{\cdots}; M \times 0)$ the limit $\varinjlim \mathrm{Homeo}(M \times D^r; M \times 0)$; first we will check that

$$
\beta \colon \widetilde{\mathscr{H}}(\mathrm{M} \times Q * M \times 0)\Big/\widetilde{\mathbb{H}\mathrm{omeo}}(M \times D \ldots * M \times 0) \to
$$

$$
\widetilde{\mathscr{H}}(M \times Q)\Big/\widetilde{\mathrm{Homeo}}(M) \text{ and}
$$

$\widetilde{\mathrm{Homeo}}(M \times Q \times I) \to \tilde{\mathscr{K}}(M \times Q \times I)$ are homotopy equivalences. For this purpose we consider the following commutative diagrams

$$\begin{array}{llll}
(1) & \tilde{\mathscr{K}}(M \times Q : M \times 0) \to & \tilde{\mathscr{K}}(M \times Q) \to & \overset{\circ}{\widetilde{\mathrm{Maps}}}(M \times 0, M \times Q) \\
& \uparrow \quad\quad \mathrm{III} & \uparrow & \uparrow \gamma \\
(2) & \widetilde{\mathbb{H}\mathrm{omeo}}(M \times D^{\cdots}; M \times 0) \to & \widetilde{\mathbb{H}\mathrm{omeo}}(M) \to & \varinjlim \overset{\circ}{\widetilde{\mathscr{E}\mathrm{mb}}}(M \times 0, M \times D^r) \\
& & & \\
(3) & \tilde{\mathscr{K}}(M \times Q \times I; M \times Q \times 0) \to & \tilde{\mathscr{K}}(M \times Q \times I) \to & \\
& & & \to \overset{\circ}{\widetilde{\mathrm{Maps}}}(M \times Q \times 0, M \times Q \times I) \\
& \uparrow & \uparrow & \uparrow \gamma' \\
(4) & C(M \times Q) \to & \widetilde{\mathrm{Homeo}}(M \times Q \times I) \to & \overset{\circ}{\widetilde{\mathscr{E}\mathrm{mb}}}{}^{\mathrm{v}}(M \times Q \times 0, M \times Q \times I)
\end{array}$$

whose horizontal lines are fibrations (1), (2), (3) from obvious reasons and (4) by Proposition I.2 above. The general position arguments for finite dimensional manifolds imply γ is a homotopy equivalence, which means that square III is homotopy cartesian therefore β is a homotopy equivalence. Since both $\tilde{C}(M \times Q)$ and $\tilde{\mathscr{K}}(M \times Q \times I; M \times Q \times 0)$ are contractible and γ' a homotopy equivalence by Proposition I.2 then $\widetilde{\mathrm{Homeo}}(M \times Q \times I) \to \tilde{\mathscr{K}}(M \times Q \times I)$ is a homotopy equivalence.

Now we observe that $\tilde{\mathscr{K}}(M \times Q; M \times 0)\big/ \mathbb{H}\mathrm{omeo}(M \times D^{\cdots}; M \times 0) \sim B\,\widetilde{\mathbb{H}\mathrm{omeo}}(M \times D^{\cdots}; M \times 0)$ and $\widetilde{\mathrm{Maps}}(M, \tilde{\mathscr{D}}_r) \to \widetilde{\mathrm{Homeo}}(M \times D^r; M \times 0)$ as shown in [3] Lemma C.11 is $(r-1-n)$ connected; then $\mathrm{Maps}(M, \widetilde{\mathbb{T}\mathrm{op}}) \sim \mathrm{Maps}(M, \mathbb{T}\mathrm{op}) \sim \mathbb{H}\mathrm{omeo}(M \times D^{\cdots}; M \times 0)$ is a homotopy equivalence and then

$$\tilde{\mathscr{K}}(M \times Q; M \times 0)\big/ \mathbb{H}\mathrm{omeo}(M \times D^{\cdots}; M \times 0) \sim$$

$$\sim B\,\mathrm{Maps}(M, \mathbb{T}\mathrm{op}) \sim B\,\mathrm{Maps}(M, \mathbb{T}\mathrm{op})_0.$$

To conclude Theorem B′(1) we notice that in the diagram below β and B(i) are homotopy equivalences.

$$\begin{array}{ccccc}
\tilde{\mathscr{K}}(M \times Q : M \times 0)\big/ \widetilde{\mathbb{H}\mathrm{omeo}}(M \times D^{\cdots}; M \times 0) & & & & \\
\downarrow{\scriptstyle \beta} & & & & \\
\tilde{\mathscr{K}}(M \times Q)\big/ \widetilde{\mathbb{H}\mathrm{omeo}}(M) & \to & B\,\widetilde{\mathbb{H}\mathrm{omeo}}(M) & \to & B\tilde{\mathscr{K}}(M \times Q) \\
\uparrow & & \uparrow & & \uparrow{\scriptstyle B(i)} \\
F & \to & B\,\widetilde{\mathbb{H}\mathrm{omeo}}(M) & \to & B\,\widetilde{\mathrm{Homeo}}(M \times Q)
\end{array}$$

To prove Theorem B′(2) we consider $\mathring{\tilde{R}}(M^n)$ the Δ-monoid of bundle representations of $T(M^n)$ whose k-simplexes are pairs $(\tilde{\psi}, \psi)$

$$\begin{array}{ccc}
T(M) \times \Delta[k] & \xrightarrow{\tilde{\psi}} & T(M) \times \Delta[k] \\
\downarrow & & \downarrow \\
M \times \Delta[k] & \xrightarrow{\psi} & M \times \Delta[k]
\end{array}$$

with $\tilde{\psi}(T(M) \times d_i\Delta[k]) \subset T(M) \times d_i\Delta[k]$, and ψ a simple homotopy equivalence. We denote by $\widetilde{\mathrm{Iso}}(TM)$ the submonoid whose simplexes have $\psi = id$. This is a Δ-group. We also consider the Δ-submonoid $\tilde{R}(M \times D^r; M \times 0)$ whose simplexes $(\tilde{\psi}, \psi)$ have $\psi | M \times 0 = id$ and we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \widetilde{\mathrm{Iso}}(T(M \times D^r)) & \to & \mathring{\tilde{R}}(M \times D^r) & \to & \mathring{\tilde{\mathscr{K}}}(M \times D^r) & \to & 0 \\
 & & \searrow{\scriptstyle \alpha_r} & & \nearrow & & & & \\
 & & \downarrow \quad \tilde{R}(M \times D^r; M \times 0) & & & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \widetilde{\mathrm{Iso}}(T(M \times D^{r+1})) & \to & \mathring{\tilde{R}}(M \times D^{r+1}) & \to & \mathring{\tilde{\mathscr{K}}}(M \times D^{r+1}) & \to & 0 \\
 & & \searrow{\scriptstyle \alpha_{r+1}} & & \nearrow & & & & \\
 & & \downarrow \quad \tilde{R}(M \times D^{r+1}; M \times 0) & & & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

whose horizontal lines are exact sequences of Δ-monoids and α_r are homotopy equivalences. Passing to the limit we obtain the exact sequence

$$\begin{array}{ccccccccc}
0 & \to & \widetilde{\mathrm{Iso}}(\mathbb{T}(M)) & \xrightarrow{i} & \mathring{\tilde{\mathbb{R}}}(M) & \to & \mathring{\tilde{\mathscr{K}}}(M) & \to & 0 \\
 & & \searrow{\scriptstyle \alpha} & & \nearrow & & & & \\
 & & & \tilde{\mathbb{R}}(M \times D; M) & & & & &
\end{array}$$

with α homotopy equivalence. If M is stably parallelizable $T(M^n \times D^r) \simeq M \times D^r \times R^{n+2}$ and this trivialisation of the tangent bundle provides a canonical inverse $\psi: \mathring{\tilde{\mathscr{K}}}(M) \to \mathring{\tilde{\mathbb{R}}}(M)$ which is an homomorphism. Consequently

$$(\square) \quad B\,\widetilde{\mathrm{Iso}}(\mathbb{T}(M) \to B\mathring{\tilde{\mathbb{R}}}(M) \to B\mathring{\tilde{\mathscr{K}}}(M)$$

has a cross section. It remains to show that with the assumption of M stably parallelizable, the fibration $(\square)$ is the same as

$$B\,\widetilde{\mathbb{H}\mathrm{omeo}}(M \times D^{\cdots}; M \times 0) \to B\,\widetilde{\mathbb{H}\mathrm{omeo}}(M) \to B\mathring{\tilde{\mathscr{K}}}(M) \times Q).$$

For this purpose we consider the diagram

$$\begin{array}{ccccc} B\,\widetilde{\mathrm{Iso}}(\mathbb{T}(M)) \subseteq B\tilde{\mathbb{R}}(M \times D^{\cdots}, M \times 0) & \to & B\mathring{\tilde{\mathbb{R}}}(M) & \to & B\mathring{\tilde{\mathscr{K}}}(M) \\ \uparrow{\scriptstyle\delta'} & & \uparrow{\scriptstyle\delta} & & \uparrow \\ (*) \quad B\,\widetilde{\mathbb{H}\mathrm{omeo}}(M \times D^{\cdots}, M \times 0) & \to & B\,\widetilde{\mathbb{H}\mathrm{omeo}}(M) & \to & B\mathring{\tilde{\mathscr{K}}}(M) \end{array}$$

where δ is the "differential" which associates to each homeomorphism $h: M \times D^r \to M \times D^r$ the induced bundle representation $\delta(h) = (h, dh)$ and δ' the restriction of δ

$$\begin{array}{ccc} T(M \times D^r) & \xrightarrow{dh} & T(M \times D^r) \\ \downarrow & & \downarrow \\ M \times D^r & \xrightarrow{h} & M \times D^r. \end{array}$$

A simple analysis shows that δ' is a homotopy equivalence and since (*) is a fibration up to homotopy Theorem B′2) follows. Q.e.d.

Section II

To prove Theorem A we recall the following:

If $E_i \xrightarrow{\pi_i} B$ are two compact ANR fibrations $i = 1, 2$ and $E_1 \xrightarrow{t} E_2$ a fibrewise cell like map $(\pi_2 \circ t = \pi_1)$ then $E_1 \to B$ and $E_2 \to B$ are concordant. Precisely if M_t denotes the mapping cylinder of $t: M_t =$

$= E_1 \times [0,1] \bigcup_{t:E_1 \times 0 \to E_2} E_2$ and $p: M_t \to B \times [0,1]$ is defined by $p(e_1, t) = (\pi_1(e_1), t)$ for $t < 1$ and $p(e_2) = \pi(e_2)$, then p is a compact ANR fibration. In particular if $E \to B$ is a compact ANR fibration and $L \to E$ is a compact ANR fibration with contractible fibres then $L \to E \to B$ is a compact ANR fibration concordant to $E \to B$.

Proof of Theorem A(1): Let $E \xrightarrow{\pi} B$ be a compact ANR fibration. $E \times Q \to E \to B$ is a bundle with fibre a compact Q-manifold [6] which by J. West's theorem can be chosen to be $M^n \times Q$ for M^n a parallelizable compact manifold. By our previous observation it is concordant to $E \to B$ and by Theorem B(2) it is equivalent to $\tilde{\tilde{E}} \times Q \to \tilde{\tilde{E}} \to B$ where $\tilde{\tilde{E}} \to B$ is a bundle with fibre $M \times I^k$ for k big enough (because for M stable parallelizable $B\,\mathbb{H}\mathrm{omeo}(M) \to B\,\mathrm{Homeo}(M \times Q)$ is homotopically surjective).

Proof of Theorem A(2): If $L \to B$ is a bundle, with fibre homeomorphic to $M \times D^n$, concordant to $E \to B$ then the tangent bundle along the fibre, ξ, is the topological euclidean bundle which makes the fibre of $E \to B$ realizable by the manifold M. Conversely, assume that the compact manifold M realizes the fibre of $E \to B$; let $N(M)$ be a regular neighborhood of M embedded in a big euclidean space. By Theorem A(1) we can choose $L \to B$ so that the fibre will be homeomorphic to $N(M)$. Regarding ξ as a bundle over L one can take the disc bundle of ξ, $\tilde{L} \to B$ which is concordant to $L \to B$. If ξ is of big enough rank (eventually after replacing ξ by $\xi \oplus \varepsilon^k$ with ε^k the trivial bundle of rank k and k big enough), the fibre of $\tilde{L} \to B$ is clearly $M \times D^N$ for some N.

To prove Theorem A(3) we need the following:

Proposition II(1): *Let $E_i \to K$, $i = 1, 2$ be two bundles over a compact ANR K with fibres the compact manifold M^n; assume they are concordant as compact ANR fibrations. Then there exists the disc bundles $L_i \to E_i$ so that the compositions (which are bundles over K) $L_1 \to E_1 \to K$ and $L_2 \to E_2 \to K$ are isomorphic.*

Proposition II(2): (1) *if $E \to K$ is a bundle over the compact ANR K with fibres the compact manifold M^n, then there exists a bundle $L \to E$ so that the bundle $L \to E \to K$ has compact parallelizable manifolds as fibres;*

(2) *if $E_i \to K$ are two bundles over a compact ANR K, with fibres compact manifolds M_i^n, $i = 1, 2$ and M_1^n homotopy equivalent to M_2^n, then there exists the disc bundles $L_i \to E_i$ so that the bundles $L_i \to K$ have homeomorphic fibres.*

Proof of Theorem A(2): If $E_i \to K$, $i = 1,2$ are two bundles over a compact ANR K, which are concordant as compact ANR fibrations then there exists the disc bundles $L_i^1 \to E_i$ so that $L_i^1 \to K$ are concordant and have homeomorphic fibres by Proposition II.2 and then the disc bundles $L_i \to L_i^1$ so that $L_i \to K$ are isomorphic. This proves Theorem A(2).

Proof of Theorem II(1): Let us denote by $\mathcal{B}(K;M)$ the simplicial set of bundles over K (K is assumed to be a polyhedron) with fibres homeomorphic to M. A k-simplex is therefore a bundle over $K \times \Delta[k]$ with fibres homeomorphic to M. The restrictions to $K \times d_i\Delta[k]$ and the projections $\sigma_i : K \times \Delta[k] \to K \times \Delta[k-1]$ induce face and degeneracy operators. Let $\mathbb{B}(K;M) = \varinjlim \mathcal{B}(K;M \times I^r)$ and $i : \mathbb{B}(K,M) \subsetneq \mathcal{B}(K; M \times Q)$ be the inclusion. For a bundle $\xi : E \to K$ with fibre homeomorphic to $M \times I^s$ hence representing a 0-simplex in $\mathbb{B}(K;M)$, we consider ${}^{\xi}\mathcal{B}'(E;I^k)$ the simplicial subset of $\mathcal{B}(E;I^k)$ consisting of those simplexes represented by bundles which restrict to trivial bundle on each fibre of E. We define ${}^{\xi}\mathbb{B}'(E,pt) = \lim\limits_{s\to\infty} {}^{\xi}\mathcal{B}'(E;I^s)$. Clearly we have the diagram

(D)
$$\begin{tikzcd} {}^{\xi}\mathbb{B}'(E;pt) \arrow[r, "j"] & \mathbb{B}(K;M) \arrow[r, "i"] & \mathcal{B}(K;M \times Q) \\ \mathrm{Maps}'(E, B\lim \mathscr{D}_r) \arrow[u] & \mathrm{Maps}(K, B\,\mathbb{H}\mathrm{omeo}(M)) \arrow[u] \arrow[r] & \mathrm{Maps}(K, B\,\mathrm{Homeo}(M \times Q)) \arrow[u] \end{tikzcd}$$

where $\mathscr{D}_r = \mathrm{Homeo}(D^r, 0)$ and $\mathrm{Maps}'(E, B\lim \mathscr{D}_r)$ is the subcomplex of $\mathrm{Maps}(E, B\lim \mathscr{D}_r)$ whose simplexes are represented by maps which restricted to the fibres of ξ are homotopic to the constant map. All vertical arrows are obviously homotopy equivalences. The inclusion j is defined as follows: If $L \to E \times \Delta[k]$ is a k-simplex in ${}^{\xi}\mathbb{B}'(E,pt)$, we consider the composition $L \to E \times \Delta[k] \xrightarrow{\xi x id} K \times \Delta[k]$ which is clearly a k-simplex in $\mathbb{B}(K;M)$. We will check that the homotopy theoretic fibre of i over the 0-simplex $i(\xi) \in \mathcal{B}(K, M \times Q)_0$ is homotopy equivalent to ${}^{\xi}\mathbb{B}'(E,pt)$ or equivalently that (*) is a fibration up to homotopy. For this purpose let us consider the assignments $K' \Rightarrow \mathbb{B}(K',M)$, $K' \Rightarrow \mathcal{B}(K', M \times Q)$ and $K' \Rightarrow {}^{\xi/K'}\mathbb{B}(E/K',pt)$ which can be regarded as functors from the category of subpolyhedra K' of K to simplicial spaces; they satisfy the Meyer–Victoris condition, which means that the commutative diagram (and the analogous diagrams for $\mathbb{B}(\ldots;M)$ $\mathcal{B}(\ldots;M \times Q)$ ${}^{\xi/\cdots}\mathbb{B}'(E/\ldots;pt)$)

$$\begin{tikzcd}
\mathbb{B}(K_i; M) \times \mathbb{B}(K_2; M) \arrow[r, "r_1 \times r_2"] & \mathbb{B}(K_1 \cap K_2; M) \times \mathbb{B}(K_1 \cap K_2 : M) \\
\mathbb{B}(K_1 \cup K_2 : M) \arrow[u, "s_{12}"] \arrow[r, "r_{12}"] & \mathbb{B}(K_1 \cap K_2 : M) \arrow[u, "\Delta"]
\end{tikzcd}$$

is homotopy cartesian; here r_1, r_2, r_{12} are the maps induced by restriction to $K_1 \cap K_2$. This happens because the vertical arrows in diagram (D) are homotopy equivalences and the Mayer–Vietoris property holds for $K' \Rightarrow \text{Maps}'(E/k, B \lim \mathscr{D}_r)$, $K' \Rightarrow \text{Maps}(K', B\,\mathbb{H}\text{omeo}(M)$ and $K' \Rightarrow \text{Maps}(K, B\,\text{Homeo}(M \times Q))$. On the other side if E/K' is trivial in the diagram (D) one can add the dotted arrow and form a homotopy commutative diagram

$$\begin{tikzcd}
{}^{\xi}\mathbb{B}'(E/K', pt) \arrow[r] & \mathbb{B}(K' : M) \arrow[r] & \mathscr{B}(K' : M \times Q) \\
\text{Maps}'(E/K', B\mathscr{D}) \arrow[u] & & \\
\text{Maps}(K', \text{Maps}(M, B\,\text{Top})) \arrow[u, "i_1"] \arrow[r, dashed] & \text{Maps}(K', B\,\mathbb{H}\text{omeo}(M)) \arrow[uu] \arrow[r] & \text{Maps}(K', B\,\text{Homeo}(M \times Q)) \arrow[uu]
\end{tikzcd}$$

hence ${}^{\xi}\mathbb{B}'(E/K', pt) \to \mathbb{B}(K' : M) \to \mathscr{B}(K : M \times Q)$ is a fibration up to homotopy. We leave the reader to check inductively (using handles or the simplexes of K) that ${}^{\xi}\mathbb{B}'(E : pt) \xrightarrow{j} \mathbb{B}(K, M) \xrightarrow{i} \mathscr{B}(K, M \times Q)$ is a fibration up to homotopy. Q.e.d.

Proof of Proposition II(2): (1) Let $E \to K$ be a bundle with fibre the compact manifold M^n. Let τ' be the tangent microbundle along the fibres which might be regarded as a disc bundle (after enough stabilization). It is given by $E \xrightarrow{\Delta} E \times_K E \xrightarrow{p_j} E$. Let $\eta : L \to E$ be a disc bundle so that $\tau' \oplus \eta$ is a trivial disc bundle; η clearly exists since E is compact. $L \to E \to K$ is a bundle whose fibres are parallelizable manifolds.

(2) Let $E_i \to K$, $i = 1, 2$ as in Proposition II(2). By Proposition II(1) one can assume the fibres are parallelizable. Then if we take r big enough $E_i \times D^r \to K$, $i = 1, 2$ have homeomorphic fibres because two compact parallelizable manifolds of the same homotopy type become homeomorphic after product with D^r for r big enough. Q.e.d.

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