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A CHARACTERIZATION OF THE SPHERICALLY COMPLETE NORMED SPACES WITH A DISTINGUISHED BASIS

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The theory of normed spaces over a trivially valued field (or valued spaces) was developed mainly by P. Robert in his series of papers [3]. He introduced the concept of distinguished basis, also called orthogonal bases in the literature, and in order to deal with spaces that possess distinguished bases, he restricted himself to V-spaces ([3], p. 16), that is, complete valued spaces E such that

\[ \|E\| = \{\|x\| : x \in E\} \subset \{0\} \cup \{\rho^n : n \in \mathbb{Z}\} \]

for some real number \( \rho > 1 \). K.-W. Yang, [5], has given a different proof of the fact that V-spaces have a distinguished basis. All V-spaces are easily shown to be spherically complete.

In this note we give a characterization of all valued spaces which are spherically complete and have a distinguished basis. These spaces need not be V-spaces. Moreover, we answer a question of Robert ([3], p. 8), by giving examples of valued spaces without a distinguished basis.

For notations, we refer to [3] and [4].

THEOREM: Let E be a complete valued space over a field K (i.e., a non-archimedean Banach space over a field with the trivial valuation). Then, the following are equivalent:

(i) E has a distinguished (or orthogonal) basis, and it is spherically complete.

(ii) Every strictly decreasing sequence in \( \|E\| \) converges to zero.

PROOF: Assume (ii). Let \( X \subset E \) be a maximal orthogonal subset of E ([3], p. 9). It is very easy to prove that our hypothesis (ii) implies the
closed linear span of \(X, [X]\), is spherically complete. Then by Ingleton's Theorem ([4], Ex. 4.H; the proof also works when \(K\) is trivially valued), if \([X] \neq \cdot E\), there is a linear projection \(P : E \to [X]\) of norm one, and for any \(z \in E \setminus [X]\), \(z - Pz\) is orthogonal to \([X]\) and different from zero, contradicting the maximality of \(X\).

Conversely, assume \(E\) has a distinguished basis \(X\) and is spherically complete, and that there is a sequence in \(\|E\|\) strictly decreasing and bounded away from zero. Since for every nonzero element of \(E\) there is some basic vector with the same norm, there must exist a sequence \((x_n)\) in \(X\) with strictly decreasing norms but not convergent to zero.

Call \(F\) the closed vector subspace \([x_n : n \in \mathbb{N}]\). Then \(F\) is linearly isometric to the quotient of \(E\) by the subspace generated by the other members of \(X\), hence it must be spherically complete (Cf. [4], Th. 4.2). But it is not: consider the sequence of closed balls

\[
B(x_1 + \ldots + x_n, \|x_n\|), \quad n \in \mathbb{N}.
\]

**Remarks:** (1) For non-archimedean Banach spaces over a non-trivially valued field, the same is true: a proof can be found in [4], Th. 5.16. That proof also works in our setting, but it is much more elaborated than the one given above; our proof is also valid when the valuation is not trivial, with a minor modification: in that case one cannot be sure that the set of norm values of a basis is the same as \(\|E\| \setminus \{0\}\), and one has to change \((x_n)\) into \((\lambda_n x_n)\) for suitable \(\lambda_n \in K\).

(2) It is not difficult to prove that a valued space is spherically complete and has a distinguished basis if and only if it is linearly isometric with a space \(c_0(I : s)\) defined as the set

\[
\{x : I \to K | \|x(i)\| s(i) \to 0 \text{ for the Frechet filter on } I\}
\]

(where \(I\) is any nonempty set) endowed with the norm

\[
\|x\|_s = \max \{s(i) | x(i) \neq 0\}
\]

where \(s : I \to [0, + \infty)\) is a function whose range does not contain any strictly decreasing sequence with a positive limit.

Consequently, one can give examples of valued spaces with a distinguished basis, apart from \(V\)-spaces.

(3) Now we can produce several examples of valued spaces without a distinguished basis:

(a) Over the real field: the fields \(pR\) introduced by A. Robinson, regarded as valued spaces over \(R\) (trivially valued), are spherically complete (see [1]), and have \(\|pR\| = [0, + \infty)\).
(b) Over any field $K$: the field $E$ of formal power series with coefficients in $K$ and rational exponents, with the set of exponents relative to nonzero coefficients well-ordered is spherically complete ([2], p. 38), and has $\|E\|$ dense in $[0, +\infty)$.

REFERENCES


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