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ON EQUIVARIANT FINITENESS

Sławomir Kwasik

1. Introduction

Let G be a finite group. Consider the class of CW complexes with a G-action which are equivariantly dominated by a finite complexes. In [1] J.A. Baglivo has defined an algebraic invariant to decide when a space in this class (under some restriction on the action of G) is equivariantly homotopy equivalent to a finite complex. The purpose of this paper is to:

(1) develop the equivariant finiteness obstruction from a more geometrical point of view;

(2) extend this obstruction theory in two directions:

(a) with respect to the class of spaces – all spaces which are equivariantly dominated by a finite complex, without restriction on the action of G,

(b) with respect to the class of groups – G arbitrary compact Lie group;

(3) compare obstruction with the algebraic Baglivo obstruction;

(4) show that in the case of G-space with simply connected fixed point sets and with $G = T^n$, $n \geq 1$, n-dimensional torus every such obstruction vanishes.

Observe that (4) shows that the action of a connected and nonconnected compact, abelian Lie group is completely different from the equivariant finiteness obstruction point of view.

I wish to thank the referee for pointing out an error in an earlier draft of this paper.

2. Definitions and notations

In this section we recall some notions which will be used in this paper.

Let G be a compact Lie group. By a G-space X we mean a space X with a given action $\theta: X \times G \to X$ of a group G on X; we will denote $\theta(x, g)$ simply by $g(x)$.

A map $f: X \to Y$ between two G-spaces is called equivariant (G-map) if $gf(x) = fg(x)$ for every $g \in G$, $x \in X$.

A subset $A \subset X$ of a G-space X is called a G-subset if $g(A) \subset A$ for every $g \in G$.

By $p: X \to X/G$ we will denote the natural projection on the orbit space.

DEFINITION 2.1: A G–CW complex is a G-space X with a decomposition

$$X = \lim_{\to} X^n, \text{ where } X^0 = \bigcup_{j \in A_0} G/H_j,$$

$$X^{n+1} = X^n \bigcup_F \Big(\bigcup_{j \in A_n} G/H_j \times D^n \Big),$$

for some G-map $F: \bigcup_{j \in A_n} G/H_j \times S^n \to X^n$ and $\{H_j\}_{j \in A_n}$ a collection of closed subgroups of G (comp. [7]). As in the nonequivariant case we have a natural notion of a cellular G-map between G–CW complexes. Observe that if $f: X \to Y$ is a cellular G-map between two G–CW complexes then the mapping cylinder M_f of the G-map f is a G–CW complex and Y is a G-deformation retract of M_f. Let X be a G-space and let $x \in X$ be a point. By $G_x = \{g \in G \mid g(x) = x\}$ we will denote the isotropy subgroup of G at x. Let $H \subseteq G$ be a subgroup of G. There are the following natural subspaces of X:

$$X^H = \{x \in X \;\; H \subset G_x\}; \; \bar{X}^H = \{x \in X \;\; H \subset G_x \text{ and } H = G_x\};$$

$$X_H = \{x \in X \;\; H = G_x\}.$$

If we denote by NH the normalizer of H in G then these spaces are in a natural manner NH/H-spaces. We will use the following special type of G–CW complex.

DEFINITION 2.2: A G-simplicial complex is a G-space X such that X

has a given G–CW decomposition such that the orbit space X/G is a simplicial complex under this structure.

For G-simplicial complexes we can define stars, links and so on by taking inverse images (under projection) from the orbit space.

3. Equivariant Whitehead torsion and obstruction to finiteness

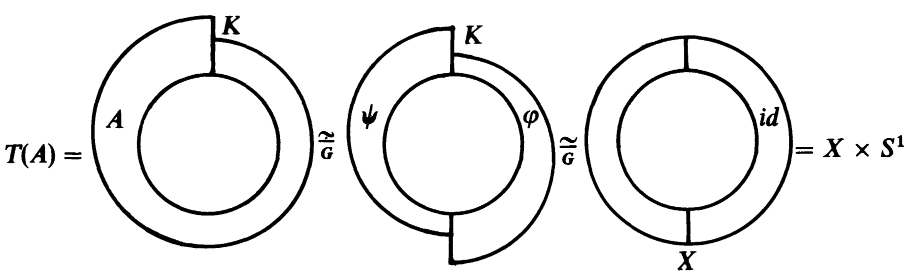
In this section we define the obstruction to equivariant finiteness and deduce its fundamental properties. Next we give the algebraic interpretation of this obstruction and compare it with the Baglivo invariant.

Our construction is based on [5] and the proofs in the equivariant case are modifications of the proofs in [5] so we only give sketches of proofs. We will assume some familiarity with simple homotopy theory and its equivariant version as for example in [4] and [7]. In [7] was defined the equivariant Whitehead group $Wh_G(X)$ of X using equivalences of G-deformation retracts $i:[X \to V]$ of finite G–CW complexes.

We will consider a more general situation, namely X will be a G-space which is only equivariantly dominated by a finite G–CW complex so we will use the definition of $Wh_G(X)$ modified as in [6].

LEMMA 3.1: *Let X be a G-space which is equivariantly dominated by a finite G–CW complex. Then the G-space $X \times S^1$ (trivial action of G on S^1) has the equivariant homotopy type of a finite G–CW complex.*

PROOF: Let K be a finite G–CW complex which G-dominates X i.e. there exist G-maps $\varphi: X \to K$, $\psi: K \to X$ such that $\psi\varphi \underset{G}{\simeq} id_X$, where $\underset{G}{\simeq}$ means G-homotopic. Let $A = \varphi\psi: K \to K$. Then A is a G-map and we may assume (see [7] p. 9) that A is cellular up to G-homotopy. Now denote by $T(A)$ the space obtained from the mapping cylinder M_A by identification of the top and bottom of M_A using the identity map. The space $T(A)$ is a finite G–CW complex and has the G-homotopy type of $X \times S^1$. This is evident (comp. [5]) from elementary properties of a mapping cylinder of a G-map and the following picture



DEFINITION 3.2: Let $B: X \times S^1 \to X \times S^1$ be the equivariant homeomorphism given by $B(x, s) = (x, -s)$, where $s \to -s$ is the complex conjugation. Let $\lambda: T(A) \to X \times S^1$ be the G-homotopy equivalence of Lemma 3.1. Denote by $\tau(\lambda^{-1}B\lambda) \in Wh_G(T(A))$ the torsion of the G-homotopy equivalence $\lambda^{-1}B\lambda: T(A) \to T(A)$. Define $O_G(X) \in Wh_G(X \times S^1)$ as $O_G(X) = \lambda_*(\tau(\lambda^{-1}B\lambda))$.

LEMMA 3.3: *The obstruction $O_G(X)$ is well defined i.e. it does not depend on φ, ψ or K.*

The proof of Lemma 3.3 *is strictly analogous to the nonequivariant case* (*see* [5]).

THEOREM 3.4: $O_G(X) = 0$ *if and only if X has the equivariant homotopy type of a finite G–CW complex.*

PROOF: $\Leftarrow$ follows easily from the nonequivariant version and from elementary properties of the Wh_G functor.

$\Rightarrow$ To prove this we will construct a G-simplicial complex W and two G-maps $f_1: W \to T(A)$, $f_2: W \to T(A)$ with contractible point inverses that the following diagram is commutative up to G-homotopy:

$$\begin{array}{ccc} & W & \\ {\scriptstyle f_1}\swarrow & & \searrow{\scriptstyle f_2} \\ T(A) & \xrightarrow[\lambda^{-1}B\lambda]{} & T(A) \end{array}$$

First observe that combining [4, 7.2] and [12, Lemma 4.3] we can prove that every finite G–CW complex has the G-simple homotopy type of a finite G-simplicial complex.

Now if $O_G(X) = 0$ then $\tau(\lambda^{-1}B\lambda) = 0$ and this means that we can go from $M_{\lambda^{-1}B\lambda}$ to $T(A)$ by a sequence of equivariant expansions and collapses. By a remark which we made before we can assume $M_{\lambda^{-1}B\lambda}$ and $T(A)$ are G-simplicial complexes.

Now embed $M_{\lambda^{-1}B\lambda}$ in some euclidean space R^n with an orthogonal action of G (this is possible because $M_{\lambda^{-1}B\lambda}$ is a compact, metric space with a finite number of orbit types).

Let N_1, N_2 be regular neighbourhoods of $M_{\lambda^{-1}B\lambda}/G$, $T(A)/G$ respectively in the simplicial complex R^n/G. If we look at the orbit spaces then the elementary equivariant expansion and collapse from $M_{\lambda^{-1}B\lambda}$ to $T(A)$ corresponds to ordinary simplicial expansion and collapse from $M_{\lambda^{-1}B\lambda}/G$ to $T(A)/G$.

From [10] we infer that the regular neighbourhoods N_1, N_2 are homotopy equivalent. Let $W_1 = p^{-1}(N_1)$ and $W_2 = p^{-1}(N_2)$, where $p: R^n \to R^n/G$ is the natural projection. The spaces W_1, W_2 are equivariantly equivalent G-simplicial complexes. Let $h: W_1 \to W_2$ be an equivariant homotopy equivalence. We define $W = W_2$. The G-map $f_1: W \to T(A)$ is given by the collapsing of W_2 on $T(A)$; $f_2: W \to T(A)$ is given by the composition r coll h^{-1}, where coll is the collapsing from W_1 on $M_{\lambda^{-1}B\lambda}$ and $r: M_{\lambda^{-1}B\lambda} \to T(A)$ is a standard retraction on the bottom of $M_{\lambda^{-1}B\lambda}$. It is easy to see that the diagram

$$\begin{array}{ccc} & W & \\ {}^{f_1}\swarrow & & \searrow^{f_2} \\ T(A) & \xrightarrow[\lambda^{-1}B\lambda]{} & T(A) \end{array}$$

commutes up to G-homotopy. The rest of our proof is analogous to that in [5] hence is omitted.

From Theorem 3.4, the Sum Theorem for equivariant Whitehead torsion and Props. (2.4), (2.5) in [5] follows:

COROLLARY 3.5: *If $X = X_1 \cup X_2$ with $X_0 = X_1 \cap X_2$ and each X_j, $j = 0, 1, 2$ is a finitely, equivariantly dominated G-CW complex so is X, and*

$$O_G(X) = i_{1*}O_G(X_1) + i_{2*}O_G(X_2) - i_{0*}O_G(X_0),$$

where $i_j: X_j \to X$, $j = 0, 1, 2$ are the natural inclusions.

COROLLARY 3.6: *If X and Y are G-spaces dominated by finite G-CW complexes and $f: X \to Y$ is a G-homotopy equivalence then $f_* O_G(X) = O_G(Y)$.*

REMARK 3.7: The Sum Theorem for equivariant Whitehead torsion formulated in [5] for a finite G–CW complex extends naturally to the case of an arbitrary G-space.

Now we describe the obstruction $O_G(X)$ from a more algebraic point of view. Let G be a group. By $E(G)$ we denote the total space of a universal G-bundle. Observe that the space $E(G) \times X$ is in a natural manner a G-space. Recall the following:

THEOREM 3.8. (Th. Iv. 1 in [6]): *There exists a natural isomorphism:*

$$Wh_G(X) \overset{J}{\approx} \prod_{H \subseteq G} Wh((E(NH/H) \times X^H)/NH/H)$$

In our situation we have the decomposition:

$$(*) \quad Wh_G(X \times S^1) \overset{J}{\approx} \prod_{H \subseteq G} Wh(((E(NH/H) \times X^H)/NH/H) \times S^1)$$

On the other hand there is the following decomposition of the functor Wh (see [2] *and* [3]*):*

$$(**) \quad Wh\pi_1(Y \times S^1) \approx Wh\pi_1(Y) \oplus \tilde{K}_0 Z(\pi_1(Y)) \oplus Nil\ term$$

hence (*) *and* (**) *yield:*

$$(***) \quad Wh_G\pi_1(X \times S^1) \approx \prod_{H \subseteq G} Wh\pi_1((E(NH/H) \times X^H)/NH/H) \oplus$$

$$\oplus \prod_{H \subseteq G} \tilde{K}_0 Z(\pi_1((E(NH/H) \times X^H)/NH/H)) \oplus Nil\ terms.$$

Recall that the obstruction $O_G(X)$ in $Wh_G(X \times S^1)$ is given by:

$$O_G(X) = \lambda_*(\tau(\lambda^{-1}B\lambda)) = \lambda_*[M_{\lambda^{-1}B\lambda}, T(A)] =$$

$$= [M_{\lambda^{-1}B\lambda} \bigcup_{T(A)} M_{\lambda^{-1}B\lambda}, X \times S^1] \in Wh_G(X \times S^1).$$

Now assume G is a finite group and X is a G-space such that for every $H \subseteq G$, X^H is connected and $X^G \neq \emptyset$. There is the following description of elements of $Wh_G(X)$ (see [6]).

Let $[V, X] \in Wh_G(X)$. For every subgroup $H \subseteq G$ consider a universal covering $p: \widetilde{V^H} \to V^H$. Let $\widetilde{X^H \cup \bar{V}^H}$ be a subcovering of $\widetilde{V^H}$ which corresponds to $X^H \cup \bar{V}^H \subset V^H$. Consider the cellular chain complex $C_*(\widetilde{V^H}, \widetilde{X^H \cup \bar{V}^H})$. There are natural cellular actions of the group $\pi_1(X^H, x) = \pi_1(V^H, x)$, $x \in X^G$ and of the group NH/H on this chain complex. Denote by πH the semidirect product of $\pi_1(X^H, x)$ and NH/H (note that πH is no other than $\pi_1((E(NH/H) \times X^H)/NH/H)$. Now the chain complex $C_*(\widetilde{V^H}, \widetilde{X^H \cup \bar{V}^H})$ is a πH complex and the lifting of cells from $V^H - \bar{V}^H = V_H$ gives a preferred base in this complex. We refer to

$$\tau(C_*(\widetilde{V^H}, \widetilde{X^H \cup \bar{V}^H})) \in Wh\pi_1((E(NH/H) \times X^H)/NH/H)$$

as the algebraic Whitehead torsion of this cellular πH complex. The obstruction $O_G(X)$ under the isomorphism J from Th. 3.8 splits as follows:

$$O_G(X) = [M_{\lambda^{-1}B\lambda} \bigcup_{T(A)} M_{\lambda^{-1}B\lambda}, X \times S^1] =$$

$$= \prod_{H \subseteq G} \tau(C_*(\overbrace{M^H_{\lambda^{-1}B\lambda} \bigcup_{T(A)^H} M^H_{\lambda^{-1}B\lambda},}$$

$$\overbrace{X^H \times S^1 \cup \bar{M}^H_{\lambda^{-1}B\lambda} \bigcup_{T(A)^H} \bar{M}^H_{\lambda^{-1}B\lambda}}) =$$

$$= \prod_{H \subseteq G} \tau(C_*(\overbrace{M_{\lambda^{-1}H_BH_\lambda H} \bigcup_{T(A^H)} M_{\lambda^{-1}H_BH_\lambda H},}$$

$$\overbrace{X^H \times S^1 \cup \bar{M}_{\lambda^{-1}H_BH_\lambda H} \bigcup_{\overline{T(A^H)}} \bar{M}_{\lambda^{-1}H_BH_\lambda H}})) \in$$

$$\in \prod_{H \subseteq G} Wh(((E(NH/H) \times X^H)/NH/H) \times S^1),$$

where $T(A^H)$ is the mapping torus of a map $A^H : K^H \to K^H$. But we know from [3] p. 1339–1340 that each element of the last sum is in the relevant summand $\tilde{K}_0 Z(\pi H)$ under the decomposition (***), so we have the following.

THEOREM 3.9: *The equivariant finiteness obstruction $O_G(X)$ has the following representation:*

$$O_G(X) = \prod_{H \subseteq G} O_H(X^H), \text{ where } O_H(X^H) \in \tilde{K}_0 Z(\pi H).$$

In particular from Theorem 3.9 follows:

COROLLARY 3.10: *If for every $H \subseteq G$, $\pi_1(X^H, x) = 0$ then X has the equivariant homotopy type of a finite G-CW complex if and only if the obstruction $O_G(X) = \prod_{H \subseteq G} O_H(X^H) \in \prod_{H \subseteq G} \tilde{K}_0 Z(NH/H)$ vanishes.*

The obstruction $O_G(X) = \prod_{H \subseteq G} O_H(X^H) \in \prod_{H \subseteq G} \tilde{K}_0 Z(\pi H)$ is precisely the Baglivo obstruction. We recall that the Baglivo obstruction is defined as follows:

Assume that G is a finite group and X is a G–CW complex such that for every $H \subseteq G$, X^H is connected and $X^G \neq \emptyset$. Let for each $H \subseteq G$

$p: \widetilde{X^H} \to X^H$ be a universal covering and write $X^{>H} = \cup X^{H'}$, where the union is over all subgroups H' such that $H \subseteq H'$ and $H \neq H'$.

DEFINITION 3.11 (Baglivo): The equivariant finiteness obstruction $E_1(X, G)$ is given by:

$$E_1(X,G) = \prod_{H \subseteq G} w_H(X) \in \prod_{H \subseteq G} \tilde{K}_0 Z(\pi H),$$

where $w_H(X) = w(C_*(\widetilde{X^H}, p_H^{-1}(X^{>H}))$ and $w(C_*(\cdot))$ is an ordinary Wall obstruction (see [11]).

It is not hard to see that each $O_H(X^H)$ is given by:

$$O_H(X^H) = w(C_*(\widetilde{X^H}, p_H^{-1}(\bar{X}^H))$$

which shows that $O_G(X)$ is the same as $E_1(X, G)$.

REMARK 3.12: The representation $O_G(X) = \prod_{H \subseteq G} O_H(X^H) \in \sum_{H \subseteq G} \tilde{K}_0 Z(\pi H)$ remains valid under weaker assumptions than connectedness of X^H and $x^G \neq \emptyset$; we need only assume that the action of NH/H on X^H leaves on each component X_i^H of X^H some point $x_i \in X_i^H$ invariant.

4. Equivariant finiteness obstruction

Let X be a G-space, where $G = T^n$ is the n-dimensional torus, $n \geq 1$. In this section we prove that if $\pi_1(X^H) = 0$ for every closed subgroup $H \subseteq T^n$ then the obstruction $O_{T^n}(X) \in Wh_{T^n}(X \times S^1)$ vanishes and X has the equivariant homotopy type of a finite T^n–CW complex. Namely we have the following:

THEOREM 4.1: *Let X be a T^n-space which is equivariantly dominated by a finite T^n-CW complex. If for every closed subgroup $H \subseteq T^n$ $\pi_1(X^H) = 0$ then X has the equivariant homotopy type of a finite T^n–CW complex.*

PROOF: We will show that the equivariant finiteness obstruction $O_{T^n}(X) \in Wh_{T^n}(X \times S^1)$ vanishes. To see it we recall the decomposition (*) of the functor Wh_G

$$(*) \quad Wh_G(X \times S^1) \overset{J}{\approx} \prod_{H \subseteq G} Wh(((E(NH/H) \times X^H)/NH/H) \times S^1)$$

Since $G = T^n$ then for every closed subgroup $H \subseteq G$ $NH = T^n$ and NH/H is connected. The trivial G-fibration $E(NH/H) \times X^H \to E(NH/H)$ induces the following fibration $(E(NH/H) \times X^H)/NH/H \to B(NH/H)$. Because $\pi_1(X^H) = 0$ we infer

$$\pi_1((E(NH/H) \times X^H)/NH/H) \approx \pi_1(B(NH/H))$$

and $\pi_1(B(NH/H)) = 0$ (since $E(NH/H)$ is contractible). Therefore for every closed subgroup $H \subseteq T^n$ we obtain:

$$Wh\pi_1(((E(NH/H) \times X^H)/NH/H) \times S^1) = Wh\pi_1(S^1) = WhZ = 0 \text{ so } Wh_G(X \times S^1) = 0.$$

This obviously implies $O_{T^n}(X) = 0$ and by Theorem 3.4 X has the equivariant homotopy type of a finite T^n–CW complex.

Remark: In the proof of Theorem 4.1 the connectivity of X^H was assumed. The proof of general case follows easily.

Corollary 4.2: *Let G be a compact, connected Lie group and let X be a G-space with a semi-free action of G. If X is equivariantly dominated by a finite G–CW complex and if $\pi_1(X) = \pi_1(X^G) = 0$ then X has the equivariant homotopy type of a finite G–CW complex.*

Corollary 4.3: *Let X be a compact, metric T^n–ANR (compact, locally smooth T^n-manifold). If for every closed subgroup $H \subseteq T^n$ $\pi_1(X^H) = 0$ then X has the equivariant homotopy type of a finite T^n–CW complex.*

Proof: Every compact, metric G–ANR is equivariantly dominated by a finite G–CW complex (it follows from [8]) so Corollary 4.3 is a consequence of Theorem 4.1.

When G is a finite group then every compact, locally smooth G-manifold has the equivariant homotopy type of a finite dimensional G–CW complex (see [9]). But the following problem remains open:

Conjecture: Every compact, metric G–ANR (compact, locally smooth G-manifold) has the equivariant homotopy type of a finite G–CW complex.(*)

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(*) Note added in proof

The negative answer was given by Frank Quinn in the paper Ends of Maps II, *Invent. Math. 68* (1982) 353–424.

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