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ARCS IN THE HILBERT CUBE (S^n) WHOSE COMPLEMENTS HAVE DIFFERENT FUNDAMENTAL GROUPS

R.J. Daverman[1] and S. Singh

Abstract

A collection of arcs $\{A_i\}_i$ in a space X is called π_1-*distinct* if and only if $\pi_1(X - A_i)$ is different from $\pi_1(X - A_j)$ for any two (distinct) arcs A_i and A_j. We prove that there exists an uncountable collection of π_1-distinct arcs in the Hilbert cube and in the n-sphere with $n \geq 5$. As an immediate application, we obtain uncountable collections of nonhomeomorphic: (a) generalized n-manifolds with $n \geq 5$, (b) various open manifolds, and (c) infinite dimensional compact absolute retracts. Furthermore, we obtain uncountable collections of nonhomeomorphic infinite dimensional compact absolute retracts and generalized n-manifolds with $n \geq 5$ each of which has trivial group of self-homeomorphisms.

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1. Introduction

The main concern of this note is wild embeddings of the closed interval [0, 1] (arcs) in the n-sphere S^n and the Hilbert cube Q. A collection of arcs $\{A_i\}_i$ in a space X is called π_1-*distinct* if and only if the fundamental group $\pi_1(X - A_i)$ is different from $\pi_1(X - A_j)$ for any two (distinct) arcs A_i and A_j. We study the following general problem:

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(1.1) Problem: *Does there exist an uncountable collection of π_1-distinct arcs in S^n, with $n \geq 3$, or in Q?*

We provide an affirmative answer to this problem for S^n with $n \geq 5$ and for Q. Our methods do not apply to handle the cases $n = 3$ and 4. There are many sporadic results concerning the fundamental groups of the complements of arcs in S^3, starting with the classic work of Artin and Fox [AF], but these groups are rather cumbersome. The same is true for known examples of wild arcs in S^n with $n \geq 4$ and in Q; see [BR] and [WO] for related references. We emphasize that it is easy to construct countable collections of π_1-distinct arcs in Q or S^n with $n \geq 3$; however, the transition from a countable such collection to an uncountable collection is not an easy one because of the algebraic difficulties.

We make full use of the fact that "the double-suspension of a homology sphere is a sphere" [C1, C2]; furthermore, in the case of Q we use the main result of [D1], which, in turn, depends on [DW] and [T], and the results of [T] depend on Edwards' theorem (cf. [CH], see Ch. XIV). Although the techniques we employ are not elementary, the framework is conceptually rather simple, and it is our hope that the methods set forth will be useful to study other embeddings in Q by utilizing the fundamental group and a similar framework.

Some applications to cell-like decompositions of S^n or Q will also be discussed; furthermore, these results are applied in [S2] to extend the main results of [S1] to decompositions of Hilbert cube manifolds. Incidentally, a solution to Problem (1.1) is of independent interest in relation to the study of open manifolds.

2. Notation and terminology

By a *decomposition* of a space X we shall always mean an upper semicontinuous decomposition of X. Let Q denote the Hilbert cube and S^n denote the n-sphere. By a *disk* we mean any space homeomorphic to the (compact) 2-cell. A k-cell will always be compact unless otherwise stated. We denote by $\sum X$ and $\sum^2 X$, respectively, the suspension and the double-suspension over a space X. For the purposes of computing the fundamental groups, we often suppress the base points. We use the symbol "$\approx$" to indicate an isomorphism of groups or homeomorphisms of spaces. By an *uncountable* set we shall really mean a set whose cardinality equals that of the power set of the integers. Most of the terminology is standard, but the following references will be useful for further clarification of notation and terminology [C2, CH, D3, L].

3. Preliminaries

(3.1) *Double Suspension of Homology Spheres and Arcs.* The main result that we use here states that "the double-suspension of a homology sphere is a sphere" [C1]; see also the survey article of Cannon [C2] for related discussions. Suppose H^n is a homology n-sphere such that the fundamental group $\pi_1 H^n \neq 1$ (of course $n \geq 3$). Choose a PL n-cell B^n in H^n and a point x in the boundary ∂B^n of B^n. Then:

(3.1.1) The double-suspension $\sum^2 H^n$ is homeomorphic to S^{n+2};

(3.1.2) the double-suspension $\sum^2 B^n$ is an $(n+2)$-cell B^{n+2} in $\sum^2 H^n$; and

(3.1.3) the double-suspension $\sum^2 \{x\}$ is a disk D in the boundary of B^{n+2}.

In this setting, we apply a "disk squeezing technique" due to Daverman and Eaton [DE] to squeeze the disk D to an arc A. More specifically, there exists a decomposition G of $\sum^2 H^n$ such that: (a) the elements of G are arcs on D whose union is D minus two points and singletons otherwise, (b) the decomposition space $\sum^2 H^n/G$ is homeomorphic to S^{n+2}, i.e., the decomposition is shrinkable, and (c) the image of D under the projection map $\pi : \sum^2 H^n \to \sum^2 H^n/G$ is an arc A. Indeed it follows from [G1, Theorem 0.2] or from [C1, cf. pp. 83–84] that the decomposition G of $\sum^2 H^n$ whose nondegenerate elements are the arcs corresponding to $\sum\{x\} \times \{t\}$, $t \in (-1, 1)$, in $\sum^2 \{x\}$ does what we want.

We assume familiarity with many results concerning cell-like decompositions; see the survey article of Lacher [L]. We observe the following:

(3.1.4) $\pi_1(\sum^2 H^n/G - B') \approx \pi_1(\sum^2 H^n/G - A) \approx \pi_1(\sum^2 H^n - D) \approx \pi_1 H^n$ where B' is the image of B^{n+2} under π;

(3.1.5) the image of $\sum^2 H^n - \operatorname{Int} B^{n+2}$ under π is a crumpled $(n+2)$-cube C and the image of B^{n+2} under π is an $(n+2)$-cell and

(3.1.6) by the Siefert-Van Kampen Theorem (cf. [M]) $\pi_1(C - A) \approx \pi_1 H^n$.

The next proposition is an immediate consequence of discussions given above.

(3.1.7) PROPOSITION: *If H^n is a homology n-sphere with $\pi_1 H^n \neq 1$, then there exists a crumpled $(n+2)$-cube C contained in S^{n+2} and an* arc *A tamely embedded in the boundary of C such that the closure of $(S^{n+2} - C)$ is $(n+2)$-cell B, ∂C is locally flat modulo A, and $\pi_1(S^{n+2} - B) \approx \pi_1(S^{n+2} - A) \approx \pi_1(C - A) \approx \pi_1 H^n$.*

A *sewing* of two crumpled n-cubes C and D is a homeomorphism $h: \partial C \to \partial D$ between their boundaries; the *sewing space* $C \bigcup_h D$ of h is the quotiënt space obtained from the disjoint union of C and D by identifying each point x in ∂C to its image $h(x)$ in ∂D; our main reference concerning crumpled cubes is Daverman's survey article [D3]. The crumpled $(n+2)$-cube in Proposition (3.1.7) satisfies the hypotheses of the following proposition.

(3.1.8) PROPOSITION: *If C is a crumpled n-cube in S^n such that $n \geq 5$, ∂C is locally flat modulo an* arc *A in S^n, and A is tamely embedded in ∂C, then $C \bigcup_{\mathrm{Id}} C$ is homeomorphic to S^n.*

This proposition follows from Theorems (8B.6) and (8B.7) of [D3].

(3.2) *A Suspension Method.* Consider the triple (S^{n+2}, C, A) obtained from Proposition (3.1.7). Apply suspension to obtain $(\sum S^{n+2}, \sum C, \sum A)$. Put $S^{n+3} = \sum S^{n+2}$, $D = \sum C$ and $E = \sum A$. We consider S^{n+3} as the quotient space obtained from $S^{n+2} \times [-1, 1]$ by identifying $S^{n+2} \times \{-1\}$ to a point and $S^{n+2} \times \{1\}$ to another point; let $\phi: S^{n+2} \times [-1, 1] \to S^{n+3}$ denote the associated quotient map. Define a decomposition G of S^{n+3} into arcs $A_t = \phi(A \times \{t\})$, $-1 < t < 1$, and singletons. Let $p: S^{n+3} \to S^{n+3}/G$ denote the associated projection. The decomposition space $S^{n+3}/G \approx S^{n+3}$ (see [AC, BR]) and the image $P(E)$ of the disk E is an arc A' in S^{n+3}. We denote the crumpled $(n+3)$-cube $p(D)$ by C'. The next result follows from repeated application of the suspension method.

(3.2.1) PROPOSITION: *If H^n is a homology n-sphere with $\pi_1 H^n \neq 1$ and $k \geq 2$ is an integer, then there exist a crumpled $(n+k)$-cube C contained in S^{n+k} and an* arc *A tamely embedded in the boundary of C such that:* (a) *the closure of $(S^{n+k} - C)$ is an $(n+k)$-cell B,* (b) *∂C is locally flat modulo A, and* (c) *$\pi_1(S^{n+k} - B) \approx \pi_1(S^{n+k} - A) \approx \pi_1(C - A) \approx \pi_1 H^n$.*

(3.3) *Arcs and Homology Spheres.* We need the following definitions for convenience of reference and formalization of some notions.

(3.3.1) DEFINITION: A group G is called *indecomposable* if G is not a free product of two non-trivial groups.

(3.3.2) DEFINITION: A collection $\{H_i^n\}$ of homology n-spheres is called *admissible* if the fundamental group $\pi_1 H_i^n$ is indecomposable for all i, and the groups $\pi_1 H_i^n$ and $\pi_1 H_j^n$ are distinct whenever $i \neq j$.

(3.3.3) DEFINITION: A collection of arcs $\{A_i\}_i$ in a space X is said to be *π_1-distinct* if the groups $\pi_1(X - A_i)$ and $\pi_1(X - A_j)$ are distinct whenever $i \neq j$.

(3.3.4) DEFINITION: A collection of arcs $\{A_i\}_i$ in S^{n+k}, $k \geq 2$, is said to be *associated* with a collection $\{H_i^n\}_i$ of homology n-spheres if each arc A_i is constructed by first applying the method of (3.1) and then applying (3.2).

(3.3.5) Observe that a collection $\{A_i\}_i$ of arcs in S^{n+k}, $k \geq 2$, associated with an admissible collection $\{H_i^n\}_i$ of homology spheres is π_1-distinct.

(3.4) *A Countable Collection of Homology 3-Spheres.* For each $n \geq 1$, let Γ_n be a group given by the presentation

$$\langle a, b \mid b^{2n+2} = (ab)^{n+1} b (ab)^{n+1}, (ab)^{n+1} = [b(ab)^{n+1} a^{-1} (b^{-1} a^{-1})^{n+1}]^n \rangle.$$

For each $n \geq 1$, there exists a contractible 4-manifold W_n^4 whose boundary is a homology 3-sphere M_n^3 such that $\Gamma_n \approx \pi_1 M_n^3$; see [G2], pages 108–119, for this and the following facts:

(3.4.1) For each $n \geq 1$, the group Γ_n is indecomposable (and perfect); and

(3.4.2) the sequence of groups $\Gamma_2, \Gamma_4, \Gamma_6, \ldots$ contains infinitely many distinct groups, say, $\Gamma_{n(1)}, \Gamma_{n(2)}, \ldots$.

(3.4.3) NOTATION: For each $i \geq 1$, put $G_i = \Gamma_{n(i)}$ and $H_i^3 = M_{n(i)}^3$. Note that $\pi_1 H_i^3 \approx G_i$ for $i \geq 1$; thus $\{H_i^3\}_{i=1}^{\infty}$ is an admissible collection of homology 3-spheres.

(3.5) *A Countable Collection of Homology 4-Spheres.* For each $n \geq 2$, let Γ'_n denote the group given by the presentation

$$\langle a, b \,|\, a^{n-2} = (ab)^{n-1}, b^3 = (ba^{-2}ba^2)^2 \rangle.$$

The following facts are proved in [G2]; see pages 104–106:

(3.5.1) For each $n \geq 2$, there is a homology 4-sphere M_n^4 such that $\pi_1 M_n^4 \approx \Gamma'_n$;

(3.5.2) each Γ'_n is indecomposable (and perfect) for $n \geq 2$; and

(3.5.3) The sequence $\Gamma'_4, \Gamma'_6, \Gamma'_8, \ldots$ contains infinitely many distinct groups, say, $\Gamma'_{k(1)}, \Gamma'_{k(2)}, \ldots$.

(3.5.4) NOTATION: For each $i \geq 1$, put $G'_i = \Gamma'_{k(i)}$ and $H_i^4 = M_{k(i)}^4$. Note that $\pi_1 H_i^4 \approx G'_i$ for $i \geq 1$; again, $\{H_i^4\}_{i=1}^{\infty}$ is an admissible collection of homology 4-spheres.

(3.6) *Arcs in S^n, $n \geq 5$.* Fix an integer $n \geq 5$. Let $\{A_i\}_{i=1}^{\infty}$ be the collection of π_1-distinct arcs associated with the admissible collection $\{H_i^3\}_{i=1}^{\infty}$ of homology 3-spheres given in (3.4.3). The following is now clear.

(3.6.1) PROPOSITION: *With notation above, for each $i \geq 1$ there exists a crumpled n-cube C_i contained in S^n such that:* (a) *the* arc A_i *is tamely embedded in the boundary of C_i;* (b) *the closure of $(S^n - C_i)$ is an n-cell B_i;* (c) *∂C_i is locally flat modulo A_i; and* (d) $\pi_1(S^n - B) \approx \pi_1(S^n - A_i) \approx \pi_1(C_i - A_i) \approx \pi_1 H_i^3 \approx G_i$.

(3.7) *Arcs in S^n, $n \geq 6$.* Fix an integer $n \geq 6$. Let $\{A'_i\}_{i=1}^{\infty}$ be the collection of π_1-distinct arcs associated with the admissible collection $\{H_i^4\}_{i=1}^{\infty}$ of homology 4-spheres. For each $i \geq 1$, there exists a crumpled n-cube C'_i contained in S^n satisfying conclusions analogous to those of Proposition (3.6.1).

(3.8) *Infinite Inflation.* We follow [D1] for notation and terminology. For each crumpled n-cube C, its infinite inflation $J^{\infty}(C)$ equals $(C \times Q)/K$ where K is the decomposition of $C \times Q$ into singletons and the sets of the form $\{c\} \times Q$ when c belongs to ∂C. Moreover, there is a natural embedding $e: \partial C \to J^{\infty}(C)$ sending c to the image of $\{c\} \times Q$ in $J^{\infty}(C)$. The crucial result of [D1] is the following:

(3.8.1) PROPOSITION: *If C is a crumpled n-cube such that $C \bigcup_{\mathrm{Id}} C$ is homeomorphic to S^n, then $J^\infty(C)$ is homeomorphic to Q.*

In the next section, we shall produce an uncountable collection of π_1-distinct arcs in S^n with $n \geq 5$ and in Q.

4. The main result

For the forthcoming geometric construction we suppose the following hypotheses are true.

(4.1) HYPOTHESES: Suppose $\{A_i\}_{i=1}^{\infty}$ is a collection of arcs in S^n satisfying the hypotheses:

(a) Each arc A_i is tamely embedded in the boundary of an n-cell B_i;

(b) the boundary ∂B_i is locally flat (in S^n) modulo the arc A_i for $i \geq 1$;

(c) for each $i \neq j$, $\pi_1(S^n - B_i)$ is not isomorphic to $\pi_1(S^n - B_j)$;

(d) for each $i \geq 1$, $\pi_1(S^n - B_i)$ is indecomposable (with respect to free products).

The following is an immediate consequence of the Siefert-Van Kampen Theorem and hypotheses (a) and (b) above.

(4.1.1) For each $i \geq 1$, the fundamental groups $\pi_1(S^n - B_i)$ and $\pi_1(S^n - A_i)$ are isomorphic.

(4.2) *A Geometric Construction.* Let Λ denote an infinite subset of the set of positive integers N. Let $\{j(1), j(2), \ldots\}$ be an enumeration of Λ. Let D be an n-cell tamely embedded in S^n. Choose a null sequence $\{D_i\}_{i=1}^{\infty}$ of tame n-cells in D such that they are pairwise disjoint, the intersection $(\partial D_i \cap \partial D)$ is a tame $(n-1)$-cell in ∂D_i and ∂D for $i \geq 1$, and, furthermore, there exists a point p in ∂D such that all but finitely many D_i's are contained in any neighborhood of p in D.

Choose n-cells $B_{j(1)}, B_{j(2)}, \ldots$, corresponding to the enumerated set $\Lambda = \{j(1), j(2), \ldots\}$, from the n-cells $B_1, B_2, \ldots$ given in (4.1.1). For each $i \geq 1$, find a homeomorphism $h_i : S^n \to S^n$ satisfying: the image $h_i(S^n - \operatorname{Int} B_{j(i)})$ is contained in D_i, and $h_i(\partial B_{j(i)}) \cap \partial D_i$ equals the $(n-1)$-cell F_i where F_i is the closure of $[\partial D_i - (\partial D_i \cap \partial D)]$. The construction of h_i is easy because, if one chooses a small $(n-1)$-cell in $\partial B_{j(i)}$, the resulting set is, if the collar is constructed with care, a tamely embedded n-cell that can be identified with the closure of $(S^n - D_i)$. Consider the n-cell B_Λ bounded by the union of $[\partial D - \bigcup_{i=1}^{\infty} (\partial D_i \cap \partial D]$ and $\bigcup_{i=1}^{\infty} cl[h_i(\partial B_{j(i)} - F_i)]$

(where "cl" denotes the "closure"). Observe that the arcs $\{A_i'\}_{i=1}^{\infty}$, $A_i' = h_i(A_{j(i)})$, are tamely embedded in ∂B_A. Now it is straightforward to find an arc A_A, by connecting any two successive arcs A_i' and A_{i+1}', such that A_A contains the union of the arcs $A_1', A_2', \ldots$, and A_A is tame in ∂B_A. Observe the following:

(4.2.1) the set $C_0 = (\operatorname{Int} D - \bigcup_{i=1}^{\infty} D_i)$ is simply connected (in fact, C_0 is an open n-cell);

(4.2.2) the set $[S^n - (B_A \cup C_0)]$ breaks up into disjoint "chambers" $C_1, C_2, \ldots$; in fact, each C_i is a crumpled n-cube with an $(n-1)$-cell removed from its boundary;

(4.2.3) $\pi_1(S^n - B_A) \approx \pi_1(S^n - A_A)$ (use the Siefert-Van Kampen Theorem [M] and the facts: (a) ∂B_A is tame modulo the union of the arcs $\{A_i'\}_{i=1}^{\infty}$ and $\{p\}$, and (b) A_A is tame in ∂B_A); and

(4.2.4) for each i, $\pi_1 C_i \approx \pi_1(S^n - B_{j(i)}) \approx \pi_1(S^n - A_{j(i)})$ (see (4.1)).

In this setting, we now prove the following.

(4.2.5) Proposition: *The fundamental group $\pi_1(S^n - B_A)$ is isomorphic to the free product $\prod_{i\geq 1}^{*} G_i$ where $G_i = \pi_1 C_i$ for $i \geq 1$.*

Proof: Put $X_n = \bigcup_{i=0}^{n} C_i$, where C_i's are as in (4.2.1) and (4.2.2). We consider the direct system of spaces $X_1 \to X_2 \to X_3 \to \ldots$ whose unlabelled maps are inclusions. Observe that $X = S^n - B_A$ is direct limit (union) of this system. By applying π_1, we obtain a direct system $\pi_1 X_1 \to \pi_1 X_2 \to \pi_1 X_3 \to \ldots$ of groups whose direct limit $\varinjlim_i \pi_1 X_i$ is isomorphic to the fundamental group $\pi_1 X$. By the Siefert-Van Kampen Theorem [M], it follows that $\pi_1 X_n = \prod_{1\leq i\leq n}^{*} G_i$, where $G_i = \pi_1 C_i$; and furthermore, it is easy to see that the inclusion $X_n \to X_{n+1}$ induces a homomorphism which is the *natural inclusion homomorphism* $\prod_{1\leq i\leq n}^{*} G_i \to (\prod_{1\leq i\leq n}^{*} G_i) * G_{n+1}$ onto the first factor. Our proof is finished by the observation that the direct limit G^{∞} of the direct system $G_1 \to G_1 * G_2 \to G_1 * G_2 * G_3 \to \ldots$, whose bonds are the natural inclusions (see above), is isomorphic to the free product $\prod_{i\geq 1}^{*} G_i$. This can be easily seen by considering the natural homomorphism $\phi: G^\infty \to \prod_{i\geq 1}^* G_i$ and observing that ϕ is $1-1$ and onto (see [K, M] for matters related to free products). This finishes our proof. ∎

(4.3) PROPOSITION: *Suppose* $G = \prod_{i\geq 1}^* G_i$ *and* $H = \prod_{i\geq 1}^* H_i$ *are two free products of groups satisfying*:

(a) *For each* $i \geq 1$, *each of the groups* G_i *and* H_i *is non-trivial, non-cyclic, and indecomposable* (*with respect to free products*); *and*

(b) *there exists an* $i(0) \geq 1$ *such that* $G_{i(0)}$ *is not isomorphic to any* H_j *for* $j \geq 1$.

Then: G *is not isomorphic to* H.

Although this can be easily deduced from Kurosh [K], see pages 26–27, we shall include a quick sketch of a proof as follows: Suppose for convenience of notation that $i(0) = 1$ and $G = \prod_{i\geq 1}^* G_i = \prod_{i\geq 1}^* H_i$. By the Kurosh Subgroup Theorem (cf. [K, M]), $G_1 \subset hH_jh^{-1}$ for some j and h in G; similarly $H_j \subset gG_ig^{-1}$ for some i and g in G. Use the fact that each conjagate of a factor intersects any other factor in the trivial group to deduce that $H_j \subset gG_1g^{-1}$, i.e., $i = 1$. Now $G_1 \subset (hg)G_1(hg)^{-1}$ implies that hg is in G_1, since xG_1x^{-1} intersects G_1 in 1 or x is in G_1 (a general fact!). Thus, $hg = g_1$ and $g = h^{-1}g_1$ for some g_1 in G_1. It follows that $H_j \subset gG_1g^{-1} = h^{-1}g_1G_1g_1^{-1}h = h^{-1}G_1 \subset H_j$, i.e., G_1 is isomorphic to H_j. This is a contradiction to hypothesis (b) and our proof is finished. ∎

We now have all the ingredients to prove our main results:

(4.4) THEOREM: *If Hypotheses* (4.1) *are satisfied, then there exist an uncountable collection of* π_1*-disjoint arcs in* S^n, *where* S^n *is as in Hypotheses* (4.1).

PROOF: Let $\mathcal{N}$ denote the set of all subsets of the set N of positive integers. For each $\Lambda \in \mathcal{N}$ we have constructed an arc A_Λ and an n-cell B_Λ such that A_Λ is tamely embedded in the boundary of the crumpled n-cube $C_\Lambda = S^n - \operatorname{Int} B_\Lambda$, and $\pi_1(S^n - A_\Lambda) \approx \pi_1(S^n - B_\Lambda) \approx \pi_1(C_\Lambda - A_\Lambda) \approx \prod_{i\geq 1}^* G_i$ where $G_i = \pi_1(S^n - B_{j(i)})$ is indecomposable (see Proposition (4.2.5) and preceding discussions); although we have only discussed the case when Λ is infinite, the case when Λ is finite can be easily deduced from our discussions. Now, if Λ and Λ' are two distinct subsets of N, then the fundamental groups $\pi_1(S^n - A_\Lambda)$ and $\pi_1(S^n - A_{\Lambda'})$ are necessarily distinct by Proposition (4.3). The collection of arcs $\{A_\Lambda : \Lambda \in \mathcal{N}\}$ forms an uncountable collection of π_1-distinct arcs in S^n. This finishes our proof. ■

(4.4.1) THEOREM: *If Hypotheses* (4.1) *are satisfied for some* S^n *with* $n \geq 5$, *then there exists an uncountable collection of* π_1*-distinct arcs in* Q.

PROOF: We refer to the proof of Theorem (4.4) for notation. Consider the uncountable collection $\{A_\Lambda : \Lambda \in \mathcal{N}\}$ (given in the proof of Theorem (4.4)). We now construct an uncountable collection $\{\hat{A}_\Lambda : \Lambda \in \mathcal{N}\}$ of π_1-distincts arcs in Q as follows.

For each Λ in $\mathcal{N}$, $C_\Lambda \bigcup_{\mathrm{Id}} C_\Lambda \approx S^n$ by Proposition (3.1.8), and thus $J^\infty(C_\Lambda)$ is homeomorphic to Q by Proposition (3.8.1). We equate $J^\infty(C_\Lambda)$ with Q and set $\hat{A}_\Lambda = e(A_\Lambda)$ where e is the embedding given in (3.8). It follows from the proof of Proposition (5) of [D1] and [AP, L] that $\pi_1(Q - \hat{A}_\Lambda) \approx \pi_1(C_\Lambda - A_\Lambda)$. Our proof is finished since $\pi_1(S^n - A_\Lambda) \approx \pi_1(C_\Lambda - A_\Lambda)$. ■

(4.4.2) COROLLARY: *There exists an uncountable collection of* π_1*-distinct arcs in* S^n *with* $n \geq 5$.

PROOF: Observe that for each integer $n \geq 5$, the collection $\{A_i\}_{i=1}^\infty$ of arcs given by Proposition (3.6.1) satisfies Hypotheses (4.1). Therefore, the corresponding collection $\{A_\Lambda : \Lambda \in \mathcal{N}\}$ of arcs in S^n has the required properties.

For each $n \geq 6$, the collection $\{A'_i\}_{i=1}^\infty$ of arcs given in (3.7) also satisfies Hypotheses (4.1), and thus we have another uncountable collection $\{A'_\Lambda : \Lambda \in \mathcal{N}\}$ of π_1-distinct arcs in S^n. (We prefer this collection since the presentations of groups Γ'_n are more manageable than the presentations of groups Γ_n). ■

(4.4.3) COROLLARY: *There exists an uncountable collection of* π_1*-distinct arcs in* Q.

PROOF: Our proof is finished by observing that each of the collections $\{\hat{A}_\Lambda : \Lambda \in \mathcal{N}\}$ and $\{\hat{A}'_\Lambda : \Lambda \in \mathcal{N}\}$, constructed as in the proof of Theorem (4.4.1) corresponding to the specific collections $\{A_\Lambda : \Lambda \in \mathcal{N}\}$ and $\{A'_\Lambda : \Lambda \in \mathcal{N}\}$ given in the proof of Corollary (4.4.2), satisfies the required properties. ■

The next three results can be derived easily by thickening the arcs of Corollary 4.4.2 to k-cells or k-spheres tamely embedded in the boundary of the appropriate n-cell arising as in (4.4).

(4.4.4) COROLLARY: *For* $n \geq 5$ *and* $k \in \{1, \dots n\}$ *there exists an uncountable collection of* π_1*-distinct k-cells in* S^n.

(4.4.5) COROLLARY: *For* $n \geq 5$ *and* $k \in \{1, \dots, n-1\}$ *there exists an uncountable collection of* π_1*-distinct k-spheres in* S^n.

(4.4.6) COROLLARY: *For* $k \geq 1$ *there exist uncountable collections of* π_1*-distinct k-cells and of* π_1*-distinct k-spheres in* Q.

(4.4.7) *A Summary of Our Method.* The following steps lead to the construction of an uncountable collection of π_1-distinct arcs in Q (or in a sphere):

Step I. Choose an admissible collection $\{H_i^n\}_{i=1}^{\infty}$ of homology n-spheres (many such collections exist; for instances, two are given in this paper).

Step II. Construct a collection $\{A_i\}_{i=1}^{\infty}$ of π_1-distinct arcs in S^{n+k}, with $k \geq 2$ but fixed, associated with the collection $\{H_i^n\}_{i=1}^{\infty}$; see Section 3.

Step III. Use the collection $\{A_i\}_{i=1}^{\infty}$ to construct an uncountable collection $\{A_\Lambda : \Lambda \in \mathcal{N}\}$ of π_1-distinct arcs in S^{n+k} with k as above, see Section 4.

Step IV. Use infinite inflations to construct an uncountable collection $\{\hat{A}_\Lambda : \Lambda \in \mathcal{N}\}$ of π_1-distinct arcs in Q corresponding to the collection $\{A_\Lambda : \Lambda \in \mathcal{N}\}$ of Step III; see (3.8).

(4.5) *Some Applications to Decomposition Spaces.* Fix an integer $n \geq 5$. Let S_Λ^n and Q_Λ denote the quotient spaces of S^n and Q obtained by identifying the arcs A_Λ and $\hat{A}_\Lambda$ to a respective point. Let $p_\Lambda : S^n \to S_\Lambda^n$ and $q_\Lambda : Q \to Q_\Lambda$ denote the quotient maps. It follows from [AP, L] that $\pi_1(S_\Lambda^n - \{x_0\}) \approx \pi_1(S^n - A_\Lambda)$ where $\{x_0\} = p_\Lambda(A_\Lambda)$, and similarly, $\pi_1(Q_\Lambda - - \{y_0\}) \approx \pi_1(Q - \hat{A}_\Lambda)$ where $\{y_0\} = q_\Lambda(\hat{A}_\Lambda)$; furthermore, $\pi_1(S^n - \{x\}) = 1$ for any $x \neq x_0$ and $\pi_1(Q_\Lambda - \{y\}) = 1$ for any $y \neq y_0$. In summary:

(4.5.1) THEOREM: *The uncountable collection of decomposition spaces* $\{S_\Lambda^n : \Lambda \in \mathcal{N}\}$ *has the following properties*: (a) *Each* S_Λ^n *is a generalized n-manifold with* $\pi_1(S_\Lambda^n - \{x_0\}) \approx \pi_1(S^n - A_\Lambda)$ *and* $\pi_1(S_\Lambda^n - \{x\}) = 1$ *when* $x \neq x_0$, *and* (b) S_Λ^n *is not homeomorphic to* $S_{\Lambda'}^n$ *whenever* $\Lambda \neq \Lambda'$.

(4.5.2) THEOREM: *The uncountable collection of decomposition spaces* $\{Q_\Lambda : \Lambda \in \mathcal{N}\}$ *has the following properties*: (a) *Each* Q_Λ *is an absolute retract with* $\pi_1(Q_\Lambda - \{y_0\}) = \pi_1(Q - \hat{A}_\Lambda)$ *and* $\pi_1(Q_\Lambda - \{y\}) = 1$ *for* $y \neq y_0$; *and* (b) Q_Λ *is not homeomorphic to* $Q_{\Lambda'}$ *whenever* $\Lambda \neq \Lambda'$.

Much more can be said about decomposition spaces of this type (since these decompositions are the simplest, i.e., there is exactly one nondegenerate element which is an arc); see [L] where other references may also be found.

A space X is said to be *rigid* if and only if the group of homeomorphisms of X is the trivial group, i.e., X has no homeomorphism other than the identity. Decompositions can be used to construct rigid spaces with additional properties; see [R, S2]. In the present setting, we construct some rigid spaces as follows. For any infinite subset $\{\Lambda_i : i \geq 1\}$ of $\mathcal{N}$ find a collection of arcs $\{B_i : i \geq 1\}$ in Q (or S^n with $n \geq 5$) such that $\pi_1(Q - B_i) \approx \pi_1(Q - \hat{A}_{\Lambda_i})$ (or $\pi_1(S^n - B_i) \approx \pi_1(S^n - A_{\Lambda_i})$) for $i \geq 1$, the arcs $\{B_i : i \geq 1\}$ form a null collection of pairwise disjoint arcs, and their union is dense in Q (or S^n). It is easy to find such a collection and we omit details. Let $G = G(\{\Lambda_i : i \geq 1\})$ denote the decomposition of Q (or S^n) whose nondegenerate elements are the arcs $B_1, B_2, \ldots$. The associated decomposition space Q/G (S^n/G) is clearly rigid; consider the fundamental groups of $(Q/G - \{\text{point}\})$ or $(S^n/G - \{\text{point}\})$. The space Q/G is an AR; this can be easily deduced from a theorem of Lelek (cf. [B], see page 213). Also, S^n/G is a generalized n-manifold [W]. In summary, we have the following:

(4.5.3) THEOREM: *There exists an uncountable collection $\mathscr{C}(Q)$ of topologically distinct (infinite dimensional) rigid compact absolute retracts such that, for any X and Y* in *$\mathscr{C}(Q)$, $X \times Y \approx Q$ (in particular, $X \times X \approx Q$ for all $X \in \mathscr{C}(Q)$).*

PROOF: This is immediate from the discussion above and from Theorem 5 of [T]. ■

(4.5.4) THEOREM: *There exists an uncountable collection $\mathscr{C}(S^n)$ of topologically distinct (compact) rigid generalized n-manifolds with $n \geq 5$ such that* (a) *$X \times Y \approx S^n \times S^n$ for any X and Y in $\mathscr{C}(S^n)$ (in particular, $X \times X \approx S^n \times S^n$, for any X in $\mathscr{C}(S^n)$) and* (b) *$X \times S^1 \approx S^n \times S^1$.*

PROOF: Consider the collection $\mathscr{C}(S^n)$ of rigid generalized n-manifolds described above. By work of C.D. Bass [BA] each product $X \times Y$ has the Disjoint Disks Property, and then by the Cell-like Approximation Theorem of R.D. Edwards [E] $X \times Y \approx S^n \times S^n$. The additional fact that $X \times S^1 \approx S^n \times S^1$ follows from a result of D.V. Meyer [ME]. ■

(4.6) *Concluding Remarks.* After A.V. Arhangel'skii [AR] asked whether there exists a nonhomogeneous compactum X whose product $X \times X$ is homogeneous, J. van Mill [VM] gave an infinite dimensional example with this property; our Theorem (4.5.3) provides uncountably many similar examples. Furthermore, our Theorem (4.5.4) provides uncountably many finite dimensional examples with this property, thereby answering a question posed by van Mill [VM]. The present versions of Theorems (4.5.3) and (4.5.4) were added following suggestions of the referee, whom we wish to thank for pertinent remarks.

REFERENCES

[AC] J.J. Andrews and M.L. Curtis: n-Space Modulo an Arc, Ann. of Math. 75 (1962), 1–7.

[AR] A.V. Arhangel'skii: Structure and classification of topological spaces and cardinal invariants, Russian Math. Surveys 33 (1978), 33–96.

[AP] S. Armentrout and T.M. Price: Homotopy Properties of Decomposition Spaces, Trans. Amer. Math. Soc. 143 (1969), 499–507.

[AF] E. Artin and R.H. Fox: Some Wild Cells and Spheres in Three-Dimensional Space, Ann. of Math. 49 (1948), 979–990.

[B] K. Borsuk: Theory of Retracts, PWN, Warsaw, 1967.

[BA] C.D. Bass: Some products of topological spaces which are manifolds, Proc. Amer. Math. Soc. 81 (1981), 641–646.

[BR] M. Brown: Wild Cells and Spheres in Higher Dimensions, Mich. Math. J. 14 (1967), 219–224.

[C1] J.W. Cannon: Shrinking Cell-like Decompositions of Manifolds. Codimension Three, Ann. of Math. 110 (1979), 83–112.

[C2] J.W. Cannon: The Recognition Problem: What is a Topological n-Manifold?, Bull. Amer. Math. Soc. 84 (1978), 832–866.

[CH] T.A. Chapman: Lectures on Hilbert Cube Manifolds, CBMS Regional Conf. Series in Math., no. 28, Amer. Math. Soc., Providence, R.I., 1976.

[D1] R.J. Daverman: Infinite Inflations of Crumpled Cubes, Topology and Its Appl. 12 (1981), 35–43.

[D2] R.J. Daverman: Sewings of Closed n-cell Complements, Trans. Amer. Math. Soc., to appear.

[D3] R.J. Daverman: Embeddings of $(n-1)$-spheres in Euclidean n-Space, Bull. Amer. Math. Soc. 84 (1978), 377–405.

[DE] R.J. Daverman and W.T. Eaton: Each Disk in E^n Can be Squeezed to An Arc, preprint.

[Ė] R.D. Edwards: The topology of manifolds and cell-like maps, in Proceedings of the International Congress of Mathematicians, Helsinki, 1978 (O. Lehto, editor), Adacemia Scientarium Fennica, Helsinki, 1980, 111–127.

[DW] R.D. Edwards and J.J. Walsh: Čech Homology Characterizations of Infinite Dimensional Manifolds, Amer. J. Math. 103 (1981), 411–435.

[G1] L.C. Glaser: Euclidean $(q+r)$-Space Modulo an r-Plane of Collapsible p-Complexes, Trans. Amer. Math. Soc. 157 (1971), 261–278.

[G2] L.C. Glaser: Geometric Combinatorial Topology, Vol. II, Van Nostrand and Reinhold Co., London–New York, 1972.

[K] A.G. Kurosh: The Theory of Groups, Vol. II, Chelsea Publishing Co., New York, 1956.

[L] R.C. Lacher: Cell-Like Mappings and Their Generalizations, Bull. Amer. Math. Soc. 83 (1977), 495–552.

[M] W.S. Massey: Algebraic Topology: An Introduction, Harcourt, Brace and World, New York, 1967.

[ME] D.V. MEYER: More decompositions of E^n which are factors of E^{n+1}, Fund. Math. 67 (1970), 49–65.

[R] H. ROSLANIEC: On Decompositions Spaces of E^n with Small Homeomorphism Group, Bull. Acad. Polon. Sciences 24 (1976), 901–904.

[S1] S. SINGH: Generalized Manifolds (ANR's and AR's) and Null Decompositions of Manifolds, Fund. Math., to appear.

[S2] S. SINGH: Decompositions of the Hilbert Cube Manifolds and ANR's, in preparation.

[T] H. TORÚNCZYK: On CE Images of the Hilbert Cube and Characterizations of Q-Manifolds, Fund. Math. 106 (1980), 31–40.

[VM] J. VAN MILL: A rigid space X for which $X \times X$ is homogeneous: an application of infinite dimensional topology, Proc. Amer. Math. Soc. 83 (1981), 597–600.

[W] R.L. WILDER: Monotone mappings of manifolds, Pacific J. Math. 7 (1957), 1519–1528.

[WO] R.Y.T. WONG: A wild cantor set in the Hilbert Cube, Pacific J. Math. 24 (1968), 189–193.

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