COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 48, nº 2 (1983), p. 185-208
<http://www.numdam.org/item?id=CM_1983__48_2_185_0>

COMPOSITIO MATHEMATICA, Vol. 48, Fasc. 2, 1983, pag. 185–208

LIE GROUPS WITH SMOOTH CHARACTERS AND WITH SMOOTH SEMICHARACTERS

Niels Vigand Pedersen*

Introduction and main results

Let G be a connected, simply connected Lie group. In a recent paper we showed that if G is solvable and of type R (i.e. all its roots are purely imaginary), then all characters of G are smooth, that is, they give rise to distributions on the group ([19] Theorem 4.1.7, p. 239). In the special case where G is nilpotent this result is due to Dixmier ([7]). If G is semisimple the same result holds as shown by Harish–Chandra ([13]) (At this point we would like to remind the reader that when dealing with solvable (or more general) Lie groups the notion of a character is more complicated than e.g. for nilpotent or semisimple groups; this is due to the fact that solvable Lie groups are not necessarily type I (for characters in general, cf. e.g. [9] §6; for characters on connected Lie groups, cf. [25])). The limits for the class of Lie groups for which such a result (i.e. the smoothness of all characters) can hold was set by Pukanszky ([26], [27], cf. also the paper [18] of Moore, Rosenberg): He showed that if all normal representations (cf. [25]) are GCCR (cf. [27], where this notion is defined; this can be rephrased by saying that the normal representation has a densely defined character, cf. Lemma 1.2.1) then G must be the direct product of a connected semisimple Lie group and of a connected Lie group whose radical is cocompact and type R. This implies in particular that groups for which all characters are smooth must be of this form. One of the purposes of this paper is to extend our result of smoothness of all characters to this largest possible class of groups. Specifically we have:

* Supported by a fellowship from the University of Copenhagen and by Det Naturvidenskabelige Forskningsråd, Denmark. This work was performed while the author was a visitor at the Department of Mathematics, University of California, Berkeley.

0010-437X/83/02/0185–24$0.20

THEOREM 1: *Let G be a connected, simply connected Lie group. The following conditions are equivalent:*

(i) *Any normal representation of G has a smooth character.*

(ii) *Any normal representation of G has a densely defined character* (*or, equivalently, is GCCR, cf. Lemma* 1.2.1).

(iii) *Prim(G) is a T_1 topological space.*

(iv) *G is the direct product of a connected, simply connected semisimple Lie group and a connected, simply connected Lie group whose radical is cocompact and type R.*

The equivalence of (ii), (iii) and (iv) is due to Pukanszky ([26], [27] Proposition 3). The equivalence of (iii) and (iv) was also proved by Moore and Rosenberg ([18] Theorem 1); cf. also the papers of Moore [1], Chap. V and Lipsman [15].

In [19] we also showed that for an arbitrary connected, simply connected solvable Lie group it is possible to associate a semicentral distribution to any normal representation of *G* ([19] Theorem 4.1.1, p. 232). Our results in [19] were formulated by aid of the notion of a semicharacter, defined to be a certain relatively invariant weight on the group C*-algebra. We said that a semicharacter is smooth if it gives rise to a distribution on the group (in which case the distribution is a semicentral distribution). The main purpose of this paper is to confine the class of Lie groups for which all normal representations have a smooth semicharacter. Specifically we prove:

THEOREM 2: *Let G be a connected, simply connected Lie group. The following conditions are equivalent:*

(i) *Any normal representation of G has a smooth semicharacter.*

(ii) *Any normal representation of G has a densely defined semicharacter.*

(iii) *Any normal representation of G is either GCCR or is induced from a normal GCCR representation of a connected, normal subgroup of codimension one* (*or both*).

(iv) *G is the direct product of a connected, simply connected semisimple Lie group and a connected, simply connected Lie group whose radical is cocompact.*

At this point it will be appropriate to mention the following new feature arising by the introduction of semicentral distributions for the description of representations of connected Lie groups: If v is a semicentral distribution of positive type there is associated a representation λ_v to v (via the GNS-construction). Let us assume that this representation is factorial. It is then a non-trivial problem whether the representation λ_v is normal and even whether λ_v generates a semifinite factor. This, however, we settled in the affirmative in a recent paper ([21]). More precisely we showed that if f is a semicharacter on a connected Lie group then the representation λ_f associated to f (via the GNS-construction) is actually normal. We can thus summarize the main features of our results as follows: Starting out with a semicentral distribution of positive type which is 'extremal' (i.e. λ_v is a factor representation) leads us to a normal representation, and all normal representations can be obtained in this fashion precisely if the group satisfies condition (iv) in Theorem 2.

As it will be seen from Theorem 1, a connected, simply connected Lie group G, fails to be GCCR if at least one of the following two phenomena occurs: 1. the radical of G is not of type R or 2. G has non-compact semisimple subgroup acting non-trivially on the radical. From Theorem 2 it is seen that if we use the broader notion of a semi-character only the phenomenon 2. cannot be effectively dealt with. This makes us venture the following remark: If π is a normal representation of a separable locally compact group and if π is not GCCR there seems to be two distinct ways in which π can fail to be so, and these are classified according to whether or not there exists a continuous homomorphism $\chi: G \to \mathbb{R}_+^*$, such that π has a densely defined semicharacter (we shall resist the temptation to call π SGCCR (= semi-GCCR) if such a χ exists).

For unexplained notation we refer to the notational conventions adopted in [19] and [21].

I would like to thank the Department of Mathematics, University of California, Berkeley, where this work was performed, for its hospitality during my stay in 1979/80.

1. Preliminaries

1.1. In this section we shall make a few remarks about smooth semi-traces. This will be useful in the following.

Let G be a Lie group and let $\chi: G \to \mathbb{R}_+^*$ be a continuous homomorphism. Recall that a χ-semitrace f on G (cf. [19] Definition 2.1.5) is called smooth if $C_c^\infty(G) \subset \mathfrak{m}_f$ and if $\varphi \to f(\varphi): C_c^\infty(G) \to \mathbb{C}$ is a distribution on G ([19] Definition 2.3.4).

We have the following converse of [19] Proposition 2.3.5:

LEMMA 1.1.1: *Let f be a smooth χ-semitrace on G. Then there exists $m \in \mathbb{N}$, such that $C_c^m(G) \subset \mathfrak{m}_f$.*

PROOF: Since a distribution is locally of finite order there exists an open neighborhood U of the identity in G, such that the restriction of f to $C_c^\infty(G, U)$ is continuous in the $C_c^m(G)$-topology for some $m \in \mathbb{N}$. Let V be an open, symmetric neighborhood of the identity in G, such that $V^2 \subset U$. Let $\varphi \in C_c^m(G, V)$. There exists a sequence $\varphi_n \in C_c^\infty(G, V)$, such that $\varphi_n \to \varphi$ in the $C_c^m(G)$-topology. In particular $\varphi_n - \varphi_m \to 0$ for $m, n \to +\infty$, in the $C_c^m(G)$-topology, and therefore $(\varphi_n - \varphi_m)^* * (\varphi_n - \varphi_m) \to 0$ in the $C_c^m(G)$-topology, and this implies that $f((\varphi_n - \varphi_m)^* * (\varphi_n - \varphi_m)) \to 0$ for $m, n \to +\infty$, since the restriction of f to $C_c^\infty(G, V)$ is continuous in the $C_c^m(G)$-topology. But this shows that $\dot{\varphi}_n \in \mathfrak{n}_f/\mathfrak{n}_f^0 \subset H_f$ is a Cauchy-sequence, hence in particular $\|\varphi_n\|^2 = f(\varphi_n^* * \varphi_n)$ is convergent. But since also $\varphi_n \to \varphi$ in $C^*(G)$ we have that $f(\varphi^* * \varphi) \leqq \liminf f(\varphi_n^* * \varphi_n) < +\infty$, and this shows that $C_c^m(G, V) \subset \mathfrak{n}_f$.

With m, V as above, let $m' \in \mathbb{N}, \psi, \psi_0 \in C_c^m(G, V)$ as in [19] Lemma 2.3.6, p. 206. From the formula in [19], p. 206, bottom, it then follows that $C_c^{2m'}(G) \subset \mathfrak{m}_f$. This proves the lemma.

Let now X be a Borel space and let μ be a measure on X. Moreover, let $\xi \to f_\xi$, $\xi \in X$, be a family of χ-semitraces on G, such that $\xi \to f_\xi(x)$ is borel for all $x \in C^*(G)^+$. Assume that there is given a χ-semitrace f on G, such that the formula

$$f(x) = \int_X f_\xi(x)\, d\mu(\xi)$$

holds for all $x \in C^*(G)^+$. We then have:

LEMMA 1.1.2: *If f is smooth, then f_ξ is smooth for almost all $\xi \in X$.*

PROOF: Choose $m \in \mathbb{N}$, such that $C_c^m(G) \subset \mathfrak{m}_f$ (Lemma 1.1.1) and choose $m' \in \mathbb{N}$ and $\psi, \psi_0 \in C_c^m(G)$ as in [19] Lemma 2.3.6., p. 206. Arguing as in the proof of [19] Proposition 2.3.5, pp. 206–207 we find that f_ξ is smooth for those ξ for which $\psi, \psi_0 \in \mathfrak{n}_{f_\xi}$, and since clearly $\psi, \psi_0 \in \mathfrak{n}_{f_\xi}$ for almost all $\xi \in X$, we have proved the lemma.

Let N be a closed, normal subgroup of G and set $\eta = \chi \cdot (\Delta_{G/N} \circ c_{G/N})^{-1}$. Assume that f is a (G, η)-semitrace on N (such that in particular f is a semitrace on N with multiplier $\chi | N$, cf. [19] 2.1). The induced semitrace $\tilde{f} = \mathrm{ind}_{N \uparrow G} f$ is then well-defined ([19] Definition 2.1.2). $\tilde{f}$ is a χ-semitrace on G. With this notation we have:

PROPOSITION 1.1.3: *If f is smooth, then $\tilde{f} = \mathrm{ind}_{N \uparrow G} f$ is smooth. Moreover, $\tilde{f}(\varphi) = f(\varphi | N)$ for all $\varphi \in C_c^\infty(G)$.*

PROOF: Since f is smooth there exists $m \in \mathbb{N}$ such that $C_c^m(G) \subset \mathfrak{m}_f$ (Lemma 1.1.1). In particular $\tilde{f}(\varphi^* * \varphi) = f(\varphi^* * \varphi | N)$ for $\varphi \in C_c^m(G)$ ([21] Proposition 2.1.3), and thus $C_c^m(G) \subset \mathfrak{n}_f$. It follows from [21] Proposition 2.3.5, p. 206 that $\tilde{f}$ is smooth. Moreover, by polarisation we get that $\tilde{f}(\varphi) = f(\varphi | N)$ for all $\varphi \in C_c^m(G) * C_c^m(G) \supset C_c^\infty(G)$ (cf. [2], p. 252, top). This proves the proposition.

1.2. Let G be a separable locally compact group. In [27], p. 40 Pukanszky defines what if means that a factor representation of G is generalized completely continuous (GCCR). The following lemma clarifies the situation:

LEMMA 1.2.1: *A normal representation π of G is* GCCR *if and only if it has a densely defined character.*

PROOF: Let $\mathfrak{A} = \pi(G)''$ and let ϕ be a faithful, normal, semifinite trace on $\mathfrak{A}$. Then $f = \phi \circ \pi | C^*(G)^+$ is a character for π. In [27], p. 40 it was shown that the linear span $C(\mathfrak{A})$ of the set of elements $T \in \mathfrak{A}^+$, such that $\phi(I - E_\lambda) < +\infty$ for all $\lambda > 0$, where $T = \int_0^\infty \lambda dE_\lambda$ is the spectral resolution of T, is a normclosed, two-sided ideal in $\mathfrak{A}$. Now it is easily seen that if $T \in \mathfrak{m}_\phi^+$, then $T \in C(\mathfrak{A})^+$, and therefore $\overline{\mathfrak{m}_\phi}^{(N)}$ (normclosure) is contained in $C(\mathfrak{A})$. Conversely, if $T \in C(\mathfrak{A})^+$, it was shown loc. cit. that T can be approximated with elements from $\mathfrak{m}_\phi^+$, and therefore $\overline{\mathfrak{m}_\phi}^{(N)} = C(\mathfrak{A})$. We have thus seen that π is GCCR if and only if $\pi(C^*(G)) \subset \overline{\mathfrak{m}_\phi}^{(N)}$. But this is equivalent to the character f being densely defined ([21] Lemma 3.4.2).

2. Proof of Theorem 2, (i) ⇒ (ii)

This is trivial.

3. Proof of Theorem 2, (ii) ⇒ (iii)

Assume that (ii) is satisfied and that the normal representation π of G is not GCCR. Let $\chi : G \to \mathbb{R}_+^*$ be a continuous homomorphism and let ω be a χ-relatively invariant weight on $\mathfrak{A} = \pi(G)''$, such that $f = \omega \circ \pi | C^*(G)^+$ is a densely defined χ-semicharacter for π. In particular $\chi \not\equiv 1$. It follows from [20] Theorem 6.2 that there exists a normal representation π_0 of $G_0 = \ker \chi$, such that π is equivalent to $\mathrm{ind}_{G_0 \uparrow G} \pi_0$. Moreover there exists a character f_0 for π_0, such that

$$f_1(x) = \int_{G/G_0} f_0 \circ \tau^G_{G_0}(s^{-1})(x)\chi(s)d\dot{s},$$

where f_1 is the (G, χ)-character on G_0 with $\mathrm{ind}_{G_0 \uparrow G} f_1 = f$ (cf. [21] Theorem 3.3.1). Now since f is densely defined it follows that f_1 is densely defined (cf. loc. cit.) and therefore f_0 is densely defined. But then π_0 is GCCR (cf. 1.2). Since G_0 is a connected, normal subgroup of codimension one we have proved that (ii) $\Rightarrow$ (iii).

4. Proof of Theorem 2, (iii) $\Rightarrow$ (iv)

Assume that the connected, simply connected Lie group G is not of the form described in (iv). We shall exhibit a normal representation of G, which does not have the property described in (iii).

4.1. For this it will suffice to exhibit a connected, normal subgroup B of G, such that $\tilde{G} = G/B$ has an irreducible, normal representation $\tilde{\pi}$ with the property that $\tilde{\pi}$ is neither CCR nor is induced from a CCR-representation of a connected, normal subgroup of codimension one. In fact, if $\tilde{\pi}$ is such a representation, define the representation π of G by $\pi(s) = \tilde{\pi}(\dot{s})$, $\dot{s} = sB$. Then π is an irreducible representation of G, which is normal, but not CCR. Assume then that π is induced from a normal representation π_0 of the connected normal subgroup G_0 whose codimension is one. Clearly π_0 is irreducible, and the stabilizer of π_0 in G is G_0, since otherwise $\pi = \mathrm{ind}_{G_0 \uparrow G}\pi_0$ would not be irreducible. Now the restriction of π to G_0 is equivalent to

$$\int^{\oplus}_{G/G_0} s\pi_0 \, d\dot{s} \tag{*}$$

(the notation explains itself) and this decomposition is central, since π_0 is GCR and since the stabilizer of π_0 is G_0. It follows from this that all elements in B leaves π_0 invariant (since $\pi(b) = I, b \in B$, and so $\pi(b)$ induces the trivial automorphism in the von Neumann algebra generated by the representation (*)). But then $B \subset G_0$ and we can define the representation $\tilde{\pi}_0$ on $\tilde{G}_0 = G_0/B$ by $\tilde{\pi}_0(\dot{s}) = \pi_0(s)$, and we have $\tilde{\pi} = \mathrm{ind}_{\tilde{G}_0 \uparrow \tilde{G}}\tilde{\pi}_0$ and $\tilde{G}_0$ is a connected, normal subgroup in $\tilde{G}$ of codimension one. Therefore, by assumption, $\tilde{\pi}_0$ is not CCR, and so π_0 is not CCR. This shows that it suffices to exhibit a B as described.

4.2. The rest of Sect. 4 will inevitably have a lot in common with [1], pp. 171–174, [15], pp. 744–747, [18], pp. 208–211 and [27], pp. 43–45, to which papers we at certain points refer for further details.

4.3. We first define some special groups which will be of interest in the following.

Let S be a connected, simply connected, non-compact, (real) semi-simple Lie group with Lie algebra $\mathfrak{s}$.

First, let σ be a locally faithful (i.e. the differential $d\sigma$ is faithful) irreducible representation of S on $\mathbb{R}^n$. We define the group $S_1(\sigma)$ to be the semidirect product $\mathbb{R}^n \times_s S$, where S acts in $\mathbb{R}^n$ via σ. We define $S_2(\sigma)$ to be the semidirect product $\mathbb{R}^n \times_s (S \times \mathbb{R})$, where the direct product $S \times \mathbb{R}$ acts in $\mathbb{R}^n$ by $(s, t)x = e^t\sigma(s)x, s \in G, t \in \mathbb{R}, x \in \mathbb{R}^n$.

Let next σ be a locally faithful, irreducible representation of S on $\mathbb{C}^n$. We define the group $S_3(\sigma)$ to be the semidirect product $\mathbb{C}^n \times_s (S \times \mathbb{C})$, where the direct product $S \times \mathbb{C}$ acts on $\mathbb{C}^n$ by $(s, t)x = e^t\sigma(s)x$, $s \in S$, $t \in \mathbb{C}$, $x \in \mathbb{C}^n$. Finally we define $S(\sigma, \mu)$ for $\mu \in \mathbb{R}$ to be the semidirect product $\mathbb{C}^n \times_s (S \times \mathbb{R})$, where the direct product $S \times \mathbb{R}$ acts in $\mathbb{C}^n$ by $(s, t)x = e^{(\mu + i)t}\sigma(s)x$, $s \in S$, $t \in \mathbb{R}$, $x \in \mathbb{C}^n$.

The groups $S_j(\sigma), j = 1, 2, 3, S(\sigma, \mu), \mu \in \mathbb{R}$, are clearly connected, simply connected Lie groups.

For the group $G = S_1(\sigma)$ we define subgroups $N = \mathbb{R}^n \times_s \{0\}$ and $A = \{e\}$, and for $G = S_2(\sigma)$ we set $N = \mathbb{R}^n \times_s (\{e\} \times \{0\})$ and $A = \{0\} \times_s (\{e\} \times \mathbb{R})$. Similarly for the other special groups. In this fashion, if G is one of the groups $S_j(\sigma), j = 1, 2, 3, S(\sigma, \mu), \mu \in \mathbb{R}$, then N is the nilradical of G and $G = NAS$. The radical R of G is AN.

We say that a Lie algebra has no semisimple direct factors, if its maximal semisimple ideal, which is a direct factor, is trivial (cf. [27], p. 43, top).

LEMMA 4.3.1: *If G is one of the groups $S_j(\sigma), j = 1, 2, 3, S(\sigma, \mu), \mu \in \mathbb{R}$, we have* ($\mathfrak{g}$ *being the Lie algebra of G*): (i) $\mathfrak{g}$ *has no semisimple direct factors,* (ii) *the radical $\mathfrak{r}$ of $\mathfrak{g}$ is not cocompact,* (iii) *the nilradical $\mathfrak{n}$ is a minimal proper ideal and* (iv) *the radical $\mathfrak{r}$ has precisely one real root or precisely two complex conjugate roots. Conversely, if G is a connected, simply connected Lie group satisfying the properties* (i)–(iv), *then G is isomorphic to one of the groups $S_j(\sigma), j = 1, 2, 3, S(\sigma, \mu), \mu \in \mathbb{R}$.*

PROOF: The special groups clearly have the properties (i)–(iv). Assume conversely that $\mathfrak{r}$ has two complex conjugate roots $\varphi = \varphi_1 + i\varphi_2$ and $\bar{\varphi} = \varphi_1 - i\varphi_2$, such that φ_1 and φ_2 are linearly independent, and let us show that G is isomorhpic to $S_3(\sigma)$. The other cases are treated in a fashion similar to this one. Let $\mathfrak{s}$ be a Levi subalgebra in $\mathfrak{g}$ and let S be the analytic subgroup corresponding to $\mathfrak{s}$. Since the representation $s \to Ad(s)|\mathfrak{r}$ is semisimple and since $\mathfrak{n}$ is an invariant subspace under this representation, there exists a supplementary subspace $\mathfrak{a}$ to $\mathfrak{n}$ in $\mathfrak{r}$, such that $\mathfrak{a}$ is $Ad(s)$-invariant. Since $[\mathfrak{r}, \mathfrak{s}] \subset \mathfrak{n}$ it follows that $[\mathfrak{a}, \mathfrak{s}] = 0$. Since $\mathfrak{n} = \ker\varphi = \ker\varphi_1 \cap \ker\varphi_2$ and since φ_1 and φ_2 are linearly independent we have that $\dim\mathfrak{a} = 2$. Pick $X_1, X_2 \in \mathfrak{a}$ with $\langle\varphi_j, X_k\rangle = \delta_{jk}$, $j, k = 1, 2$. Then set $V = \{Z \in \mathfrak{n}_{\mathbb{C}} | \operatorname{ad} X(Z) = \varphi(X)Z$ for all $X \in \mathfrak{r}\}$. Then V is a nonzero subspace in $\mathfrak{n}_{\mathbb{C}}$ and it is invariant under G (since $\mathfrak{a}$ and $\mathfrak{s}$ commute) and since $\mathfrak{n}$ is a minimal ideal it follows that $\mathfrak{n}_{\mathbb{C}} = V \oplus \bar{V}$. For all $Z \in V$ we have in particular $\operatorname{ad} X_1(Z) = Z$ and $\operatorname{ad} X_2(Z) = iZ$, from which $\operatorname{ad}([X_1, X_2]) = [\operatorname{ad} X_1, \operatorname{ad} X_2] = 0$. But then $[X_1, X_2] = 0$. Let $e_j^1 + ie_j^2$, $j = 1, \ldots, p$, be a basis in V. Then e_j^1, e_j^2, $j = 1, \ldots, p$, is a basis in $\mathfrak{n}$. Define the map $\mathfrak{n} \to \mathbb{C}^n$ by $x_1e_1^1 + \ldots + x_ne_n^1 + y_1e_1^2 + \ldots + y_ne_n^2 \to (x_1 + iy_1, \ldots, x_n + iy_n) = (z_1, \ldots, z_n) = z$. Identifying $\mathfrak{n}$ with $\mathbb{C}^n$ in this way we find $\operatorname{Ad}(\exp(t_1X_1 + t_2X_2))z = e^{t_1 + it_2}z$. Let $\tilde{\sigma}$ be the representation of S on $\mathfrak{n}$ given by $s \to \operatorname{Ad}(s)|\mathfrak{n}$. Since $\tilde{\sigma}$ commutes with $\operatorname{ad} X_2|\mathfrak{n}$ and since $\operatorname{ad} X_2$ is multiplication by i on $\mathbb{C}^n$, it follows that the representation $\tilde{\sigma}$ actually arises from a representation σ on the complex vector space $\mathbb{C}^n$. The irreducibility and the local faithfulness of σ are clear. We have thus shown that G is isomorphic to $S_3(\sigma)$.

For a proof of the following lemma, cf. [27], proof of lemma 23, Ad (A) (b), pp. 43–44.

LEMMA 4.3.2: *The groups* $S_j(\sigma)$, $j = 1, 2, 3$, $S(\sigma, \mu)$, $\mu \in \mathbb{R}$, *are all type I.*

LEMMA 4.3.3: *None of the groups* $S_j(\sigma)$, $j = 1, 2, 3$, $S(\sigma, \mu)$, $\mu \in \mathbb{R}$, *have the property* (iii) *in Theorem* 2.

PROOF: Let G be one of the groups in question. Recall that $G = NAS$, and set $K = AS$. Let $\mathfrak{s}_1$ be a subalgebra in $\mathfrak{s}$, isomorphic to $\mathrm{sl}(2, \mathbb{R})$, and let σ' be the representation contragredient to σ. Let $f \in \mathfrak{n}'$, $f \neq 0$, and let $X \in \mathfrak{s}_1$, $X \neq 0$, such that $\sigma'(X)f = \lambda f$, where λ is a negative real number. Let then χ be the character on N given by $\chi(\exp Y) = \exp i\langle f, Y\rangle$, $Y \in \mathfrak{n}$, and let K_χ be the stabilizer of χ in K. There exists a character φ of $G_\chi = K_\chi N$, such that $\varphi|K_\chi \equiv 1$ and such that $\varphi|N = \chi$. By Mackey-theory $\pi = \operatorname{ind}_{G_\chi \uparrow G}\varphi$ is an irreducible representation of G. Now $(\exp tX)\chi(\exp(Y) = \exp i\langle\sigma'(\exp tX)f, Y\rangle = \exp i\langle(e^{\lambda t}f, Y\rangle$, and this means that $(\exp tX)\chi$ corresponds to the functional $e^{\lambda t}f$, and therefore $(\exp tX)\chi \to 1$ for $t \to +\infty$. Since $sK_\chi s^{-1} = K_{s\chi}$ and since the stabilizer of $(\exp tX)\chi$ is the same as that of χ we see that $\exp tX$ normalizes K_χ and that $(\exp tX)\varphi$ extends $(\exp tX)\chi$ on N and that $(\exp tX)\varphi|K_\chi \equiv 1$. It follows that $(\exp tX)\varphi \to 1$ for $t \to +\infty$. Let us then show that

π is not CCR (cf. the references mentioned in 4.2): We have $\pi \simeq (\exp tX)\pi \simeq \text{ind}_{G_\chi \uparrow G}(\exp tX)\varphi \to \text{ind}_{G_\chi \uparrow G}1$. If $\ker \pi$ was a maximal ideal we would have that $\ker(\text{ind}_{G_\chi \uparrow G}1) = \ker \pi$ and thus $\ker(1|N) = \ker(\pi|N)$, which is false. We next show that π is not induced from a CCR-representation of a connected subgroup of codimension one. So let G_0 be such a subgroup and let π_0 be an irreducible representation of G_0, such that $\pi = \text{ind}_{G_0 \uparrow G}\pi_0$. We have to show that π_0 is not CCR. Since $SN \subset G_0 \subset G$, G_0 is isomorphic to one of the groups $S_j(\sigma)$, $j = 1, 2, 3$, with nilradicalisé SN, so SN is regularly imbedded in G_0 (cf. [27] proof of Lemma 23 Ad (A) (b), pp. 43–44). Therefore we can assume without loss of generality that π_0 lives on the orbit $G_0 \cdot \chi$ in $\hat{N}$. There exists by Mackey-theory a character φ_1 on $(G_0)_\chi$, such that $\varphi_1 | N = \chi$ and such that $\pi_0 = \text{ind}_{(G)_\chi \uparrow G_0}\varphi_1$. But then $\pi = \text{ind}_{(G_0)_\chi \uparrow G}\varphi_1$ which, by the irreducibility of π, implies that $(G_0)_\chi = G_\chi$ and that $\varphi_1 = \varphi$. We can then argue precisely as above to show that π_0 is not CCR. This ends the proof of the lemma.

LEMMA 4.3.4: *Assume that G is a connected, simply connected Lie group not satisfying property* (iv) *in Theorem* 2. *Then G has a quotient isomorphic to one of groups.* $S_j(\sigma)$, $j = 1, 2, 3, S(\sigma, \mu)$, $\mu \in \mathbb{R}$.

PROOF: Imitating the procedure in [27] proof of Lemma 23, Ad (B) (a), p. 45 and Ad (A) (a), p. 43 one can show that the Lie algebra $\mathfrak{g}$ of G has a quotient satisfying the properties (i)–(iv) in Lemma 4.3.1, and the result then follows from this lemma.

It follows from Lemma 4.3.4 and Lemma 4.3.3 that if a connected, simply connected Lie group G fails to satisfy property (iv) in Theorem 2, then G has a connected, simply connected type I quotient not satisfying the property (iii) in Theorem 2. Bearing in mind our remarks in 4.1 we have then shown that G does not have property (iii) in Theorem 2. This ends the proof of Theorem 2, (iii) $\Rightarrow$ (iv).

5. Proof of Theorem 2, (iv) $\Rightarrow$ (i)

We do this in several steps. It is a routine matter (using [13]) to reduce to the case where G actually has cocompact radical.

5.1. So, let G be a connected, simply connected Lie group with Lie algebra $\mathfrak{g}$ whose radical is cocompact. We let $\mathfrak{r}$, $\mathfrak{h}$, $\mathfrak{n}$ denote the radical, the nilradical (= maximal nilpotent ideal) and the nilradicalisé (= $[\mathfrak{g}, \mathfrak{g}] + \mathfrak{g}$), respectively, and let R, H, N denote the corresponding closed, connected, normal subgroups.

5.2. By Ado's theorem ([6] Théorème 5, p. 153) we can without loss of generality assume that $\mathfrak{g}$ is a Lie subalgebra of the Lie algebra $\mathfrak{gl}(V)$ of endomorphisms of some real finite dimensional space V, in such a manner that the nilradical $\mathfrak{h}$ of $\mathfrak{g}$ consists of nilpotent endomorphisms. But then $\mathfrak{h}$ is algebraic ([6] Proposition 14, p. 123) and therefore also $\mathfrak{n}$ is algebraic ([10] 1.2. (ii) Lemma, p. 424 and [6] Proposition 19, p. 129). Let $\tilde{\mathfrak{g}}$ be the algebraic closure of $\mathfrak{g}$. Then $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}] = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{r}, \mathfrak{h}, \mathfrak{n}$ are ideals in $\tilde{\mathfrak{g}}$ ([5] Théorème 13, p. 173). Let $\tilde{\mathfrak{r}}$ be the radical of $\tilde{\mathfrak{g}}$. Then $\tilde{\mathfrak{r}}$ is algebraic, and, in fact, $\tilde{\mathfrak{r}}$ is the algebraic closure of $\mathfrak{r}$ ([6] Proposition 19, p. 129). We can write $\tilde{\mathfrak{r}} = \mathfrak{a} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is the ideal of nilpotent elements in $\tilde{\mathfrak{r}}$ and where $\mathfrak{a}$ is an algebraic abelian subalgebra of $\tilde{\mathfrak{g}}$ ([6] Proposition 20, p. 130). Also $\tilde{\mathfrak{g}} = \tilde{\mathfrak{r}} \oplus \mathfrak{s}$ and $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$, where $\mathfrak{s}$ is a Levi subalgebra of $\mathfrak{g}$, and $\mathfrak{a}$ and $\mathfrak{s}$ can be chosen such that $[\mathfrak{a}, \mathfrak{s}] = 0$ ([6] Proposition 5, p. 144). By [21], p. 477 b we can write $\mathfrak{a}$ as a direct sum of abelian, algebraic algebras $\mathfrak{t}$ and $\mathfrak{k}$, such that all elements in $\mathfrak{t}$ have purely imaginary eigenvalues and such that all elements in $\mathfrak{k}$ have real eigenvalues.

Let $\tilde{G}_1$ be the analytic subgroup of $GL(V)$ corresponding to $\tilde{\mathfrak{g}}$. $\tilde{G}_1$ is the connected component of an algebraic group in $GL(V)$; in particular $\tilde{G}_1$ is closed. Let $\tilde{G}$ be the universal covering group of $\tilde{G}_1$. We then have $H \subset N \subset G \subset \tilde{G}$, all as closed, normal subgroups, and $[\tilde{G}, \tilde{G}] = [G, G]$. By [24] Theorem, p. 379 N and H are both regularly imbedded in $\tilde{G}$.

Let T, K and M be the analytic subgroups in $\tilde{G}_1$ corresponding to $\mathfrak{t}, \mathfrak{k}$ and $\mathfrak{m}$, respectively. T and K are the connected components of the identity of algebraic groups, hence closed. T is clearly a compact, connected Lie group and K is an abelian connected and simply connected Lie group. Set $\mathfrak{e} = \mathfrak{k} \oplus \mathfrak{m}$. $\mathfrak{e}$ is a solvable Lie subalgebra of $\tilde{\mathfrak{g}}$ with real roots. Set $E = KM$. E is a closed, connected, simply connected subgroup in $GL(V)$ with Lie algebra $\mathfrak{e}$ (in fact, choosing a proper basis, $\mathfrak{e}$ is a subalgebra of the solvable Lie algebra of upper triangular matrices), so E is an exponential, connected, simply connected solvable Lie algebra with real roots.

5.3. Using the notation from above, let $\pi \in \hat{H}$ and set $N(\pi) = \{\rho \in \hat{N} | \rho | H$ lives on the orbit $N \cdot \pi\}$.

LEMMA 5.3.1: *$\hat{N}(\pi)$ is countable and* $\operatorname{ind}_{H \uparrow N}\pi$ *is quasi-equivalent to* $\bigoplus_{\rho \in \hat{N}(\pi)} \rho$.

PROOF: Let N_π be the stabilizer of π in N. For the first assertion it suffices to show ([16] Theorem 8.1, p. 297) that the set of elements in N whose restriction to H is a multiple of π is countable. But this is clear, since this set can be identified with a subset of the dual of a certain separable, compact group (namely an extension of N_π/H by the circle group, cf. [16]). That $\mathrm{ind}_{H\uparrow N}\pi$ is quasi-equivalent to $\oplus_{\rho\in\hat{N}(\pi)}\rho$ is contained in [14] Lemma 4.2 (using [16] Theorem 8.1, p. 297).

Let O be a $\tilde{G}$-orbit in $\hat{H}$. Set $\hat{N}(O) = \{\rho\in\hat{N} \mid \rho|H \text{ lives on } O\}$ Clearly $\hat{N}(O)$ is $\tilde{G}$-invariant.

Lemma 5.3.2: *The set of $\tilde{G}$-orbits in $\hat{N}$ contained in $\hat{N}(O)$ is countable.*

Proof: First we observe that any element in $\hat{N}$ lives on an N-orbit in $\hat{H}$, since H is regularly imbedded in N (cf. 5.2). Then pick $\pi\in O$. Clearly each $\tilde{G}$-orbit in $\hat{N}(O)$ contains at least one element from $\hat{N}(\pi)$. The lemma then follows from Lemma 5.3.1.

Lemma 5.3.3: *Each $\tilde{G}$-orbit O in $\hat{H}$ carries a non-zero, positive, invariant Radon measure.*

Proof: Let $\tilde{R}$ be the analytic subgroup in $\tilde{G}$ corresponding to $\mathfrak{r}$. Pick $\pi\in O$. First we show that $\tilde{R}\tilde{G}_\pi$ is a closed subgroup in $\tilde{G}$: Let $\alpha: \tilde{G}/\tilde{G}_\pi \to O$ be the map $\dot{s}\to s\pi$. Then α is a homeomorphism ([12] Theorem 1, p. 124). Also, the $\tilde{R}$-orbit $\tilde{R}\cdot\pi \subset O$ is locally closed (loc. cit.), hence locally compact in the topology induced from H. But then also $\tilde{R}\tilde{G}_\pi/\tilde{G}_\pi$ is locally compact in the topology induced from $\tilde{G}/\tilde{G}_\pi$, hence $\tilde{R}\tilde{G}_\pi$ is locally compact ([17] Theorem, p. 52). But then $\tilde{R}\tilde{G}_\pi$ is a closed subgroup. We next observe that since $[\tilde{R},\tilde{R}] \subset H \subset \tilde{G}_\pi$ we have that $\tilde{R}\tilde{G}_\pi/\tilde{G}_\pi$ carries the structure of an abelian locally compact group, hence it carries a non-zero, positive, invariant Radon measure ν. Let then S be the analytic subgroup corresponding to the Levi subalgebra $\mathfrak{s}$. S is a compact group. Define the map $\varepsilon: S\times \tilde{R}/\tilde{R}_\pi \to \tilde{G}/\tilde{G}_\pi$ by $\varepsilon: (s,\dot{r}) \to (s,r)^{\boldsymbol{\cdot}}$. This map is clearly welldefined, surjective and continuous. Moreover it is proper. In fact, if C is a compact subset of $\tilde{G}/\tilde{G}_\pi$, then $\varepsilon^{-1}(C)$ is closed and contained in $S\times(SC\cap \tilde{R}\tilde{G}_\pi/\tilde{G}_\pi)$, which is compact since $\tilde{R}\tilde{G}_\pi/\tilde{G}_\pi$ is closed in $\tilde{G}/\tilde{G}_\pi$ (cf. above). It follows that $\varepsilon^{-1}(C)$ is compact. We have thus proved that ε is proper. Define then the measure μ to be the image of $ds\otimes\nu$, where ds is a Haar measure on S. This μ is easily seen to be a non-zero, positive, invariant Radon measure on $\tilde{G}/\tilde{G}_\pi$. This proves the lemma.

Lemma 5.3.4: *Each $\tilde{G}$-orbit in $\hat{N}$ carries a non-zero, positive, invariant Radon measure.*

PROOF: This is clear, since $[\tilde{G}, \tilde{G}] \subset N \subset \tilde{G}_\rho$ for all $\rho \in \hat{N}$, and therefore $\tilde{G}/\tilde{G}_\rho$ carries the structure of an abelian locally compact group.

For a $\tilde{G}$-orbit O in $\hat{H}$, ket ν_O be the (essentially unique) non-zero, positive invariant Radon measure on O (Lemma 5.3.3.). We define the equivalence class of representations Π_O of H by

$$\Pi_O = \int_{\hat{H}}^{\oplus} \pi d\nu_O(\pi)$$

(cf. [9] 8.6.3. Définition). Similarly, if $\mathcal{O}$ is a $\tilde{G}$-orbit in $\hat{N}$ we let $\nu_{\mathcal{O}}$ denote the (essentially unique) non-zero, positive, invariant Radon measure on $\mathcal{O}$ (Lemma 5.3.4). We define the equivalence class of representations $P_{\mathcal{O}}$ of N by

$$P_{\mathcal{O}} = \int_{\hat{N}}^{\oplus} \rho d\nu_{\mathcal{O}}(\rho)$$

(cf. loc. cit.).

LEMMA 5.3.5: $\mathrm{ind}_{H \uparrow N} \Pi_O$ *is quasi-equivalent to* $\oplus_{\mathcal{O} \subset \hat{N}(O)} P_{\mathcal{O}}$.

PROOF: Set $X = \tilde{G}/\tilde{G}_\pi$, $Y = \tilde{G}/\tilde{G}_\pi N$ and $Z = \tilde{G}_\pi N/\tilde{G}_\pi$, observing that $\tilde{G}_\pi N$ is a closed subgroup in G. Let dx be the invariant measure ν_O on X and let dy be an invariant Radon measure on Y (existing since Y carries the structure of an abelian locally compact group). But then also Z carries an invariant measure dz, say, which can be normalized such that

$$\int_X \varphi(x) dx = \int_Y dy(\dot{s}) \int_Z \varphi(sz) dz$$

(cf. [2], p. 95).

Let $\sigma : \tilde{G}/\tilde{G}_\pi N = Y \rightarrow \tilde{G}$ be a Borel section with $\sigma(\tilde{G}_\pi N) = e$ and let $\sigma_1 : \tilde{G}_\pi N/\tilde{G}_\pi = Z \rightarrow \tilde{G}_\pi N$ be a Borel section with $\sigma_1(\tilde{G}_\pi) = e$ and such that $\sigma_1(z) \in N$ for all $z \in Z$ (this is possible since $\tilde{G}_\pi N/\tilde{G} \sim N/N_\pi$). Now there is defined a Borel isomorphism $Y \times Z \rightarrow X$ by $(y, z) \rightarrow c(y)z$. We can extend σ_1 to $\tilde{G}/\tilde{G}_\pi = X$ as a Borel section, also called σ_1, such that $\sigma_1(x) = \sigma(y)\sigma_1(z)$ if $x = \sigma(y)z$.

Having this setup we can describe Π_O by

$$\Pi_O = \int_X^{\oplus} \sigma_1(x)\pi \, dx$$

on the Hilbert space $\int_X^{\oplus} H_x\, ds$, where $x \to H_x$ is the constant field of Hilbert spaces with $H_x = H_\pi$ (here we identify $\tilde{G}/\tilde{G}_\pi$ with O). Therefore $\mathrm{ind}_{H \uparrow N} \Pi_O$ can be described by the direct integral

$$\tilde{\Pi}_O = \int_X^{\oplus} \sigma_1(x)\tilde{\pi}\, dx$$

on the Hilbert space $\int_X^{\oplus} \tilde{H}_x\, dx \simeq L^2(X, dx, H_{\tilde{\pi}})$, where $\tilde{\pi} = \mathrm{ind}_{H \uparrow N}\pi$ and where $x \to \tilde{H}_x$ is the constant field of Hilbert spaces with $\tilde{H}_x = H_{\tilde{\pi}}$. Define the unitary R on $L^2(X, dx, H_{\tilde{\pi}})$ by $RF(x) = \tilde{\pi}(\sigma_1(z))F(x)$, where $x = c(y)z$. Then $R\tilde{\Pi}_O(s)R^{-1}F(x) = \tilde{\pi}(\sigma_1(z))(\sigma_1(x)\tilde{\pi})(s)\tilde{\pi}(\sigma_1(z)^{-1})F(x) = \tilde{\pi}(\sigma_1(z)\sigma_1(x)^{-1}s\sigma_1(x)\sigma_1(z)^{-1})F(x) = (\sigma(y)\tilde{\pi})(s)F(x)$, where $x = c(y)z$. Now identifying $L^2(X, dx, H_{\tilde{\pi}})$ with $L^2(Y, dy, H_{\tilde{\pi}}) \otimes L^2(Z, dz, \mathbb{C})$ via the above Borel isomorphism we see that $\tilde{\Pi}_O$ is quasi-equivalent to the representation

$$\int_Y^{\oplus} \sigma(y)\tilde{\pi}\, dy.$$

Let then $\rho \in \hat{N}(\pi)$. We have $\tilde{G}_\pi N \supset \tilde{G}_\rho$ and $\tilde{G}_\pi N/\tilde{G}_\rho$ is abelian. Let $\bar{X} = \tilde{G}/\tilde{G}_\rho$ and $\bar{Z} = \tilde{G}_\pi N/\tilde{G}_\rho$ and let $\bar{\sigma}_1 : \bar{Z} \to \tilde{G}_\pi N$ be a Borel section with $\bar{\sigma}_1(\tilde{G}_\pi) = e$. As above we define a Borel isomorphism $Y \times \bar{Z} \to \bar{X} : (y, \bar{z}) \to \sigma(y)\bar{z}$ and extend $\bar{\sigma}_1$ to $\bar{X}$ so that $\bar{\sigma}_1(\bar{x}) = \sigma(y)\bar{\sigma}_1(\bar{z})$ if $\bar{x} = \sigma(y)\bar{z}$. Also, there are invariant measures $d\bar{x}, d\bar{z}$, such that $d\bar{x}$ on $\bar{X}$ corresponds to $dy \otimes d\bar{z}$ on $Y \times \bar{Z}$ via the above isomorphism.

Let us then observe that there exists a Borel function $\bar{z} \to U_{\bar{z}}$ from $\bar{Z}$ into the unitary group of $H_{\tilde{\pi}}$ such that $U_{\bar{z}}\tilde{\pi}(s)U_{\bar{z}}^{-1} = \bar{\sigma}_1(\bar{z})\tilde{\pi}(s)$. To see this we first note that since $\tilde{G}_\pi$ stabilizes the irreducible representation π of H there exists a Borel function $t \to v(t)$ from $\tilde{G}_\pi$ into the unitary group of H_π, such that $v(t)\pi(s)v(t)^{-1} = (t\pi)(s)$. But then there exists a Borel function $t \to V(t)$ from $\tilde{G}_\pi$ into the unitary group of $H_{\tilde{\pi}}$, such that $V(t)\tilde{\pi}(s)V(t)^{-1} = (t\tilde{\pi})(s)$. Next we find Borel functions $a : \tilde{G}_\pi N \to \tilde{G}_\pi$ and $b : \tilde{G}_\pi N \to N$, such that $s = a(s)b(s)$, $s \in \tilde{G}_\pi N$. The existence of such functions a and b are easily verified using the existence of Borel sections on coset spaces of closed subgroups. Define then finally $U_{\bar{z}} = \tilde{\pi}(b(\bar{\sigma}_1(\bar{z})))V(a(\bar{\sigma}_1(\bar{z})))$. Then $\bar{z} \to U_{\bar{z}}$ is a Borel function and it is easily seen that $U_{\bar{z}}\tilde{\pi}(s)U_{\bar{z}}^{-1} = \bar{\sigma}_1(\bar{z})\tilde{\pi}(s)$ for $\bar{z} \in \bar{Z}$, $s \in N$.

Now arguing as above we find that

$$\int_X^{\oplus} \sigma_1(\bar{x})\tilde{\pi}\, d\bar{x}$$

is quasi-equivalent to

$$\int_Y^{\oplus} \sigma(y)\tilde{\pi}\, dy,$$

and comparing this to what we saw above we find that $\tilde{\Pi}_O$ is quasi-equivalent to

$$\int_{\tilde{G}/\tilde{G}_\rho}^{\oplus} \bar{\sigma}_1(\bar{x})\tilde{\pi}\, d\bar{x}.$$

We then apply the above to a fixed element $\rho_0 \in \hat{N}(\pi)$ and recall that $\tilde{\pi}$ is quasi-equivalent to $\oplus_{\rho\in\hat{N}(\pi)}\rho$ (Lemma 5.3.1), from which $\tilde{\Pi}_O$ is quasi-equivalent to

$$\oplus_{\rho\in\hat{N}(\pi)} \int_{\tilde{G}/\tilde{G}_{\rho_0}}^{\oplus} \bar{\sigma}_1(\bar{x})\rho\, d\bar{x}.$$

Letting $\mathcal{O}_0$ be the G-orbit through ρ_0 we have that $P_{\mathcal{O}_0}$ is represented by

$$\int_{G/G_{\rho_0}}^{\oplus} \bar{\sigma}_1(\bar{x})\rho_0\, d\bar{x}.$$

The conclusion of all this is then that $\tilde{\Pi}_O$ is quasi-equivalent to a representation, which contains $P_{\mathcal{O}_0}$ as a subrepresentation, and this is so for all $\tilde{G}$-orbits $\mathcal{O}_0$ contained in $\hat{N}(O)$. Now taking a multiple of $\tilde{\Pi}_O$ we then arrive at the following preliminary result: *$\tilde{\Pi}_O$ is quasi-equivalent to a representation which contains $\oplus_{\mathcal{O}\subset\hat{N}(O)}P_{\mathcal{O}}$ as a subrepresentation.*

We then consider the other direction of the lemma. As we saw $\tilde{\Pi}_O$ is quasi-equivalent to

$$\int_Y^{\oplus} \sigma(y)\tilde{\pi}\, dy,$$

which, by Lemma 5.3.3, is quasi-equivalent to

$$\oplus_{\rho\in\hat{N}(\pi)} \int_Y^{\oplus} \sigma(y)\rho\, dy.$$

For a given $\tilde{G}$-orbit $\mathcal{O}\subset\hat{N}(O)$ and $\rho\in\mathcal{O}\cap\hat{N}(\pi)$ we also saw, with the notation from above, that $P_{\mathcal{O}}$ is described by

$$\int_{Y\times\bar{Z}}^{\oplus} \sigma(y)\bar{\sigma}_1(\bar{z})\rho\, dy\, d\bar{z}. \tag{*}$$

Let us then observe that $\bar{Z}$ is countable. In fact, $\tilde{G}_\rho N$ acts as a transformation group in $\hat{N}(\pi)$, which is countable (lemma 5.3.1), and therefore $\tilde{G}_\rho N/\tilde{G}_\pi = \bar{Z}$ is countable. It follows that we can write the representation (*) as

$$\oplus_{\bar{z}\in\bar{Z}} \int_Y^{\oplus} \sigma(y)\bar{\sigma}_1(\bar{z})\rho\, dy.$$

The summand corresponding to $\bar{z} = \tilde{G}_\rho$ is $\int_Y^{\oplus} \sigma(y)\rho\, dy$ from which we conclude that $\int_y^{\oplus} \sigma(y)\rho\, dy$ is a subrepresentation of $P_{\mathcal{O}}$. But then $\oplus_{\rho\in\hat{N}(\pi)} \int_Y^{\oplus} \sigma(y)\rho\, dy$ is equivalent to a subrepresentation of $\oplus_{\rho\in\hat{N}(\pi)} P_{\mathcal{O}(\rho)}$, where $\mathcal{O}(\rho)$ is the $\tilde{G}$-orbit through ρ. Now clearly $\oplus_{\rho\in\hat{N}(\pi)} P_{\mathcal{O}(\rho)}$ is quasi-equivalent to $\oplus_{\mathcal{O}\subset\hat{N}(O)} P_{\mathcal{O}}$. We thus arrive at the following result: *$\oplus_{\mathcal{O}\subset\hat{N}(O)} P_{\mathcal{O}}$ is quasi-equivalent to a representation which has a subrepresentation quasiequivalent to $\tilde{\Pi}_O$.* Comparing this with our previous result we finally arrive at the conclusion that $\oplus_{\mathcal{O}\subset\hat{N}(O)} P_{\mathcal{O}}$ is quasi-equivalent to $\tilde{\Pi}_O$. This proves the lemma.

Assume now that $\chi: \tilde{G} \to \mathbb{R}_+^*$ is a continuous homomorphism with $\ker\chi \supset N$ and that μ_O is a non-zero, positive, χ-relatively invariant Radon measure on O. Then also each $\tilde{G}$-orbit $\mathcal{O}$ in $\hat{N}(O)$ carries a non-zero, positive, relatively invariant Radon measure with multiplier χ, $\mu_{\mathcal{O}}$, say. Let f_O be the $(\tilde{G}, \chi)$-character on H given by

$$f_O(x) = \int_O \operatorname{Tr}(\pi(x))\, d\mu_O(\pi), x \in C^*(H)^+,$$

(cf. [21] Proposition 5.1.4) and let $f_{\mathcal{O}}$ be the $(\tilde{G}, \chi)$-character on N given by

$$f_{\mathcal{O}}(x) = \int_{\mathcal{O}} \operatorname{Tr}(\rho(x))\, d\mu_{\mathcal{O}}(\rho), x \in C^*(N)^+,$$

(cf. loc. cit.)

LEMMA 5.3.6: *The (non-zero) measures $\mu_{\mathcal{O}}$ and μ_O can be normalized such that*

$$\operatorname{ind}_{H\uparrow N} f_O(x) = \textstyle\sum_{\mathcal{O}\subset\hat{N}(O)} f_{\mathcal{O}}(x), x \in C^*(N)^+.$$

PROOF: Set $\tilde{f}_O = \mathrm{ind}_{H\uparrow N} f_O$. Since λ_{f_O} is quasi-equivalent to Π_O (cf. above) we get that $\lambda_{\tilde{f}_O} \approx \tilde{\lambda}_{f_O} = \mathrm{ind}_{H\uparrow N}\lambda_{f_O}$ is quasi-equivalent to $\tilde{\Pi}_O = \mathrm{ind}_{H\uparrow N}\Pi_O$. Similarly $\lambda_{f_{\mathcal{O}}}$ is quasi-equivalent to $P_{\mathcal{O}}$ and we then get from Lemma 5.3.5. that $\lambda_{\tilde{f}_O}$ is quasi-equivalent to $\oplus_{\mathcal{O}\subset\hat{N}(O)}\lambda_{f_{\mathcal{O}}}$. Let us then observe that the latter decomposition is central. This follows from the fact that the $\lambda_{f_{\mathcal{O}}}$'s are mutually disjoint, since they live on disjoint subsets in $\hat{N}$, and therefore they are mutually strongly disjoint (since H is type I, cf. [11] Corollary 3.10, p. 102), and from [11] Theorem 4.3, p. 105. This also implies that $\lambda_{\tilde{f}_O}$ and $\oplus_{\mathcal{O}\subset\hat{N}(O)}\lambda_{f_{\mathcal{O}}}$ are in fact equivalent, since they generate the von Neumann algebras $U_{\tilde{f}_O}$ and $\oplus_{\mathcal{O}\subset\hat{N}(O)}U_{f_{\mathcal{O}}}$, respectively, and these are on standard form (cf. [8] Theorem 6, p. 225). Now $U_{\tilde{f}_O}$ carries the faithful, normal, semifinite trace $\omega_{\tilde{f}_O}$, and disintegrating this trace we obtain traces $\omega_{\mathcal{O}}$ on $U_{f_{\mathcal{O}}}$ which are proportional to $\omega_{f_{\mathcal{O}}}$, and normalizing the $\omega_{\mathcal{O}}$'s properly we can assume that $\omega_{f^{\mathcal{O}}} = \omega_{\mathcal{O}}$, that is, that $\omega_{\tilde{f}_O} = \oplus_{\mathcal{O}\subset\hat{N}(O)}\omega_{f_{\mathcal{O}}}$. From this the lemma follows.

5.4. The purpose of this section is to prove the following lemma:

LEMMA 5.4.1: *Let O be a $\tilde{G}$-orbit in $\hat{H}$. There exists a continuous homomorphism $\chi: \tilde{G} \to \mathbb{R}_+^*$ with* $\ker\chi \supset N$ *and a non-zero, positive, χ-relatively invariant Radon measure μ on O, such that the $(\tilde{G}, \chi)$-character f_μ on H given by*

$$f_\mu(x) = \int_O \mathrm{Tr}(\pi(x))\, d\mu(\pi), x \in C^*(H)^+,$$

(*cf.* [21] *Proposition* 5.1.4) *is smooth.*

PROOF: First we prove:

LEMMA 5.4.2: *For each $\tilde{G}$-orbit Ω in $\mathfrak{h}'$ there exists a continuous homomorphism $\chi: \tilde{G} \to \mathbb{R}_+^*$ with* $\ker\chi \supset N$, *such that Ω carries a non-zero, positive, tempered* (*cf.* [19] *Definition* 3.3.6) *χ-relatively invariant Radon measure.*

PROOF: Let $f \in \Omega$. We first consider the E-orbit through f (cf. 5.2). The space $\mathfrak{e} = \mathfrak{k} \oplus \mathfrak{m}$ is a solvable subalgebra of $\mathrm{gl}(V)$ consisting of elements with real eigenvalues and its image in $\mathrm{gl}(\mathfrak{h})$ via the representation $X \to \mathrm{ad}\, X|\mathfrak{h} : \mathfrak{e} \to \mathrm{gl}(\mathfrak{h})$ is a subalgebra of the same kind. Therefore there exists a finite sequence $\mathfrak{h}_j$, $j = 0, \ldots, m$, of subspaces in $\mathfrak{h}$ with

$$\mathfrak{h} = \mathfrak{h}_m \supset \mathfrak{h}_{m-1} \supset \ldots \supset \mathfrak{h}_1 \supset \mathfrak{h}_0 = \{0\},$$

such that $\dim \mathfrak{h}_j/\mathfrak{h}_{j-1} = 1$, $j = 1,\dots,m$, and such that $[\mathfrak{e},\mathfrak{h}_j] \subset \mathfrak{h}_j$, $j = 0,1,\dots, m$. Let $\mathfrak{h}_j^\perp$ be the orthogonal subspace of $\mathfrak{h}_j$ in $\mathfrak{h}'$. We then have the sequence

$$\mathfrak{h}' = \mathfrak{h}_0^\perp \supset \mathfrak{h}_1^\perp \supset \dots \supset \mathfrak{h}_{m-1}^\perp \supset \mathfrak{h}_m^\perp = \{0\},$$

and we have $\dim \mathfrak{h}_{j-1}^\perp/\mathfrak{h}_j^\perp = 1$ and $s\mathfrak{h}_j^\perp \subset \mathfrak{h}_j^\perp, j = 1,\dots,m, s \in E$.

Let $v_j \in \mathfrak{h}_{j-1}^\perp \backslash \mathfrak{h}_j^\perp$, $j = 1,\dots,m$. From [19] p. 235 *e.* we then conclude that there exists a subset $\{j_1 < \dots < j_d\} \subset \{1,\dots,m\}$, such that the E-orbit $\Omega_1 = Ef$ can be described by

$$\Omega_1 = \{\textstyle\sum_{j=1}^m F_j(x)v_j | x \in D\},$$

where F_j are C^∞-functions on the open subset $D \subset \mathbb{R}^d$ with $F_{j_k}(x) = x_k$ and such that $F_j(x)$ only depends on $x_1,\dots,x_k$, where $k = \max\{r | j_r \leq j\}$. The map $\alpha: D \to \Omega_1 : x \to \sum_{j=1}^m F_j(x)v_j$ is a homeomorphism between D and Ω_1.

Let $\lambda_j : \mathfrak{e} \to \mathbb{R}$ be the homomorphisms with $Xv_j = \lambda_j(X)v_j (\mathrm{mod}\, \mathfrak{h}_j^\perp)$, and let $\Lambda_j : E \to \mathbb{R}_+^*$ be the continuous homomorphism with $\Lambda_j(\exp X) = \exp \lambda_j(X)$. Λ_j is identically one on M, since λ_j vanishes on $\mathfrak{m}$ (because $\mathfrak{m}$ is a nilpotent ideal). Set $\Lambda = \prod_{k=1}^d \Lambda_{j_k}$ and let S be the analytic subgroup corresponding to the Levi subalgebra $\mathfrak{s}$. Arguing like [19], p. 238 and using that $\mathfrak{s}$ and $\mathfrak{a} = \mathfrak{t} \oplus \mathfrak{k}$ commute (cf. 5.2) and that S is compact we find that the measure

$$\nu(\varphi) = \int_{S \times T \times D} \varphi(st\alpha(x))\, ds\, dt\, dx, \varphi \in \mathscr{K}(\Omega),$$

is a non-zero, positive, tempered, relatively invariant Radon measure on Ω with multiplier $\Lambda : \tilde{G}_1 \to \mathbb{R}_+^*$. Here Λ is an extension to $\tilde{G}_1$ of the Λ defined above.

Lifting Λ to $\tilde{G}$ and calling it χ we have proved the lemma, since clearly $\ker \chi \supset N$.

We then turn to the proof of Lemma 5.4.1: Let $\pi \in O$ and let $f \in \mathfrak{h}'$ be an element in the orbit in $\mathfrak{h}'$ associated to π by the Kirillov theory. Let Ω be the $\tilde{G}$-orbit through f. Since $\tilde{G}$ is the universal covering group of the connected component of an algebraic group it follows that Ω is locally closed, and thus the map $\tilde{G}/\tilde{G}_f \to \Omega : \dot{s} \to sf$ is a homeomorphism. Arguing like in the proof of [19] Proposition 3.1.7, p. 218 one finds that $\tilde{R}/\tilde{R}_f$ admits a non-zero, positive, invariant Radon measure. Using the fact that also the orbit $\tilde{R}f$ is locally closed we can conclude, by an argument completely analogous to the one given in the proof of Lemma 3.3, that $\tilde{G}/\tilde{G}_f$, and therefore Ω, carries a non-zero positive, invariant Radon measure dx, say. We set $X = \tilde{G}/\tilde{G}_f$. Also we set $Y = \tilde{G}/\tilde{G}_\pi$ and recall that $\tilde{G}_\pi = \tilde{G}_f H$. Y carries a non-zero, positive, invariant Radon measure dy, say (Lemma 5.3.4). Finally, set $Z = \tilde{G}_f H/\tilde{G}_f = \tilde{G}_\pi/\tilde{G}_f$. Z carries an invariant measure dz, say, such that

$$\int_X \varphi(x)\,dx = \int_Y dy(\dot{s}) \int_Z \varphi(sz)\,dz$$

for $\varphi \in L^1(X, dx)$ ([2], p. 95).

Now let ω_y, $y \in Y$, be the H-orbit in $\mathfrak{h}'$ given by $\omega_y = Hsf$, where $y = s\tilde{G}_\pi$. ω_y is clearly a well-defined H-orbit, and as such it is closed in $\mathfrak{h}'$. Let β_y be the measure on ω_y given by

$$\int_{\omega_y} \psi(l)\,d\beta_y(l) = \int_Z \psi(shf)\,dz(\dot{h})$$

for $\psi \in \mathscr{K}(\omega_y)$. β_y is then a non-zero, positive, H-invariant Radon measure. We then claim that if dz is normalized such that β_y is "canonical" (i.e. the Kirillov character formula is valid with β_y as the invariant measure on the orbit) for one $y \in Y$, then it is "canonical" for all $y \in Y$. In fact, this is seen by an argument similar to the one used in the proof of [19] Lemma 3.3.14, pp. 230–231.

By Lemma 5.4.2 there exists a continuous homomorphism $\chi : \tilde{G} \to \mathbb{R}_+^*$ with $\ker\chi \supset N$ and a non-zero, positive, χ-relatively invariant Radon measure ν on Ω, such that ν is tempered. Since Ω carries an invariant measure we have $\ker\chi \supset \tilde{G}_f$ and thus $\ker\chi \supset \tilde{G}_f N \supset \tilde{G}_\pi$, which shows that O carries a non-zero, positive, χ-relatively invariant Radon measure μ, and this is given by

$$\int_O h(x)\,d\mu(x) = \int_Y h(s\pi)\chi(s)\,dy(\dot{s}).$$

Let then $m \in \mathbb{N}$ be chosen such that the Fourier transform of all functions in $C_c^{2m}(\mathfrak{h})$ are ν-integrable over Ω (this is possible, since ν is tempered). For $\varphi \in C_c^m(G)$ we then get

$$\int_\Omega (\varphi^* * \varphi \circ \exp)^\wedge (l)\,d\nu(l) = \int_{\tilde{G}/\tilde{G}_f} (\varphi^* * \varphi \circ \exp)^\wedge (sf)\chi(s)\,dx(\dot{s})$$

$$= \int_Y dy(\dot{s}) \int_Z (\varphi^* * \varphi \circ \exp)^\wedge (shf)\chi(s)\,dz(\dot{h})$$

$$= \int_Y \chi(s)\,dy(\dot{s}) \int_{\omega_y} (\varphi^* * \varphi \circ \exp)^\wedge(l)\,d\beta_y(l)$$

$$= \int_Y \mathrm{Tr}(s\pi(\varphi^* * \varphi))\chi(s)\,dy(\dot{s}) = f_\mu(\varphi^* * \varphi).$$

Here we have applied the Kirillov character formula (see e.g. [22]). It follows from the above that $C_c^m(G) \subset \mathfrak{n}_{f_\mu}$. But then f_μ is smooth by [19] Proposition 2.3.5, p. 206. This ends the proof of Lemma 5.4.1.

5.5. Let us recall that we are dealing with a connected, simply connected Lie group G with cocompact radical and with nilradicalisé N. In particular we have the equivalence relation $\mathscr{E}_N^G$ in $\hat{N}$ (cf. [21] Lemma 6.1.1 and [21] Sect. 5.1).

Proposition 5.5.1: *Let O be an orbit of $\mathscr{E}_N^G$ in $\hat{N}$. There exists a continuous homomorphism $\chi: G \to \mathbb{R}_+^*$ with* $\ker \chi \supset N$ *and a non-zero, positive G-relatively invariant Radon measure ν on O, such that the multiplier of ν is χ and such that the (G,χ)-character f_ν on N given by*

$$f_\nu(x) = \int_O \mathrm{Tr}(\pi(x))\,d\nu(\pi)$$

(*cf.* [21] *Proposition* 5.1.4) *is smooth.*

Proof: We use all the notation from the previous sections. The orbit O is contained in a unique $\tilde{G}$-orbit $\mathcal{O}$ in $\hat{N}$ and there is a well-defined $\tilde{G}$-orbit O_0 in $\hat{H}$, such that $\mathcal{O} \subset \hat{N}(O_0)$ (cf. 5.3). From Lemma 5.4.1 we get that there exists a continuous homomorphism $\chi: \tilde{G} \to \mathbb{R}_+^*$ with $\ker \chi \supset N$ and a non-zero, positive, χ-relatively invariant Radon measure μ on O, such that the $(\tilde{G},\chi)$-character f_μ on H given by

$$f_\mu(x) = \int_O \mathrm{Tr}(\pi(x))\,d\mu(\pi), \quad x \in C^*(H)^+,$$

is smooth. It follows from Proposition 1.1.3 that $\mathrm{ind}_{H\uparrow N} f_\mu$ is smooth, and using Lemma 5.3.6 and Lemma 1.1.2 we then get that the $(\tilde{G},\chi)$-character $f_{\mathcal{O}}$ is smooth. Here $f_{\mathcal{O}}$ is the $(\tilde{G},\chi)$-character associated to a non-zero, positive, χ-relatively invariant Radon measure $\nu_{\mathcal{O}}$ on $\mathcal{O}$, cf. 5.3.

Let then $\rho \in O$ be a fixed element and set $G_1 = \overline{\tilde{G}_\rho G}$. Then $G_1 \cdot \rho = O$. Moreover there exists a $\chi | G_1$-relatively invariant Radon measure ν on $G_1/\tilde{G}_\rho$, such that

$$\int_{\tilde{G}/\tilde{G}_\rho} h(s\rho)\,d\nu_{\mathcal{O}}(s) = \int_{\tilde{G}/G_1} d\dot{s} \int_{G_1/\tilde{G}_\rho} h(st\rho)\,d\nu(\dot{t}),$$

where $d\dot{s}$ is an invariant measure on $\tilde{G}/G_1$. We then get

$$\begin{aligned} f_{\mathcal{O}}(x) &= \int_{\tilde{G}/\tilde{G}_\rho} \operatorname{Tr}(s\rho(x))\,d\nu_{\mathcal{O}}(x) = \int_{\tilde{G}/\tilde{G}_\rho} \operatorname{Tr}(\rho\circ\tau_N^{\tilde{G}}(s^{-1})(x))\,d\nu_{\mathcal{O}}(s) \\ &= \int_{\tilde{G}/G_1} d\dot{s} \int_{G_1/\tilde{G}_\rho} \operatorname{Tr}(\rho\circ\tau_N^{\tilde{G}}(t^{-1})\circ\tau_N^{\tilde{G}}(s^{-1})(x))\,d\nu(\dot{t}) \\ &= \int_{\tilde{G}/G_1} f_\nu\circ\tau_N^{\tilde{G}}(s^{-1})(x)\,ds. \end{aligned}$$

It then follows from Lemma 1.1.2 that $f_\nu\circ\tau_N^{\tilde{G}}(s^{-1})$ is smooth for almost all $\dot{s}$, and therefore for all $\dot{s}$, since $\tau_N^{\tilde{G}}(s^{-1})$ is an automorphism of $C_c^\infty(G)$. In particular f_ν is smooth and this proves the lemma.

5.6. In this section we shall derive a result which will be needed in the final step of our proof of (iv) $\Rightarrow$ (i). Actually we shall for later reference consider a situation which is a little more general than actually needed. We shall go back to the notation used in [21] Sect. 5. Specifically we shall consider a separable, locally compact group $\tilde{M}$ and closed, normal subgroups $M \supset G \supset N$ of $\tilde{M}$, such that (i) N is type I, (ii) G/N is central in M/N, (iii) N is regularly imbedded in $\tilde{M}$, (iv) G/N is central in $\tilde{M}/N$ and (v) $[\tilde{M},\tilde{M}] = [M,M]$.

Let O be an orbit of $\mathscr{E}_N^M$ in $\hat{N}$ (cf. loc. cit.) and let $\eta: M \to \mathbb{R}_+^*$ be a continuous homomorphism. We assume that O carries a non-zero, positive, η-relatively invariant Radon measure ν. We form the (M,η)-trace f_ν on N given by

$$f_\nu(x) = \int_O \operatorname{Tr}(\pi(x))\,d\nu(\pi), \qquad x\in C^*(N)^+$$

(cf. [21] Proposition 5.1.4).

Let $F = F(O)$ be the group associated to O as in [21] Section 5.1 and set $\chi = \eta\cdot(\Delta_{M/N}\circ c_{M/N})(\Delta_{M/F}\circ c_{M/F})^{-1}$.

We then consider the induced trace $\tilde{f}_\nu = \operatorname{ind}_{N\uparrow F} f_\nu$, which is an (M,χ)-trace on F, cf. [21] Sect. 2.3. Now clearly $\tilde{f}_\nu$ lives on O so by [21] Proposition 5.1.4 $\tilde{f}_\nu$ has the form

$$\tilde{f}_\nu(x) = \int_{\mathscr{U}} \operatorname{Tr}(\rho(x))\,d\bar{\mu}(\rho), \qquad x\in C^*(F)^+,$$

for some M-invariant, open subset $\mathscr{U} \subset X(O)$ (cf. [21] Sect. 5.1) and some positive, χ-relatively invariant Radon measure $\bar{\mu}$ on $\mathscr{U}$.

We shall determine $\mathscr{U}$ and $\bar{\mu}$. To do this, let $\mathscr{M}_1$ be the group $M_1 \times \hat{H}$ acting transitively in the locally compact Hausdorff subspace $X(O)$ of $\hat{F}$, cf. [21] Sect. 5.1. We define a group of automorphisms a of $\mathscr{M}_1$ on $C^*(F)$ by $a(m,p) = \tau_F^{M_1}(m)\hat{\tau}_N^F(p|F)$ (cf. [21] Sect. 1.3). This is possible since the actions of $\tau_F^{M_1}(m)$ and $\hat{\tau}_N^F(p|F)$ commute. The automorphism group a gives rise to an action of $\mathscr{M}_1$ on $\hat{F}$ *by* $(m,p)\rho = \rho \circ a(m,p)^{-1}$, and the restriction of this action to $X(O)$ is precisely the action considered above. Now as showed in the proof of [21] Lemma 5.1.2 the measure v is actually M_1-relatively invariant with a multiplier $\eta_1 : M_1 \to \mathbb{R}_+^*$ extending η. Set $\chi_1 = \chi \cdot (\Delta_{M_1/N} \circ c_{M_1/N})(\Delta_{M_1/F} \circ c_{M_1/F})^{-1}$ and extend χ_1 to $\mathscr{M}_1$ by the requirement that it is trivial on $\hat{H}$ (and clearly the restriction of χ_1 to M is then χ). It then follows from [21] Sect. 1.3 and Theorem 3.2.1 that $\tilde{f}_v$ is relatively invariant under the automorphism group $(m,p) \to a(m,p)$ with multiplier χ_1. But then it follows in particular from the proof of [21] Proposition 5.1.4 that $\mathscr{U}$ can be chosen to be $\mathscr{M}_1$-invariant, hence $\mathscr{U} = X(O)$, since $\mathscr{M}_1$ acts transitively in $X(O)$, and therefore $\bar{\mu}$ is a $\mathscr{M}_1$-relatively invariant Radon measure on $X(O)$ with multiplier χ_1.

Let then ρ be a fixed element in $X(O)$ and let $\mathcal{O}$ be the orbit of $\mathscr{L} = \mathscr{L}_N^M$ through ρ, cf. [21] Sect. 5.1. Let $\mathscr{M}$ be the closure of $M(\mathscr{M}_1)_\rho$ in $\mathscr{M}_1$, cf. loc. cit. We have $\mathcal{O} = \mathscr{M} \cdot \rho$ and the stabilizer in $\mathscr{M}_1$ of the orbit $\mathcal{O}$ is $\mathscr{M}$. Finally let $d\dot{m}$ be an invariant measure on $\mathscr{M}_1/\mathscr{M}$ (existing since $[\mathscr{M}_1, \mathscr{M}_1] \subset$ M and therefore $\mathscr{M}_1/\mathscr{M}$ carries the structure of an abelian locally compact group). With all these ingredients we then have:

LEMMA 5.6.1: *The orbit $\mathcal{O}$ carries a non-zero, positive, χ-relatively invariant Radon measure μ, which can be normalized such that*

$$\tilde{f}_v(x) = \int_{\mathscr{M}_1/\mathscr{M}} f_\mu \circ a(m^{-1})(x)\chi_1(m)\,d\dot{m}, \qquad x \in C^*(G)^+$$

(*here f_μ is the (M,χ)-trace associated to the measure μ, cf.* [21] *Proposition* 5.1.4).

PROOF: As we saw above the measure $\bar{\mu}$ can be considered as a Radon measure on $\mathscr{M}_1/\mathscr{M}_\rho$. From [3] it is then easily derived that $\mathscr{M}/\mathscr{M}_\rho$ carries a relatively invariant Radon measure with multiplier $\chi_1|\mathscr{M}$, such that

$$\bar{\mu} = \int_{\mathscr{M}_1/\mathscr{M}} \mu_m\, d\dot{m},$$

where $\dot{m} \to \mu_{\dot{m}}$ is the $d\dot{m}$-adequate (cf. [4]) family of Radon measures on $X(O)$ given by

$$\mu_{\dot{m}}(\varphi) = \chi_1(m) \int_{\mathcal{M}/\mathcal{M}_\rho} \varphi(mm_1\rho)\, d\mu(\dot{m}_1).$$

Considering μ as a Radon measure on $\mathcal{O}$ we then get

$$\begin{aligned}
\tilde{f}_\nu(x) &= \int_{\mathcal{M}_1/\mathcal{M}_\rho} \mathrm{Tr}(m\rho(x))\, d\bar{\mu}(\dot{m}) \\
&= \int_{\mathcal{M}_1/\mathcal{M}} d\dot{m} \int_{\mathcal{M}/\mathcal{M}_\rho} \chi_1(m)\, \mathrm{Tr}(mm_1\rho(x))\, d\mu(\dot{m}_1) \\
&= \int_{\mathcal{M}_1/\mathcal{M}} d\dot{m} \int_{\mathcal{M}/\mathcal{M}_\rho} \chi_1(m)\, \mathrm{Tr}(m_1\rho \circ a(m^{-1})(x))\, d\mu(\dot{m}_1) \\
&= \int_{\mathcal{M}_1/\mathcal{M}} f_\mu(a(m^{-1})x)\chi_1(m)\, d\dot{m},
\end{aligned}$$

and this proves the lemma.

5.7. We shall now end the proof of (iv) $\Rightarrow$ (i). So let π be a normal representation of G. The character of π arise as $\mathrm{ind}_{F \uparrow G} f_\mu$, where μ is an invariant measure on an orbit $\mathcal{O}$ of $\mathcal{L}_N^G$ in $X(O)$, O being an orbit of $\mathcal{E}_N^G$ in $\hat{N}$ ([21] Theorem 6.2.1). Assume then first that O carries a G-invariant measure ν, such that f_ν is smooth. Then $\mathrm{ind}_{N \uparrow F} f_\nu = \tilde{f}_\nu$ is smooth by proposition 1.1.3 and therefore, with the notation from Lemma 5.6.1, $f_\mu \circ a(m^{-1})$ is smooth for almost all $\dot{m}$ (Lemma 1.1.2) and therefore for all $\dot{m}$. This shows that f_μ is smooth and therefore also $\mathrm{ind}_{F \uparrow G} f_\mu$ is smooth (Proposition 1.1.3). Therefore, in this special case π has actually a smooth character.

Assume then that the invariant measure on O does not give rise to a smooth G-character on N. Then there exists a continuous homomorphism $\chi\,(\not\equiv 1)$ from G into $\mathbb{R}_+^*$ with $\ker\chi \supset N$ and a non-zero, positive, χ-relatively invariant Radon measure ν on O, such that f_ν is smooth (Proposition 5.5.1). Set $G_0 = \ker\chi$. Let us then observe that $F = F(O) \subset G_0$. In fact, $G_\pi \subset \ker\chi$ for $\pi \in O$, since O admits a $\tilde{G}$-invariant measure (Lemma 5.3.4). From Proposition 1.1.3 we get that $\tilde{f}_\nu = \mathrm{ind}_{N \uparrow F} f_\nu$ is a smooth (G,χ)-trace on F, and from Lemma 5.6.1 we get that the orbit $\mathcal{O}$ carries a relatively invariant measure μ_1 whose mutiplier is χ, and using Lemma 1.1.2 we get that $f_{\mu_1} \circ a(m^{-1})$ is smooth for almost all $\dot{m}$, and therefore for all $\dot{m}$. But this shows that f_{μ_1} is smooth, and therefore $\mathrm{ind}_{F \uparrow G} f_{\mu_1}$ is a smooth χ-semicharacter (Proposition 1.1.3) and it is a semicharacter for π ([21] Proposition 5.3.1). This ends the proof of Theorem 2, (iv) $\Rightarrow$ (i)

6. Proof of Theorem 1

For the equivalence of (ii), (iii) and (iv), see [27] Proposition 3, p. 47. Let us then assume that G satisfies condition (iv) in Theorem 1. We have to prove that (i) is satisfied. As mentioned in the beginning of Sect. 5 it is easy to reduce to the case where the radical of G is cocompact. Assume that this is so. We then have the following analog of Proposition 5.5.1 (using the notation from loc. cit.):

PROPOSITION 6.1: *Let O be an orbit of $\mathscr{E}_N^G$ in $\hat{N}$ and let ν be the non-zero, positive, invariant measure on O. Then the $(G, 1)$-character f_ν on N given by*

$$f_\nu(x) = \int_O \mathrm{Tr}(\pi(x))\, d\nu(\pi)$$

is smooth.

PROOF: Since the algebraic closure of a type R solvable Lie algebra of endomorphisms in $\mathrm{gl}(V)$ is also solvable and type R, it follows that Lemma 5.4.2 is valid with $\chi \equiv 1$ when the radical of G is of type R, and therefore Lemma 5.4.1 is valid with $\chi \equiv 1$. And from this Proposition 6.1 follows in the same way as Proposition 5.5.1 follows from Lemma 5.4.1.

Theorem 1 now follows from Proposition 6.1 and the remarks made in 5.7. This ends the proof of Theorem 1.

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(Oblatum 20-VII-1981)

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