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THE NON RATIONALITY OF THE GENERIC ENRIQUES' THREEFOLD

Luciana Picco Botta and Alessandro Verra

The Enriques threefold, i.e. the hypersurface of $\mathbb{P}^4$ having as hyperplane sections the classical Enriques surfaces (i.e. the surfaces of degree 6 in $\mathbb{P}^3$, passing through the edges of a tetrahedron), was studied classically by several Authors.

Fano suggested that it was not unirational ([**10**] p. 94), but Roth proved that it was unirational and, in order to prove the non-rationality, he gave an argument involving the Severi torsion. This point was in disagreement with Serre [**12**], where it is shown that a non singular unirational variety cannot have torsion. Tyrrel [**13**] pointed out that Roth's argument was not correct because of the existence of some not ordinary singular points.

In this note we find, for a generic Enriques threefold V, a non singular model $\tilde{V}'$ containing an open set W, which is a conic bundle (in the sense of [**1**]) over a suitable surface, with a complete non singular curve Δ of genus 5 as curve of the degenarate conics. By analyzing $\tilde{V}' - W$ explicitly we prove, as in the case of standard conics bundles, that the Chow group $A^2(\tilde{V}')$ is isomorphic to the Prym variety $Prym(\tilde{\Delta}/\Delta)$.

At the end, since Δ has genus 5 and so is not included in th. 4.9 of [**1**], we need some careful analysis about its halfcanonical series, to conclude that $Prym(\tilde{\Delta}/\Delta)$ is not a Jacobian of a curve and therefore V is not rational.

In the complex projective space $\mathbb{P}^4$ of homogeneous coordinates $(x_0 : x_1 : x_2 : x_3 : x_4)$ we consider the irreducible generic hypersurface V of equation:

(*) Lavoro eseguito nell'ambito del G.N.S.A.G.A. del C.N.R.

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$$x_1x_2x_3x_4\{x_0^2 + x_0 \sum_{i=1}^{4} a_ix_i + \sum_{i,j=1}^{4} b_{ij}x_ix_j\}$$
$$+ c_1x_2^2x_3^2x_4^2 + c_2x_3^2x_4^3x_1^2 + c_3x_4^2x_1^2x_2^2 + c_4x_1^2x_2^2x_3^2 = 0.$$

In particular, $c_i \neq 0$, $i = 1,2,3,4$.

It is known ([**11**] p. 44, [**13**] p. 897) that its generic hyperplane section is an Enriques surface and that V gets the following singularities:

(i) six double planes π_{ij} of equations $x_i = x_j = 0$, $1 \leq i < j \leq 4$,

(ii) four triple lines of equations $x_i = x_j = x_k = 0$, $1 \leq i < j < k \leq 4$,

(iii) one quadruple point at $0(1,0,0,0,0)$ and other two non ordinary quadruple points on each triple line.

It is also known that V is unirational ([**10**] p. 97).

In order to prove that V is non rational, we consider the following rational map:

$$\varphi : \mathbb{P}^4(x_0 : x_1 : x_2 : x_3 : x_4) - - \rightarrow \mathbb{P}^3(x : y : z : t)$$

given by

$$x : y : z : t = x_1x_3 : x_1x_4 : x_2x_3 : x_2x_4.$$

φ is not defined over the planes π_{12} and π_{34}, moreover the image of φ is the quadric surface $Q \subseteq \mathbb{P}^3$ of equation $xt = yz$.

Lemma 1: *For all $q \in Q$, let $E_q = \varphi^{-1}(q)$ be the inverse image of q. The Zariski closure of E_q is a plane in $\mathbb{P}^4$ passing through the point $0(1,0,0,0,0)$ and intersecting each plane π_{12} and π_{34} along a line.*

In other words, Q parametrizes the planes in $\mathbb{P}^4$ cutting these two fixed planes along a line.

Proof: Let $q = (\bar{x}, \bar{y}, \bar{z}, \bar{t}) \in Q$ and $p \in E_q$. Since p doesn't belong to the planes π_{12} and π_{34}, there exist only two hyperplanes passing through p and containing one of them. Their equations are precisely:

$$\begin{cases} \alpha x_1 - \beta x_2 = 0 \\ \gamma x_3 - \delta x_4 = 0 \end{cases} \qquad (*)$$

where

$$\alpha : \beta = \bar{t} : \bar{y} = \bar{z} : \bar{x}$$
and $\gamma : \delta = \bar{y} : \bar{x} = \bar{t} : \bar{z}$.

So the equations (*) define exactly the Zariski closure of E_q.

It follows immediately:

LEMMA 2: *The following (not linearly independent) equations in* $\mathbb{P}^4 \times \mathbb{P}^3$

$$\begin{cases} xt = yz \\ zx_1 - xx_2 = 0 \\ tx_1 - yx_2 = 0 \\ yx_3 - xx_4 = 0 \\ tx_3 - zx_4 = 0 \end{cases}$$

define the Zariski closure Γ_φ *of the graphe of* φ.

Note. Γ_φ can be obtained by blowing $\mathbb{P}^4$ up along the ideal of the planes π_{12} and π_{34}.

LEMMA 3: *The equations of lemma* 2, *together with the following ones*:

$$x^2(c_4x_2^2 + c_2x_4^2) + t^2(c_1x_3^2 + c_3x_1^2) + xt(x_0^2 + x_0 \sum_{i=1}^{4} a_ix_i + \sum_{i,j=1}^{4} b_{ij}x_ix_j) = 0$$

$$y^2(c_3x_2^2 + c_2x_3^2) + z^2(c_4x_1^2 + c_1x_4^2) + yz(x_0^2 + x_0 \sum_{i=1}^{4} a_1x_i + \sum_{i,j=1}^{4} b_{ij}x_ix_j) = 0$$

define the strict transforma V' *of* V *in* Γ_φ.

PROOF: Immediate.

REMARK: Let $\pi: V' \to Q$ be the restriction to V' of the canonical projection. For the fibre $\pi^{-1}(q)$ of a point $q \in Q$ we have three possibilities:

(1) if all coordinates of q are different from zero, $\pi^{-1}(q)$ is a (possibly degenerate) conic. In fact it is the residual conic cut out on V by the plane E_q, apart from the two (double) lines lying on the plane π_{12} and π_{34}.

In the above notations, if E_q has equations

$$\alpha x_1 - \beta x_2 = 0$$
$$\gamma x_3 - \delta x_4 = 0$$

($\alpha, \beta, \gamma, \delta$: fixed), on E_q we may assume homogeneous coordinates $v:u:r$ such that, for a point $p \in E_q$

$$\begin{aligned} x_0 &= v \\ x_1 &= \beta u \\ x_2 &= \alpha u \\ x_3 &= \delta r \\ x_4 &= \gamma r. \end{aligned}$$

In this coordinate system the conic $\pi^{-1}(q)$ has equation:

$$\begin{aligned} \alpha\beta\gamma\delta\{v^2 &+ v[(a_1\beta + a_2\alpha)u + (a_3\delta + a_4\gamma)r] \\ &+ (b_{11}\beta^2 + b_{12}\alpha\beta + b_{22}\alpha^2)u^2 \\ &+ (b_{33}\delta^2 + b_{34}\gamma\delta + b_{44}\gamma^2)r^2 \\ &+ (b_{13}\beta\delta + b_{14}\beta\gamma + b_{23}\alpha\delta + b_{24}\alpha\gamma)ur \\ &+ (c_1\alpha^2\gamma^2\delta^2 + c_2\beta^2\gamma^2\delta^2)r^2 \\ &+ (c_3\alpha^2\beta^2\gamma^2 + c_4\alpha^2\beta^2\delta^2)u^2 = 0. \end{aligned}$$

(2) if exactly two coordinates are zero, then $\pi^{-1}(q)$ is a double line.
(3) if three coordinates are zero, $\pi^{-1}(q) = E_q$.

(2) and (3) follow immediately from the equations.

We want to prove that V is birationally equivalent to a conic bundle.

Let $X = (1,0,0,0)$, $Y = (0,1,0,0)$, $Z = (0,0,1,0)$, $T = (0,0,0,1)$ be the four points of the case (3), and blow Q up in these points, or, equivalently, take the strict transform G of Q in the blowing up of $\mathbb{P}^3$ along the two lines of equations $x = t = 0$ and $y = z = 0$. So we realize G in $\mathbb{P}^1(\lambda:\mu) \times \mathbb{P}^1(v:\rho) \times \mathbb{P}^3(x:y:z:t)$ by the equations

$$\begin{aligned} \lambda x - \mu t &= 0 \\ v y - \rho z &= 0 \\ xt - yz &= 0. \end{aligned}$$

If $\varepsilon: G \to Q$ denotes the structure map, by base-change we obtain a birational morphism $\tilde{\varepsilon}: \tilde{\Gamma}_\varphi = \Gamma_\varphi \times G \to \Gamma_\varphi$ and a structure map $\tilde{\pi}: \tilde{\Gamma}_\varphi \to G$.

The strict transform $\tilde{V}$ of V' in $\tilde{\Gamma}_\varphi$ has equations in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3 \times \mathbb{P}^4$

$$\begin{aligned} \lambda x - \mu t &= 0 \\ v y - \rho z &= 0 \qquad\qquad (**) \\ xt - yz &= 0 \end{aligned}$$

$$
\begin{aligned}
&zx_1 - xx_2 = 0\\
&tx_1 - yx_2 = 0\\
&yx_3 - xx_4 = 0\\
&tx_3 - zx_4 = 0\\
&\mu^2(c_4x_2^2 + c_2x_4^2) + \lambda^2(c_1x_3^2 + c_3x_1^2) + \lambda\mu F = 0\\
&\sigma^2(c_3x_2^2 + c_2x_3^2) + \nu^2(c_4x_1^2 + c_1x_4^2) + \nu\sigma F = 0
\end{aligned}
\qquad (**)
$$

where

$$F = x_0^2 + x_0 \sum_{i=1}^{4} a_i x_i + \sum_{i,j=1}^{4} b_{ij}x_ix_j.$$

It follows that $\tilde{\pi}: \tilde{V} \to G$ is a "conic bundle" birationally equivalent to V.

In fact, if $g \in G$ $\varepsilon(g) = X$ (or Y, Z, T) it follows from those equations that $\tilde{\pi}^{-1}(g)$ is still a conic, precisely, if $\varepsilon(g) = X$, it is:

$$
\begin{aligned}
y = z = t = \lambda = x_2 = x_4&\\
= \rho^2 c_2 x_3^2 + \nu^2 c_4 x_1^2 + \nu\rho[x_0^2 + x_0(a_1x_1 + a_3x_3)&\\
+ (b_{11}x_1^2 + b_{13}x_1x_3 + b_{33}x_3^2)] = 0&
\end{aligned}
$$

Nevertheless, we'll see that $\tilde{V}$ still gets some singularities.

First of all, we want to study the locus of the degenarate conics. We have the following

PROPOSITION 1: *The locus of the degenerate conics for* $\tilde{\pi}: \tilde{V} \to G$ *is given by*:

– *a non singular curve* Δ *parametrizing the conics of rank* 2,

– *four lines* (*disjoint from* Δ *and not intersecting each other*), *parametrizing the double lines.*

PROOF: At first we study $\pi: V' \to Q$.

The condition for a conic $\pi^{-1}(q)$ in order to be degenerate is the following:

$$
\begin{aligned}
&\alpha^2\beta^2\gamma^2\delta^2\{4[\gamma\delta(b_{11}\beta^2 + b_{12}\alpha\beta + b_{22}\alpha^2) + \alpha\beta(c_3\gamma^2 + c_4\delta^2)]\\
&\cdot[\alpha\beta(b_{33}\delta^2 + b_{34}\gamma\delta + b_{44}\gamma^2) + \gamma\delta(c_1\alpha^2 + c_2\beta^2)] + \alpha\beta\gamma\delta(a_1\beta + a_2\alpha)\\
&\cdot(a_3\delta + a_4\gamma)(b_{13}\beta\delta + b_{14}\beta\gamma + b_{23}\alpha\delta + b_{24}\alpha\gamma) - \alpha\beta(a_3\delta + a_4\gamma)^2\\
&\cdot[\gamma\delta(b_{11}\beta^2 + b_{12}\alpha\beta + b_{22}\alpha^2) + \alpha\beta(c_3\gamma^2 + c_4\delta^2)] - \gamma\delta(a_1\beta + a_2\alpha)^2\\
&\cdot[\alpha\beta(b_{33}\delta^2 + b_{34}\gamma\delta + b_{44}\gamma^2) + \gamma\delta(c_1\alpha^2 + c_2\beta^2)]\\
&- \alpha\beta\gamma\delta(b_{13}\beta\delta + b_{14}\beta\gamma + b_{23}\alpha\delta + b_{24}\alpha\gamma)^2\} = 0.
\end{aligned}
$$

Therefore the curve of the degenerate conics on Q is given by:

(i) four lines, parametrizing the double lines

$$\begin{aligned} &z = t = 0 && (\text{for } \alpha = 0)\\ &x = y = 0 && (\text{for } \beta = 0)\\ &y = t = 0 && (\text{for } \gamma = 0)\\ &x = z = 0 && (\text{for } \delta = 0)\end{aligned}$$

(ii) the curve $C \subseteq Q$ of type $(4, 4)$ of equations

$$\begin{aligned} &xt - yz = 0\\ &\quad 4[(b_{11}xy + b_{12}xt + b_{22}zt + c_3yt + c_4xz)\\ &\quad \times (b_{33}xz + b_{34}xt + b_{44}yt + c_1zt + c_2xy)]\\ &\quad + xt(a_1a_3x + a_1a_4y + a_2a_3z + a_2a_4t)\\ &\quad \times (b_{13}x + b_{14}y + b_{23}z + b_{24}t)\\ &\quad - (a_3^2xz + 2a_3a_4xt + a_4^2yt)\\ &\quad \times (b_{11}xy + b_{12}xt + b_{22}zt + c_3yt + c_4xz)\\ &\quad - (a_1^2xy + 2a_1a_2xt + a_2^2zt)\\ &\quad \times (b_{33}xz + b_{34}xt + b_{44}yt + c_1zt + c_2xy)\\ &\quad - xt(b_{13}x + b_{14}y + b_{23}z + b_{24}t)^2 = 0\end{aligned}$$

C is the complete intersection of Q and a quartic surface R.

A simple direct computation shows that X, Y, Z, T are ordinary double points of C, and C is not tangent to the four foundamental lines lying on Q.

Moreover, we may see that $\pi^{-1}(q)$ has rank 2, $\forall q \in C - \{X, Y, Z, T\}$ (it follows from considerations on the minors of order 2 in the discriminant of $\pi^{-1}(q)$). Hence C must be non singular in q (cf. [**1**] prop. 1.2).

Therefore, for a generic V, the strict transform Δ of C in the blowing up $\varepsilon: G \to Q$ is non singular and doesn't intersect the strict transform of the four foundamental lines of Q.

Now, in order to examine the singularities of $\tilde{V}$, we denote by $\tilde{H}$ the section of $\tilde{\Gamma}_\varphi$ with $x_0 = 0$.

$\tilde{H}$ is birationally equivalent to the hyperplane H of $\mathbb{P}^4$ of equation $x_0 = 0$. By projecting from the point $0(1, 0, 0, 0, 0)$, we obtain a rational map $\eta: V \to H$, which is $2-1$ outside the double planes of V. By base-change we get a fibre-diagram

$$\begin{array}{ccc} \tilde{V} & \xrightarrow{\tilde{\eta}} & \tilde{H} \\ \downarrow & & \downarrow \\ V & \xrightarrow{\eta} & H \end{array}$$

where the vertical arrows are birational morphisms.

LEMMA 4: *The ramification locus $R_{\tilde{\eta}}$ of $\tilde{\eta}$ has equations on $\tilde{V}$*

$$\lambda\mu(2x_0 + \sum_{i=1}^{4} a_i x_i) = 0$$

$$\nu\rho(2x_0 + \sum_{i=1}^{4} a_i x_i) = 0$$

PROOF: The restriction of $\tilde{\eta}$ to $k_g = \tilde{\pi}^{-1}(g) \subseteq E_g$ coincides with the projection of k_g on the "line at infinity" of E_g. So we get the ramification points of $\tilde{\eta}|_{k_g}$ by intersecting k_g with the polar line of 0 to k_g, or, equivalently, with the polar hyperplane of 0 to the quadric hypersurfaces obtained by fixing the coordinates of g in the equations (**) of $\tilde{V}$.

In this way we find exactly the required equations.

COROLLARY: *$R_{\tilde{\eta}}$ is the union of the following sections of $\tilde{V}$:*

$$\tilde{V} \cap \{\lambda = \nu = 0\} = A_{\lambda\nu}$$
$$\tilde{V} \cap \{\lambda = \rho = 0\} = A_{\lambda\rho}$$
$$\tilde{V} \cap \{\mu = \nu = 0\} = A_{\mu\nu}$$
$$\tilde{V} \cap \{\mu = \rho = 0\} = A_{\mu\rho}$$
$$\tilde{V} \cap \{2x_0 + \sum_{i=1}^{4} a_i x_i\} = B$$

PROPOSITION 2: *A_{ij} $(i,j = \lambda, \mu, \nu, \rho)$ is a smooth quadric surface. B is a smooth Enriques surface.*

PROOF: 1) Let's consider, for example, $A_{\lambda\nu}$. Its equations in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3 \times \mathbb{P}^4$ are given by:

$$\lambda = \nu = t = z = x_2 = x_3 = x_4 = 0,$$

so they determine a smooth surface isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$:

$$A_{\lambda\nu} = \{t = z = 0\} \times \{x_2 = x_3 = x_4 = 0\}.$$

2) B is an Enriques surface.

In fact the ramification divisor of the map $\eta: V \longrightarrow H$ has equation, in the coordinates $(x_1 : x_2 : x_3 : x_4)$, the discriminant of the polinomial of degree two in x_0 defining V in $\mathbb{P}^4$, i.e.

$$x_1x_2x_3x_4[x_1x_2x_3x_4(\sum_{i=1}^{4} a_ix_i)^2 - 4\sum_{i,j=1}^{4} b_{ij}x_ix_j) - 4c_1x_2^2x_3^2x_4^2 \\ - 4c_2x_3^2x_4^2x_1^2 - 4c_3x_4^2x_1^2x_2^2 - 4c_4x_1^2x_2^2x_3^2] = 0.$$

The expression contained in the square brackets is the canonical equation of an Enriques surface in $\mathbb{P}^3$, which is birationally equivalent to B by means of η.

It is possible to verify on the equations that the blowing up defining $\tilde{V}$ induces a desingularization of this Enriques surface, but we can also observe directly that $\tilde{\pi}|_B : B \to G$ is a double covering with ramification divisor

$$R_{\tilde{\pi}} = (A_{\lambda\nu} + A_{\lambda\rho} + A_{\mu\nu} + A_{\mu\rho})\cdot B + \Delta'$$

where $\tilde{\pi}(\Delta') = \Delta$ is the irreducible smooth curve of G studied in prop. 1. So $A_{\lambda\nu}\cdot B = L'_{\lambda\nu}$ is a line of equations

$$\lambda = \nu = t = z = x_2 = x_3 = x_4 = 2x_0 + \sum_{i=1}^{4} a_ix_i = 0$$

and

$$\tilde{\pi}(L'_{\lambda\nu}) = L_{\lambda\nu} = \{t = z = 0\}$$

is a foundamental line on G parametrizing the conics of rank 1. It follows that $R_{\tilde{\pi}}$ is a (reducible) smooth curve, and therefore B is non singular.

PROPOSITION 3: *$\tilde{V}$ is non singular, except for four couples of lines contained in $A_{\lambda\nu}$, $A_{\lambda\rho}$, $A_{\mu\nu}$, $A_{\mu\rho}$, having equations*

$$\begin{aligned} \lambda = \nu = t = z = x_2 = x_3 = x_4 = (x_0^2 + a_1x_0x_1 + b_{11}x_1^2) = 0 \\ \lambda = \rho = y = t = x_1 = x_2 = x_4 = (x_0^2 + a_3x_0x_3 + b_{33}x_3^2) = 0 \\ \mu = \nu = x = z = x_1 = x_2 = x_3 = (x_0^2 + a_4x_0x_4 + b_{44}x_4^2) = 0 \\ \mu = \rho = x = y = x_1 = x_3 = x_4 = (x_0^2 + a_2x_0x_2 + b_{22}x_2^2) = 0. \end{aligned}$$

Moreover all these points are ordinary double points.

PROOF: Let $D = A_{\lambda\nu} + A_{\lambda\rho} + A_{\mu\nu} + A_{\mu\rho}$. Then $\tilde{\eta}: \tilde{V} - D \longrightarrow \tilde{H} - D$ is a double covering with smooth ramification locus, therefore it is non singular and all singular points of $\tilde{V}$ are necessarily belonging to D.

It suffices to consider one of the connected components of D, for example $A_{\lambda\nu}$, and to argue locally. So, taking the open set $U \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3 \times \mathbb{P}^4$ where $\mu = \rho = x = x_0 = 1$ and assuming affine coordinates $(\lambda, \nu, y, z, t, x_1, x_2, x_3, x_4)$, we easily see that the tangent space at any point $p = (0, 0, \bar{y}, 0, 0, \bar{x}_1, 0, 0, 0) \in A_{\lambda\nu} \cap U$ to $\tilde{V} \cap U$ is given by the following (not linearly independent) equations:

$$\begin{aligned}
& t - \bar{y}z = 0 \\
& \lambda - t = 0 \\
& \nu\bar{y} - z = 0 \\
& z\bar{x}_1 - x_2 = 0 \\
& t\bar{x}_1 - \bar{y}x_2 = 0 \\
& \bar{y}x_3 - x_4 = 0 \\
& \lambda(1 + a_1\bar{x}_1 + b_{11}\bar{x}_1^2) = 0 \\
& \nu(1 + a_1\bar{x}_1 + b_{11}\bar{x}_1^2) = 0
\end{aligned}$$

Since $\dim T_{\tilde{V},p} \geq 3$, at most six of them are linearly independent. Two different cases are possible:

1) $1 + a_1\bar{x}_1 + b_{11}\bar{x}_1^2 \neq 0$

Then $\lambda = \nu = t = z = x_2 = \bar{y}x_3 - x_4 = 0$ are six independent equations, so $\dim T_{\tilde{V},p} = 3$ and $\tilde{V}$ is non singular at p.

2) $1 + a_1\bar{x}_1 + b_{11}x_1^2 = 0$.

In this case the last two equations are identically zero, and the first six are related by the (unique) relation

$$\bar{x}_1(t - \bar{y}z) = (t\bar{x}_1 - \bar{y}x_2) - \bar{y}(\bar{x}_1 z - x_2)$$

(Note that $\bar{x}_1 \neq 0$). So $\dim T_{\tilde{V},p} = 4$.

To determine the tangent cone at p to $\tilde{V}$, we assume as local parameters at p to $\tilde{\Gamma}_\varphi$ (which is a non singular four-dimensional variety), for example, $\nu, x_3, y' = y - \bar{y}, x_1' = x_1 - \bar{x}_1$ and we obtain a term of lower degree of the kind

$$A\nu^2 + B\nu x_1' + C\nu x_3 + c_2 x_3^2 = 0 \qquad (***)$$

where

$$\begin{aligned} A &= c_4\bar{x}_1^2 + c_3\bar{x}_1^2\bar{y}^2 + a_2\bar{x}_1\bar{y} + b_{12}\bar{x}_1^2\bar{y} \\ B &= a_1 + 2b_{11}\bar{x}_1 \\ C &= a_3 + a_4\bar{y} + b_{13}\bar{x}_1 + b_{14}\bar{y}\bar{x}_1 \end{aligned}$$

It is a quadric cone over the conic (***) with discriminant

$$H = c_2(a_1 + 2b_{11}\bar{x}_1)^2$$

(remember that $c_2 \neq 0$). For a generic $V, H \neq 0$, so p is an ordinary double point. It is immediate to see that from $H = 0$ it follows that the solutions of the equation 2) (and so either the two corresponding lines on $\tilde{V}$ and the two quadruple points on the line $x_2 = x_3 = x_4 = 0$ on V) coincide.

COROLLARY: *A non singular model for V is given by the strict transform $\tilde{V}'$ of $\tilde{V}$ in the blowing up of $\tilde{\Gamma}_\varphi$ along these eight singular lines.*

PROOF: It can be done directly in the above local coordinates. In particular, for each line blown up we get an exceptional quadric.

REMARK: In conclusion, we have got a non singular model $\tilde{V}'$, of V and a map $f = \tilde{\pi}' : \tilde{V}' \to G$ whose fibres are:
i) a non singular conic if $g \notin \Delta \cup \{L_{ij}\}$
ii) two different lines if $g \in \Delta$
iii) a double line and two conics if $g \in L_{ij}$.

Therefore the inverse image $\tilde{Y}$ of the four foundamental lines L_{ij} is a union of quadrics (the A_{ij}'s and the exceptional ones).

$W = \tilde{V}' - \tilde{Y}$ is isomorphic to $\tilde{V} - \cup A_{ij}$, so that it is a non singular conic bundle over $G - \cup L_{ij}$.

Δ is the complete non singular curve of degenerate conics of the bundle.

We can construct in a standard way (see [1] 1.5) a double covering $q : \tilde{\Delta} \to \Delta$ such that every point $t \in \tilde{\Delta}$ parametrizes one of the two lines contained in the conic $k_{q(t)}$. Let us call this line $L(t)$ and look to it as an element of $C^2(W)$. By similar arguments as in [1] 3.1, and considering also [1] 3.1.9 one can prove the following

PROPOSITION 4: *The map $t \mapsto L(t)$ extends to a surjective homomorphism*

$$\varphi : J(\tilde{\Delta}) \to A^2(W)$$

whose kernel is $q^*J(\Delta)$. *Taking the quotient, we obtain an isomorphism*

$$\psi : P = Prym(\tilde{\Delta}/\Delta) \to A^2(W).$$

COROLLARY: $P \xrightarrow{\sim} A^2(\tilde{V}')$.

PROOF: Let $\bar{Y}$ be the desingularization of $\tilde{Y}$. We have the exact sequence

$$\begin{array}{ccccccc} A^1(\bar{Y}) & \to & A^2(\tilde{V}') & \to & A^2(W) & \to & 0 \\ & & \nwarrow & & \uparrow \wr & & \\ & & & & P & & \end{array}$$

and $A^1(\bar{Y}) = 0$ since $\tilde{Y}$ is the union of quadric surfaces. We observe that this is a group isomorphism.

PROPOSITION 5: *P is the algebraic representative of* $A^2(\tilde{V}')$ (cfr. [**1**] def. 3.2.3.), *and the principal polarization* ϑ *of P is the incidence polarization relative to X* ([**1**] def. 3.4.2.).

PROOF: It can be shown by the same arguments as [**1**] Prop. 3.3 and [**1**] Prop. 3.5.

LEMMA 5: *Let C be a canonical curve in* $\mathbb{P}^4$ *which is a complete intersection:*

(1) *C has a half-canonical* g^1_4 *if and only if C is contained in a quadric U of rank three;*

(2) *the unique ruling of two-planes of U cuts out the half-canonical* g^1_4 *on C.*

PROOF: (1) Let us suppose that C has a half-canonical g^1_4; if D is an effective divisor belonging to the g^1_4 then, by Riemann–Roch Theorem and the hypothesis $2D \sim K$, it follows

$$h^\circ(D) = 2 = h^\circ(K - D).$$

Since K is cut out by the hyperplane sections, $h^\circ(K - D) = 2$ means that $\operatorname{Supp} D \subset \pi$ where π is a two-plane; now we can take another effective divisor D' which is linearly equivalent to D and, without loss of generality, we can assume $\operatorname{Supp} D \cap \operatorname{Supp} D' = \emptyset$ (if not C will have a g_3^1 and will not be a complete intersection). Let π' be the two-plane containing $\operatorname{Supp} D'$: since $D + D' \sim K$ it follows that $\pi \cup \pi'$ is contained in an hyperplane H of $\mathbb{P}^4$ and that π, π' intersect along a line u. Moreover there is only one net of quadric surfaces in H passing through $\operatorname{Supp} D \cup \operatorname{Supp} D'$, so that they are sections by H of the quadric hypersurfaces through C. Since $\pi \cup \pi'$ belongs to the net, there is a quadric hypersurface U containing $C \cup \pi \cup \pi'$.

U *is* singular because it contains some 2-plane, then its rank may be equal to 3 or 4. If U had rank 4 then U will be a cone over a quadric surface S in $\mathbb{P}^3$. In this case π will be a plane through the vertex of the cone and a line l of S; then D will be cut out on C by the two-planes through the vertex of U and a line in the same ruling of l; which is absurd since π' belongs clearly to the other ruling of two-planes of U. Then U must have rank 3 and its singular locus has to be the line $u = \pi \cap \pi'$.

Viceversa and (2) follow easily by the above arguments.

LEMMA 6: *Let V be a generic Enriques' threefold, then a canonical model of Δ is the complete intersection in $\mathbb{P}^4$ of three quadric hypersurfaces Q_1, Q_2, Q_3 where Q_1, Q_2 have rank 3 and Q_3 is generic.*

PROOF: We considered a singular model C of Δ given by the following equations in $\mathbb{P}^3$ $(x:y:z:t)$, (see Prop. 1):

$$
\begin{aligned}
&xt - zy = 0\\
F_4(x,y,z,t) = {}& 4\cdot[(b_{11}xy + b_{12}xt + b_{22}zt + c_3yt + c_4xz)\\
&\times (b_{33}xz + b_{34}xt + b_{44}yt + c_1zt + c_2xy)]\\
&+ xt\cdot[(a_1a_4y + a_2a_3z)(b_{13}x + b_{14}y + b_{23}z + b_{24}t)\\
&- (b_{14}y + b_{23}z)^2] + zy[(a_1a_3x + a_2a_4t)\\
&\times (b_{13}x + b_{14}y + b_{23}z + b_{24}t) - (b_{13}x + b_{24}t)^2]\\
&- 2(b_{13}x + b_{24}t)(b_{14}y + b_{23}z)xt - (a_3^2xz + 2a_3a_4xt + a_4^2yt)\\
&\times (b_{11}xy + b_{12}xt + b_{22}zt + c_3yt + c_4xz)\\
&- (a_1^2xy + 2a_1a_2xt + a_2^2zt)\\
&\times (b_{33}xz + b_{34}xt + b_{44}yt + c_1zt + c_2xy) = 0.
\end{aligned}
$$

C is of type (4,4) in the quadric surface $Q = \{xt - yz = 0\}$ and has four ordinary double points: $X(1:0:0:0)$, $Y(0:1:0:0)$, $Z(0:0:1:0)$, $T(0:0:0:1)$. The linear system of quadric surfaces in $\mathbb{P}^3$ containing the four double points and distinct from Q cuts out on C the canonical system. Moreover it defines a rational morphism

$$\Phi: \mathbb{P}^3 \longrightarrow \mathbb{P}^4$$

which desingularizes C and embeds it canonically in $\mathbb{P}^4$. Indeed the equations of Φ can be defined by setting

$$u_0 = xy,\ u_1 = xz,\ u_2 = xt,\ u_3 = yt,\ u_4 = zt$$

where $(u_0:u_1:u_2:u_3:u_4)$ are projective coordinates in $\mathbb{P}^4$. Then the strict transform of Q is the intersection of the two quadric hypersurfaces of rank three:

$$Q_1: u_2^2 - u_0u_4 = 0, \qquad Q_2: u_2^2 - u_1u_3 = 0.$$

Moreover the quartic form $F_4(x,y,z,t)$ can also be written as a quadratic form $F(xy,xz,xt,yt,zt)$ in xy,xz,xt,yt,zt. It follows immediately that the strict transform Δ' of C in $\mathbb{P}^4$ has equations:

$$u_2^2 - u_1u_3 = u_2^2 - u_0u_4 = 0$$

$$F(u_0,u_1,u_2,u_3,u_4) = 0$$

where $F(u_0,u_1,u_2,u_3,u_4)$ is a quadratic form.

Then the affine space of the coefficients of the equation of V maps on the affine space of the coefficients of a quadratic form $F \in C[u_0,u_1,u_2,u_3,u_4]$. One can compute directly that this map is of maximal rank and surjective.

From this fact we can argue that Δ' is the complete intersection of $\mathbb{P}^4$ of Q_1, Q_2 and a third generic quadratic hypersurface Q_3. In particular Δ' is smooth and canonically embedded in $\mathbb{P}^4$.

COROLLARY 3: *Let V be a generic Enriques' threefold:*
(i) *Δ is not hyperelliptic, trigonal, nor elliptic-hyperelliptic;*
(ii) *Δ has two half-canonical g_4^1's L_1, L_2;*
(iii) *Δ does not contain a half-canonical divisor N such that $N \not\sim L_i$, $(i = 1,2)$, $h^0(N) \neq 0$, $h^0(N)$ even.*

PROOF: (i) follows from the proof of the above lemma and from the fact that Q_3 is generic. (ii) follows from Lemma 5. Now we show (iii): by (i) and Lemma 6 Δ is the base locus of a net Σ of quadric hypersurfaces containing the 2 quadrics Q_1, Q_2 of rank 3. Moreover, being Q_3 generic, Σ is generic in the family of nets as above, so that Σ does not contain a third quadric of rank 3 different from Q_1, Q_2. Then, by Lemma 5, Δ cannot carry a half-canonical divisor N with $h^\circ(N) = 2$ and $N \not\sim L_i$. In the end, if $h^\circ(N) = 4$, Δ will be clearly elliptic or rational which is absurd.

REMARK: By the corollary above Δ is generic among the curves of genus 5 having 2 and only 2 half-canonical g_4^1's. Thinking of Δ as a singular curve of type $(4, 4)$ in the quadric Q (see Prop. 1) these g_4^1's arise by intersection with the two rulings of lines in Q.

Let us consider now the étale double covering of Δ:

$$q: \tilde{\Delta} \to \Delta$$

(see the remark before Prop. 4); we will compute the semiperiod giving such a covering.

We have seen (see the remark before Prop. 1) that there is a birational morphism of V with a (singular) conic bundle $\tilde{V}$ on the surface G. G is the blowing up

$$\varepsilon: G \to Q$$

of the quadric surface $Q = \{xt - yz = 0\}$ in the four fundamental points of $\mathbb{P}^3(x:y:z:t)$. Let $\tilde{\pi}: \tilde{V} \to G$ be the map fibering $\tilde{V}$ in conics; $\forall g \in G$ the conic $\tilde{\pi}^{-1}(g)$ is obtained, via the birational morphism from V to $\tilde{V}$, from a conic K_g in V contained in a 2-plane E_g meeting both the 2-plane $\pi_{12} = \{x_1 = x_2 = 0\}$, $\pi_{34} = \{x_3 = x_4 = 0\}$ along a line (Lemma 1).

LEMMA 7: *The locus in G:*

$$\{g \in G / K_g \cap (E_g \cap \pi_{12}) \text{ is exactly one point}\}$$

is given by:

(i) *a non singular elliptic curve $\nabla \subset G$ which is the strict transform, via $\varepsilon: G \to Q$, of a quartic elliptic curve in Q passing through the four fundamental points of ε^{-1};*

(ii) *two rational curves l_2, l'_2 which are the strict transforms of the lines $\{y = t = 0\}$, $\{x = z = 0\}$ belonging to the same ruling in Q.*

PROOF: The equation of a conic $K_g \subset E_g = \{\alpha x_1 - \beta x_2 = \gamma x_3 - \delta x_4$

$= 0\} \subset \mathbb{P}^4$ is given in the remark following Lemma 3. The coefficients of such a equation depend on $(\alpha:\beta) \times (\gamma:\delta)$, the projective coordinates on E_g are $(u:v:r)$ and the line $E_g \cap \pi_{12}$ is given by setting $u = 0$. It turns out easily that, if K_g satisfies the required condition, then $(\alpha:\beta) \times (\gamma:\delta)$ annihilates the following equation:

$$(\alpha\beta\gamma^2\delta^2)\cdot[\alpha\beta(a_3\delta + a_4\gamma)^2 - 4\alpha\beta(b_{33}\delta^2 + b_{34}\gamma\delta + b_{44}\gamma^2) - 4(c_1\alpha^2\gamma\delta + c_2\beta\gamma\delta)] = 0.$$

With the same notations of Lemma 1 we have $\alpha:\beta = t:y = z:x$; $\gamma:\delta = y:x = t:z$ so that the set of zeroes of the second factor of the above equation becomes the locus in $\mathbb{P}^3(x:y:z:t)$:

$$\begin{aligned} & xt - yz = 0 \\ & \quad (a_3^2 - 4b_{33})xz + (a_4^2 - 4b_{44})ty \\ & \quad + 2(a_3a_4 - 2b_{34})xt - 4c_1tz - 4c_2xy = 0. \end{aligned}$$

If V is generic this is clearly a smooth quartic elliptic curve in Q, passing through the four fundamental points of $\mathbb{P}^3$, that is through the fundamental points of ε^{-1}; this shows (i). To show (ii) we observe that the fibers of $\tilde{\pi}$ on $l_2(l'_2)$ are double lines (see Prop. 1) and that these double lines arise, by the birational morphism quoted above, from the line $\{x_3 = x_4 = x_1 = 0\}(\{x_3 = x_4 = x_2 = 0\})$ counted twice. This one meets π_{12} twice in the point $(1:0:0:0:0)$ and this shows (ii); moreover it is clear from the geometric situation that the locus we are considering cannot have other components.

LEMMA 8: *We have on G:*
(i) $(\nabla, l_2) = (\nabla, l'_2) = 0$
(ii) Δ *and* ∇ *does not meet along the four exceptional divisors of G*
(iii) $(\Delta, \nabla) = 8$ *and, for every* $p \in \Delta \cap \nabla$, $i(p; \Delta \cap \nabla) = 2$.

PROOF: $\varepsilon(\Delta)$, $\varepsilon(\nabla)$, $\varepsilon(l_2)$, $\varepsilon(l'_2)$ pass all through the four fundamental points of ε^{-1}; since V is generic it is clear from the equation of $\varepsilon(\nabla)$ written in Lemma 7 that $\varepsilon(l_2)$, $\varepsilon(l'_2)$ are not tangent to $\varepsilon(\nabla)$; in the same way one can also see that, for every fundamental point 0, the tangent line in 0 to $\varepsilon(\nabla)$ cannot be a component of the tangent cone to $\varepsilon(\Delta)$ in 0. This shows (i) and (ii). Now we have on $Q:(\varepsilon(\Delta), \varepsilon(\nabla)) = 16$; moreover $\varepsilon(\nabla)$ meets the four singular points of $\varepsilon(\Delta)$ and these are also the fundamental ones for ε^{-1}. Then, by (ii), $(\Delta, \nabla) = 8$.

Another direct computation shows that $i(p; \Delta \cap \nabla) = 2$ for every $p \in \Delta \cap \nabla$.

Let us consider now the double covering:

$$f : \tilde{G} \to G$$

branched over $\nabla \cup l_2 \cup l'_2 : \tilde{G}$ is smooth since $\nabla \cup l_2 \cup l'_2$ is smooth. Moreover the open set $\tilde{G} - (l_2 \cup l'_2)$ parametrizes the couples (g, x) where $g \in G$ and $x \in K_g \cap \pi_{12}$. It follows that $f^{-1}(\Delta)$ parametrizes the lines being components of the degenerate conics K_g of rank 2. Then $f^{-1}(\Delta)$ is a (singular) model of $\tilde{\Delta}$. Indeed $f^{-1}(\Delta)$ is singular exactly in the four points of the set $f^{-1}(\Delta \cap \nabla)$: this can be obtained, with a local computation, by observing that, for every such a point x, $i(f(x); \Delta \cap \nabla) = 2$ and ∇ belongs to the branch locus of f.

Clearly we have the commutative diagram:

$$\begin{array}{ccc} \tilde{\Delta} & \xrightarrow{q} & \Delta \\ {\scriptstyle \nu}\downarrow & \nearrow_{f/f^{-1}(\Delta)} & \\ f^{-1}(\Delta) & & \end{array}$$

where ν is the normalization morphism.

Let us call L_1 a divisor on Δ belonging to the halfcanonical g^1_4 cut out on $\varepsilon(\Delta)$ by the lines of Q not in the ruling of $\varepsilon(l_2)$; let us call L_2 a divisor in the other half-canonical g^1_4 of Δ, (see corollary 3), we have the following

Proposition 6: *If* $\{p_1, p_2, p_3, p_4\} = \Delta \cap \nabla$ *and* $D = p_1 + p_2 + p_3 + p_4$ *on* Δ *then*

$$\eta = D - L_1$$

is the semiperiod giving the étale double covering $q : \tilde{\Delta} \to \Delta$.

Proof: On G we have $\nabla \sim 2l_1 + l_2 + l'_2$ where l_1 is the (global) transform of a line of Q not in the ruling of $\varepsilon(l_2)$.

Then $O_\Delta(\nabla - 2l_1 - l_2 - l'_2) \cong O_\Delta(2D - 2L_1) \cong O_\Delta$ so that $\eta = D - L_1$ is a semiperiod.

Observe now that $\tilde{G}$ is a (smooth) rational surface: let m be the transform of ageneric line $\varepsilon(m) \sim \varepsilon(l_2)$; since $(m, \nabla + l_2 + l'_2) = 2$ then $f : f^{-1}(m) \to m$ is a double covering of $\mathbb{P}^1$ branched on two points. It follows that $\tilde{G}$ carries a pencil of rational curves so that, by Noether's theorem, it is a rational surface.

Since $\nabla + l_2 + l'_2$ is the branch locus of f it turns out that

$$2f^{-1}(\nabla) - l_2 - l'_2) \sim f^*(2l_1) \sim 2f^*(l_1)$$

and, since Pic $\tilde{G}$ has no torsion (being $\tilde{G}$ rational),

$$f^{-1}(\nabla - l_2 - l'_2) \sim f^*(l_1).$$

By setting $\tilde{\Delta}_s = f^{-1}(\Delta)$ we have:

$$O_{\tilde{\Delta}_s}(f^{-1}(\nabla - l_2 - l'_2) - f^*(l_1)) \cong O_{\tilde{\Delta}_s}$$

that is:

$$O_{\tilde{\Delta}} \cong O_{\tilde{\Delta}}(v^* f^*(D - L_1)) \cong O_{\tilde{\Delta}}(q^*\eta).$$

Then $q^*\eta$ is trivial on $\tilde{\Delta}$: this happens if and only if $q : \tilde{\Delta} \to \Delta$ is given by η.

REMARK: $\eta \not\sim L_1 - L_2$: since $2D$ is cut out on $\varepsilon(\Delta)$ by an elliptic curve of type $(2, 2)$ on Q, it follows that Supp D cannot be contained in a line of Q, so that $D \not\sim L_i$. This shows also that η cannot be trivial.

COROLLARY 4: *$\eta \sim D' - L_2$ where $2D'$ is cut out on Δ by a smooth elliptic curve ∇' parametrizing the conics K_g of rank ≥ 2 such that $K_g \cap \pi_{34}$ is exactly one point.*

PROOF: Exactly as to show $\eta \sim D - L_1$: it suffices to substitute π_{12} with π_{34} and l_2, l'_2 with the corresponding rational curves l_1, l'_2 strict transforms of the lines $\{z = t = 0\}$, $\{x = y = 0\}$.

COROLLARY 5: *If V is generic, on Δ there is no effective even theta characteristic N such that $h^\circ(N + \eta)$ is even.*

Moreover $h^\circ(L_i + \eta) = 1$.

PROOF: If V is generic on Δ there are only two effective even theta characteristics: namely L_1, L_2, (see Corollary 3). Since $\eta \sim D - L_1 \sim D' - L_2$ it follows that $L_i + \eta$ is effective so that $h^\circ(L_i + \eta) \neq 0$. Now we cannot have $h^\circ(L_i + \eta) > 2$ unless Δ is elliptic or rational which is absurd, nor $h^\circ(L'_i + \eta) = 2$ since $L_1 + \eta \not\sim L_2$. Then $h^\circ(L_i + \eta) = 1$.

Proposition 7: *A generic Enriques' threefold V is not rational.*

Proof: Let us consider the ètale double covering $q: \tilde{\Delta} \to \Delta$: the Prym variety associated to q is an abelian variety P with principal polarization ϑ. Moreover P is the algebraic representant of $A^2(\tilde{V}')$ and ϑ is the incidence polarization (see Prop. 5). Then, by **[1]** Prop. 4.6, it suffices to show that (P, ϑ) as a principally polarized abelian variety, is not isomorphic to a product of jacobians of curves.

To get this result we observe that, by Corollary 3, Δ cannot be hyperelliptic, trigonal nor elliptic-hyperelliptic. Moreover Δ has 2 and only 2 even effective theta characteristics: L_1, L_2. By Proposition 6 and Corollary 4 q is given by $\eta \sim D - L_1 \sim D' - L_2$; and by Corollary 5 Δ cannot carry an even effective theta characteristic N such that $h^\circ(N + \eta)$ is even. Then it follows from **[6]** Theorem 7 (d) pag. 344 that (P, ϑ) cannot be a jacobian nor a product of jacobians of curves.

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