Compositio Mathematica

Toshikazu Sunada

Mean-value theorems and ergodicity of certain geodesic random walks

Compositio Mathematica, tome 48, nº 1 (1983), p. 129-137

<http://www.numdam.org/item?id=CM_1983__48_1_129_0>

COMPOSITIO MATHEMATICA, Vol. 48, Fasc. 1, 1983, pag. 129–137

MEAN-VALUE THEOREMS AND ERGODICITY OF CERTAIN GEODESIC RANDOM WALKS

Toshikazu Sunada

Abstract

We give some geometric conditions which guarantee that all the invariant functions of the spherical mean operator with certain radius on a Riemannian manifold are necessarily constant. A geometric model of a Markov process, so-called geodesic random walks, whose transition operator is the spherical mean, plays a fundamental role in our argument.

1. Introduction

Let M be a connected complete Riemannian manifold without boundary. Throughout we assume $\dim M \geq 2$. The spherical mean (operator) with radius r (≥ 0) on M is the operator L_r defined by

$$(L_r f)(x) = \int_{S_x M} f(\exp rv)\, dS_x(v),$$

where dS_x is the normalized uniform density on the unit sphere $S_x M = \{v \in T_x M; \|v\| = 1\}$. If $M = \mathbf{R}^n$ (with the standard metric), L_r is the classical spherical mean, and invariant functions of L_r are just harmonic functions. To be exact, a locally integrable function f on $\mathbf{R}^n$ is harmonic if and only if $L_r f = f$ for sufficiently small $r < \varepsilon$. A direct generalization of this classical mean-value theorem is the following.

THEOREM A: *There exists a family of self-adjoint elliptic operators $\{P_k\}_{k=1,2,\cdots}$ with $P_1 = \Delta$ such that, if $L_r f = f$ for sufficiently small $r < \varepsilon$,*

0010-437X/83/01/0129-09$0.20

then $P_k f \doteq 0$ for all k, and conversely if M is a real analytic Riemannian manifold, and if $P_k f = 0$ for all k, then $L_r f = f$ for $r \geq 0$.

This has been essentially proven in [8]. In fact this is almost equivalent to the formal expansion of L_r with respect to r;

$$L_r \sim I + \frac{\Delta}{2n} r^2 + \sum_{k=2}^{\infty} P_k r^{2k},$$

which, in the classical case, reduces to the Pizzetti's formula

$$L_r \sim \Gamma(\tfrac{1}{2}n) \sum_{k=0}^{\infty} \left(\frac{r}{2}\right)^{2k} \frac{1}{k!\Gamma\left(\frac{n}{2}+k\right)} \Delta^k.$$

We should point out that this kind of infinitesimal properties of L_r is usefull in characterizing Riemannian manifolds in terms of mean-value properties. But our discussion will not enter into this direction because the global character of L_r is our concern.

Suppose now M is compact. Since harmonic functions on such a M are constant, any integrable function f such that $L_r f = f$ for sufficiently small $r < \varepsilon$ is necessarily constant. An interesting thing is that, as we have showed in the previous paper [8], this is true even for a function with $L_r f = f$ for a fixed $r > 0$. For instance, if the restriction of the exponential mapping $\exp_x$ to the sphere rS_xM of radius r is an immersion for every point x in M, then the number 1 is a simple eigenvalue of the operator $L_r: L^2(M) \to L^2(M)$. For brevity, we call L_r *ergodic* if one can conclude the simplicity of the eigenvalue 1, which, as is known, is equivalent to the ergodicity of the Markov process on M whose transition operator is L_r. The primary purpose of this paper is to give a somewhat relaxed criterion of ergodicity, which, in some sense, resembles the criterion in the case of finite Markov chains.

THEOREM B: *Let M be a compact Riemannian manifold. If there exists a point x in M such that almost all points can be joined to x by r-geodesic chains of finite length, then L_r is ergodic. In particular, if any two points in M can be joined by an r-geodesic chain, then L_r is ergodic.*

Here r-geodesic chains of length k, k being a natural number, are continuous mappings $c: [0, k] \to M$ such that all restrictions $c|[i-1, i]$ $(i = 1, 2, \ldots, k)$ are geodisic curves with the same length r. Two points

x and y are said to be joined by an r-geodesic chain $c: [0, k] \to M$ if $c(0) = x$ and $c(k) = y$.

Our proof of Theorem B is quite elementary and supercedes the previous one [8] which relies heavily on regularity of Fourier integral operators and can be applied to only the limited case.

In connection with the above theorem, a natural question arises here. What kind of geometric condition guarantees that any two points are joined by r-geodesic chains? As was shown in [8], this is the case if $\exp_x: rS_xM \to M$ is an immersion for every point x. We will see in §3 that this condition is relaxed in the following way.

THEOREM C: *Let M be a complete Riemannian manifold. Suppose that for any point x in M there exist a natural number k and a vector $v \in krS_xM$ such that $\exp_x: krS_xM \to M$ is an immersion in a neighborhood of v. Then any two points in M can be joined by an r-geodesic chain of even length.*

As is illustrated by the example $M = S^n(1)$, $r = \pi$ or 2π, our assumption for the exponential mapping can not be omitted. On the other hand, if M is non compact, then the assumption in Theorem C is always satisfied, since one can find a geodesic ray through a point. Hence, if the fundamental group of a compact M is infinite, then one concludes that every two points are joined by r-geodesic chains. Together with Theorem B, one has

THEOREM D: *If $\pi_1(M)$ is infinite, then L_r is ergodic for any $r > 0$.*

In the last part of our discussion, we will see that two dimensional manifolds for which ergodicity of L_r is not satisfied have very remarkable properties.

REMARK: There are several references which are concerned with different kind of mean-value operators ([3] [4] [5] [7] [10]).

2. Proof of Theorem B

In view of ergodic theory of Markov processes, it is enough to prove that, for every pair of Borel sets A and B in M with positive measure, there exists a natural number k such that

$$\int_A L_r^k \chi_B dx > 0$$

(see [11]). We set

$$S_x^k M = S_x M \times \ldots \times S_x M, \text{ the } k\text{-ple product,}$$

here k is possibly infinite. We let $S^k M$ be the fiber bundle on M with fiber $S_x^k M$. The product probability measure on $S_x^k M$ and the canonical measure on M give rise to a fiber product measure P_k on $S^k M$. We identify $S_x^k M$ with the set of all r-geodesic chains of length k issued from x, by using parallel translations. This identification allows us to define a mapping

$$\pi_l : S^k M \to M \times M \qquad (0 \leq l \leq k)$$

by $\pi_l(c) = (c(0), c(l))$. The assumption in Theorem B is then equivalent to the union $\bigcup_{k=1}^{\infty} \pi_k(S^k M)$ having full measure in $M \times M$. As was shown in [8], the process $\tilde{\omega}_k : S^\infty M \to M$ defined by $\tilde{\omega}_k(c) = c(k)$ is a Markov process with the transition operator L_r, hence we have

LEMMA 1:

$$P_k(\pi_k^{-1}(A \times B) = \int_A L_r^k \chi_B dx$$

Therefore what we have to prove reduces to the following general lemma.

LEMMA 2: *Let* $\{\varphi_k : X_k \to Y, k = 1, 2, \ldots\}$ *be a family of smooth mappings of smooth paracompact manifolds such that the union* $\cup\, \varphi_k(X_k)$ *has full measure in* Y. *Then for any Borel subset* A *in* Y *with positive measure, there exists some* k *such that* $\varphi_k^{-1}(A)$ *has positive measure.*

PROOF: Let K_k be the set of critical value of φ_k, which, by the Sard's theorem (see [6]), has measure zero. The countable union $\cup\, K_k$ has also measure zero. One can choose a point y in $\cup\, \varphi_k(X_k) \backslash \cup\, K_k$ such that any open neighborhood of y and A have intersection with positive measure. Let $x_k \in X_k$ with $\varphi_k(x_k) = y$. Since φ_k is a submersion in a neighborhood of x_k, the inverse image $\varphi_k^{-1}(A)$ has positive measure, as desired.

Instead of L_r, consider the iterated operator L_r^2, which is also regarded as a transition operator of cerain Markov process. Applying a similar argument to L_r^2, we observe that 1 is a simple eigenvalue of L_r^2 whose eigen-functions are constant, provided that there exists a point x in M such that almost all points are joined to x by an r-geodesic chain of *even* length. In particular, we have

THEOREM E: -1 *is not an eigenvalue of* L_r *provided that there exists a point* x *in* M *such that the set of points joined to* x *by* r*-geodesic chains of even length has full measure in* M.

3. Geometry of geodesic chains

If an r-geodesic chain c corresponds to $(v_1, \ldots, v_k) \in S^k M$, we call c the chain associated with $(v_1, \ldots, v_k)$, and put $\tilde{\omega}_k(v_1, \ldots, v_k) = c(k)$. Let h and k be positive integers. Define a mapping $\tilde{\omega}_{h,k}: S_x M \times S_x M \to M$ by setting

$$\tilde{\omega}_{h,k}(u, v) = \exp_{\exp(hru)} P_{hru}(krv)),$$

where $P_{hru}: T_x M \to T_{\exp(hru)} M$ is the parallel translation along the geodesic curve: $t \mapsto \exp(thru)$ $(0 \leq t \leq 1)$. Then the diagram

$$\begin{array}{ccc} S_x M \times S_x M & \xrightarrow{\tilde{\omega}_{h,k}} & \\ \downarrow & & M \\ S^{h+k} M & \xrightarrow{\tilde{\omega}_{h+k}} & \end{array}$$

is commutative, where the vertical arrow is given by

$$(u, v) \mapsto (\underbrace{u, \ldots, u}_{h}, \underbrace{v, \ldots, v}_{k}).$$

From the assumption in Theorem C, one may choose vectors u and v in $S_x M$ such that

$$\exp: hrS_x M \to M$$

$$\exp: krS_{\exp(hru)} M \to M$$

are immersion around the points hru and $krP_{hru}(v)$ respectively. Note that one may choose such vectors with $u \neq \pm v$.

LEMMA 3: *$\tilde{\omega}_{h,k}: S_xM \times S_xM \to M$ is a submersion around the point (u, v) provided that $u \neq \pm v$.*

PROOF: From the Gauss' lemma it follows that

$$d\tilde{\omega}_{h,k}(0 \oplus T_vS_xM) = \text{the orthogonal complement of } \varphi_{kr}(P_{hru}(v)) \text{ in } T_{\tilde{\omega}_{h,k}(u,v)},$$

where $\varphi_t: SM \to SM$ is the geodesic flow. Given a $X \in T_uS_xM$, there is a Jacobi field J_x along the curve

$$t \mapsto c(t) = \exp(tP_{hru}(krv))$$

such that

$$J_X(1) = d\tilde{\omega}_{h,k}(X \oplus 0)$$
$$J_X(0) = (d_{hrv}(\exp_x))(hrX)$$
$$(\nabla_{P_{hru(v)}} J_X(0), P_{hru}(v)) = 0.$$

In fact, J_X is given as the infinitesimal variation of c associated with the variation

$$c_s(t) = \exp(tP_{hru(s)}(krv)), \qquad -\varepsilon < s < \varepsilon,$$

where $s \mapsto u(s)$ is a curve in S_xM with $u(0) = u$, $du(0)/ds = X$. We show that there exists some vector X in T_uS_xM such that $(J_X(1), \varphi_{kr}(P_{hru}(v))) \neq 0$, which certainly implies the assertion. Suppose it is not the case. Since

$$\frac{d^2}{dt^2}(J_X(t), \dot{c}(t)) = 0$$
$$(\nabla_{\dot{c}} J_X(0), \dot{c}(0)) = k^2r^2(\nabla_{P_{hru(v)}} J_X(0), P_{hru}(v)) = 0,$$

we find that

$$0 = (J_X(1), \varphi_{kr}(P_{hru}(v))) \equiv \frac{1}{kr}(J_X(t), \dot{c}(t)) \equiv \frac{1}{kr}(J_X(0), \dot{c}(0)) = (J_X(0), P_{hru}(v)).$$

Using again the Gauss' lemma, we have

$\{J_X(0);\ X \in T_u S_x M\}$ = the orthogonal complement of $\varphi_{hr}(u)$ in $T_{\exp(hru)}M$,

from which it follows that $P_{hru}(v) = \pm\varphi_{hr}(u) = \pm P_{hru}(u)$, or equivalently $u = \pm v$, contradicting our choice of u and v.

PROOF OF THEOREM C: Take $(u, v) \in S_xM \times S_xM$ as above. For brevity we set

$$y = \tilde{\omega}_{h,k}(u, v)$$

$$v^* = -\varphi_{kr}(P_{hru}(v))$$

$$u^* = -P_{krv}(\varphi_{hr}(u)).$$

It is easy to see that the associated chain to the $k + h - ple$ vectors

$$(\underbrace{v^*, \ldots, v^*}_{k}, \underbrace{u^*, \ldots, u^*}_{h}) \in S_y^{k+h}M$$

is just the chain obtained by traversing the chain associated to $(u, \ldots, u, v, \ldots, v)$ in the opposite direction. Since, in general, exp: $hrS_xM \to M$ is an immersion around the point hru if and only if x and $\exp(hru)$ is not conjugate along the geodesic : $t \mapsto \exp(hrtu)$ $(0 \leq t \leq 1)$, we observe that

$$\exp: hrS_{\exp(hru)}M \to M$$

$$\exp: krS_yM \to M$$

are immersions around the points $-hr\varphi_{hr}(u)(= hrP_{krv^*}(u^*))$ and krv^* respectively. Since $v^* \neq \pm u^*$ if and only if $u \neq \pm v$, we may apply the above lemma to the mapping $\tilde{\omega}_{k,h}: S_yM \times S_yM \to M$, that is, $\tilde{\omega}_{k,h}$ is a submersion around the point (v^*, u^*). From the commutative diagram $(*)$, it follows that $\tilde{\omega}_{k+h}: S_y^{k+h}M \to M$ is a submersion around the point $(v^*, \ldots, v^*, u^*, \ldots, u^*)$, so that the image of $\tilde{\omega}_{k+h}$ contains an open neighborhood U of x. Connecting the chain associated to $(u, \ldots, u, v, \ldots, v)$ with the chains issued from y associated to the $(k + h)$-ple vectors of the form $(v_1, \ldots, v_1, u_1, \ldots, u_1)(u_1, v_1 \in S_yM)$, we obtain r-geodesic chains of length $2(h + k)$ whose end points fill up U. In other words, any point in U can be joined to x by an r-geodesic chain of length $2(h + k)$. Note that the relation given by setting $x \sim y$ iff x and y are joined by an r-geodesic chain of even length is an equivalence relation. What we have proved is that each equivalence class is open. Since M is connected, this completes the proof.

REMARK: Under the assumption of Theorem C, we may further prove that, in the case M is compact, there is a positive integer k_0 such that any two points can be joined by an r-geodesic chain of length k_0.

Suppose M is not compact. For each point x in M, one may find a geodesic ray $c:[0,\infty)\to M$ with $c(0)=x$ (see [2]). The point x is not conjugate to $c(r)$ along c for any $r>0$. Therefore the assumption in Theorem C is always satisfied in this case. We should note the argument in §2 is valid to complete manifolds with finite volume, since the total space $S^\infty M$ has also finite volume and one can apply the ergodic theory. Thus we obtain the following which is the contrast to compact cases.

THEOREM F: *If M is a complete non compact Riemannian manifold with finite volume, then L_r is ergodic, and -1 is not eigenvalue for any $r>0$.*

We now apply Theorem D to the case of surfaces. Since compact 2-dimensional manifolds with finite $\pi_1(M)$ are S^2 or $P^2(R)$, we have

THEOREM G: *If M is a 2-dimensional compact manifold, not diffeomorphic to S^2 nor $P^2(R)$, then L_r is ergodic for $r>0$. A metric on S^2 or $P^2(R)$ for which L_r is not ergodic must be a Y_l^m-metric ($l=2r$) in the sense of A. L. Besse* [1]. *Namely, if M is not ergodic, then there must be a point m in M such that all the geodesic issued from m come back to m at length $2r$.*

It remains only to prove the last part. From Theorem B and C, it follows that, if L_r is not ergodic, we may find a point m in M such that the rank of $\exp_m|rS_mM$ is zero, that is, $\exp_m(rS_mM)=n\in M$. Thus it suffices to show

LEMMA 4: *Let M be a complete Riemannian manifold such that there are points m, n in M with $\exp(rS_mM)=n$. Then all the geodesics issued from m come back to m at length $2r$, and $\exp(rS_nM)=m$.*

PROOF: Since $\varphi_r(S_mM)\subset S_nM$, the restriction $\varphi_r|S_mM$ is necessarily a diffeomorphism of S_mM onto S_nM, thus for any $u\in S_mM$, there exists a vector $v\in S_mM$ such that $\varphi_r(v)=-\varphi_r(u)$. Since $-\varphi_r(u)=\varphi_{-r}(-u)$, we get $\varphi_{2r}u=\varphi_r(-\varphi_r v)=-v$, which implies $\exp(2rS_mM)=m$.

REMARK: If M is a Y_l^m-manifold for each point $m\in M$, then $L_l=Id$.

REFERENCES

[1] A.L. Besse: Manifolds all of whose Geodesics are Closed, Springer-Verlag, New York, 1978.

[2] J. Cheeger and D. Ebin: Comparison Theorems in Riemannian Geometry, North Holland, Amsterdam, 1975.

[3] A. Gray and T.J. Willmore: Mean-value theorems for Riemannian manifolds, preprint.

[4] P. Günther: Sphärische Mittelwerte in kompakten harmonischen Riemannschen Mannigfaltigkeiten, *Math. Ann. 165* (1966), 281–296.

[5] E. Jorgensen: The central limit problem for geodesic random walks, *Z. Wahrsch. Verw. Gebiete 32* (1975), 1–64.

[6] B. Malgrange: Ideals of Differential Functions, Tata Institute Bombay 1966.

[7] M. Pinsky: Moyenne stochastique sur une variété riemannienne, *C. R. Acad. Sc. Paris 292* (1981), 991–994.

[8] T. Sunada: Spherical means and geodesic chains on a Riemannian manifold, *Trans. Amer. Math. Soc. 267* (1981), 483–501.

[9] T. Sunada: Geodesic chains on a Riemannian manifold, Proc. Colloq. Global Differential Geometry/Global Analysis, *Lecture Notes in Math. 838*, Springer, New York, 1981.

[10] T. Tsujishita: Integral transformations associated with double fiberings, *Osaka J. Math. 15* (1978), 391–418.

[11] K. Yosida: Functional Analysis, Springer-Verlag, New York, 1980.

(Oblatum 28-I-1982 & 1-V-1982)

Department of Mathematics
Nagoya University
Nagoya
464 Japan