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An asymptotic series expansion of the multidimensional renewal measure

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1. Introduction and main theorem

Let $p(x)dx$ be an absolutely continuous probability distribution on Euclidean $d$-dimensional space with non-vanishing mean vector $\mu$. As usual we define the renewal measure $v$ by the formula

$$v(E) = \sum_{n=0}^{\infty} \int_{E} p^{n*}(x)dx$$

for any Borel set $E$. Then it is well known that, if say $Q$ is the unit cube,

$$v(Q + \lambda\mu) \sim \frac{C}{\lambda^{\rho}}, \quad \lambda \to +\infty,$$

where $\rho = \frac{1}{2}(d-1)$. See [1], [2], [4], and [6]. We are concerned here with the error

$$E(\lambda) = v(Q + \lambda\mu) - \frac{C}{\lambda^{\rho}}.$$

In one dimension the decay of $E(\lambda)$ is to a large extent independent of the distribution $p(x)$, provided $p(x)$ has sufficiently many moments. For example if $\int |x|^k p(x)dx < +\infty, \ k = 2, 3, 4, \ldots$, then

$$E(\lambda) = o(\lambda^{-k+1}), \quad \lambda \to +\infty.$$

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(See however Stone [7] where it is shown that

\[ E(\lambda) = R(\lambda) + o(\lambda^{-k}), \quad \lambda \to +\infty, \]

where \( R(\lambda) \) does depend on \( p(x) \). In contrast to this, in more than one dimension it is not true in general that

\[ E(\lambda) = o(\lambda^{-(p+1)}), \quad \lambda \to +\infty, \]

no matter how many moments are assumed finite. In fact \( E(\lambda) \) has an asymptotic series expansion which is very much dependent on the complete structure of \( p(x) \).

One may get a feeling for this phenomena by considering the following special example. Suppose \( dP \) is a singular measure in the plane supported on the line \( x = 1 \) and smooth on that line. So for a \( C^\infty \) function \( f \),

\[ dP(f) = \int f(1, y) g(y) dy, \quad \int g(y) dy = 1 \]

with \( g \in C^\infty \). Suppose \( \int y g(y) dy = 0 \). Then \( dP^* \) will be supported on the line \( x = n \) and its distribution on this line is governed by the local central limit theorem which is very distributional dependent. It follows obviously that \( v(Q + n) \) is also very dependent on the distribution of \( g \).

To state our theorem we need some notation. Let \( \pi \) be the hyperplane through the origin perpendicular to \( \mu \). For \( x \) in \( \mathbb{R}^d \) let \( x_1 \) be the projection of \( x \) on \( \mu \) and let \( x' \) be the projection on \( \pi \). We let \( x_2, \ldots, x_d \) denote the standard coordinates in \( \pi \) of \( x' \). Then we consider the covariance matrix

\[ B = (E[X_i X_j])_{i,j=2, \ldots, d} \]

where \( X = (X_1, \ldots, X_d) \) is a random vector with distribution \( p(x) dx \). By \( \omega_j \) we mean an expression of the form

\[
Q_0(x') + \frac{Q_1(x')}{x_1^{1/2}} + \ldots + \frac{Q_n(x')}{x_n^{n/2}} \tag{1}
\]

where each \( Q_k \) is a homogeneous polynomial of degree \( k \) in \( x' \) whose coefficients are determined by the moments of \( p(x) \).

We shall prove the following result.
THEOREM: If $p(x)$ has a sufficient number of finite moments then

$$E_k(x) = v(Q + x) - \int_{Q + x} h_k(y) \, dy = o \left( \frac{1}{x_1^{\rho + \frac{k}{2}}} \right), \quad x_1 \to +\infty,$$

(2)

where

$$h_k(x) = \frac{\|\mu\|^{-1}}{(\det B)^{1/2} (2\pi x_1)^\rho} \exp \left( - \frac{\|\mu\|^{-1}(x', x')}{2x_1} \right) \cdot \left( 1 + \frac{\omega_1(x)}{x_1^{1/2}} \frac{\omega_2(x)}{x_1} + \ldots + \frac{\omega_k(x)}{x_1^{k/2}} \right).$$

The estimate is uniform in $x'$.

REMARK 1: $\omega_j$ is bounded in any paraboloid $|x'|^2 \leq Cx_1$ and thus the behavior of $v(Q + x)$ in such a paraboloid is determined by (2) up to "order" $\rho + \frac{k}{2}$.

REMARK 2: The explicit form of (1) in terms of the moments of $p(x)$ is rather complicated. For instance in $\mathbb{R}^2$ we have

$$\omega_1(x) = \frac{1}{2} \frac{x_2}{x_1^{1/2}} \left( \frac{\mu_{0,3}}{\mu_2} - \frac{\mu_1}{\mu_2} \right) + \frac{x_2^2}{x_1^{3/2}} \left( \frac{1}{2} \frac{\mu_{1,1}}{\mu_2} - \frac{1}{6} \frac{\mu_{0,3}}{\mu_2} \right),$$

where $\mu_i = E[X_i^i]$ and $\mu_{i,j} = E[X_i^i X_j^j]$.

REMARK 3: We will prove (2) if

$$E[|X_1|^{\max(1, \rho) + \frac{k}{2}(k + 3)}] < +\infty$$

and

$$E[|X'|^{k + 5}] < +\infty.$$
for any $\alpha < 1$ if one assumes that

$$E[|X_t|^{\max(1, \rho) + \frac{3}{2}(k + \alpha) + \varepsilon}] < +\infty$$

and

$$E[|X_t|^{2 + k + \alpha + \varepsilon}] < +\infty$$

for some $\varepsilon > 0$. For the stronger result one needs to use more delicate techniques such as those developed in [1].

In proving the theorem we need the "correct" method of expanding $f(t) = \tilde{F}(t)$. We write

$$f(t) = \sum_{j=0}^{m} p_j(t) + E_m(t),$$

where $p_j(\lambda t_1, \lambda t') = \lambda^j p_j(t_1, t')$, for $\lambda > 0$. This method of counting degrees has been employed in several complex variables and operators on nilpotent groups. See [3] and [5].

2. Proof of the theorem

Let $\phi(x)$ be $C^\infty$ and have support in the unit cube $Q = \{x; |x_i| \leq 1\}$ and put $\phi_\epsilon(x) = \epsilon^{-d} \phi(x/\epsilon)$. Let $\chi_E$ denote the indicator function of the set $E$, and put $Q_r = \{x; |x_i| \leq r\}$. Let $\Omega_k$ be the measure with density $h_k$, let $N$ denote the integer $[\rho + \frac{1}{2}k] + 1$ and put $M = 2(N - \rho)$. We shall prove

$$|\phi_\epsilon \ast \chi_{Q_r} \ast (v - \Omega_M)(x)| \leq C \frac{\log^{d} \frac{1}{\epsilon}}{x_1^N}, \quad (3)$$

where $C$ can be chosen uniformly for $r$ bounded, and

$$\phi_\epsilon \ast \chi_{Q_r} \ast (v - \Omega_M)(x) - C\epsilon \leq (v - \Omega_M)(Q + x) \leq \phi_\epsilon \ast \chi_{Q_r} \ast (v - \Omega_M)(x) + C\epsilon. \quad (4)$$

(2) follows easily from (3) and (4) by putting $\varepsilon = x_1^{-m}$ with $m$ large enough. (As $N \geq \rho + \frac{1}{2}(k + 1)$ and $M \geq k + 1$, we obtain a sharper result than (2). This is possible as we assume a stronger moment condition than necessary; compare Remark 3.)
We assume without loss of generality that $|\mu| = 1$ and that $B$ is the identity matrix.

Let us turn to the estimate (3). Set $f(t) = \int e^{-it\cdot x} p(x) dx$. As observed in [1], $\hat{f} = (1 - f)^{-1} \in L^1_{\text{loc}}(\mathbb{R}^d)$. So

$$\phi_e \ast \chi_{Q_r} \ast \nu(x) = \frac{1}{(2\pi)^d} \int e^{it\cdot x} \frac{\hat{\phi}(t) \hat{\chi}_{Q_r}(t)}{1 - f(t)} dt. \quad (5)$$

Let $\psi(t)$ be $C^\infty$ and 1 near the origin. Then, as $(1 - f(t))^{-1}$ is bounded if $t$ is bounded away from the origin and $\chi_{Q_r}$ and its derivatives are bounded by a constant times $\prod_{i=1}^d |t_i|^{-1}$, $|t| \to +\infty$, $N$ integrations by parts gives

$$\left| \int e^{it\cdot x} \frac{\hat{\phi}(t) \hat{\chi}_{Q_r}(t)}{1 - f(t)} (1 - \psi(t)) dt \right| \leq \frac{C \log^d \frac{1}{e}}{x_1^N}. \quad (6)$$

It remains to consider

$$I(x) = \frac{1}{(2\pi)^d} \int e^{it\cdot x} \frac{\hat{\phi}(t) \hat{\chi}_{Q_r}(t) \psi(t)}{1 - f(t)} dt. \quad (7)$$

The general idea is to use the Taylor expansion of $f$ at the origin to prove that

$$I(x) = \phi_e \ast \chi_{Q_r} \ast \Omega_M(x) + O(x_1^{-N}), \quad x_1 \to +\infty. \quad (8)$$

By expanding $e^{-it\cdot x}$ in a Taylor series and integrating term by term, we get

$$f(t) = 1 - \sum_{j=2}^n P_j(t) + R_n(t), \quad (9)$$

where $P_j(t) = P_j(t_1, t')$ is a polynomial homogeneous in the sense that

$$P_j(\lambda^2 t_1, \lambda t') = \lambda^j P_j(t).$$

The coefficients of $P_j$ are determined by the moments of $p(x)$. From the Taylor expansion of $e^{-it\cdot x}$ and the moment condition imposed on $p(x)$, we get if $n \leq M + 2$ that

$$R_n(t) = \int \left( e^{-it\cdot x} - \sum_{\|\alpha\| \leq n} \frac{(-it\cdot x)^\alpha}{\alpha!} \right) p(x) dx = O((|t_1| + |t'|^2)^{(n+1)\frac{1}{2}}), \quad t \to 0, \quad (10)$$
where \( \|x\| = 2\alpha_1 + \alpha_2 + \ldots + \alpha_d \). Furthermore, if \( l \leq n/2 \)

\[
\frac{\partial^l R_n^n}{\partial t_1^l}(t) = \int (-i x)^l \left( e^{-it \cdot x} - \sum_{\|x\| \leq n-2l} \frac{(-it \cdot x)^l}{l!} \right) p(x) dx = \\
= O((|t_1| + |t|^2)^{\frac{n+1}{2}-l}), \quad t \to 0, \tag{11}
\]

and \( \frac{\partial^l R_n}{\partial t_1^l} \) is bounded if \( n/2 < l \leq N \).

Write

\[
\frac{1}{1 - f(t)} = \frac{1}{P_2(t)} + \frac{1}{1 - f(t)} - \frac{1}{P_2(t)} = \\
= \frac{1}{P_2(t)} + \frac{R_2(t)}{P_2(t)(1 - f(t))}.
\]

By iteration, we find

\[
\frac{1}{1 - f(t)} = \sum_{j=0}^{M} \frac{R_2^j(t)}{P_2^{j+1}(t)} + \frac{R_2^{M+1}(t)}{P_2^{M+1}(t)(1 - f(t))}. \tag{12}
\]

We write

\[
R_2(t) = -\sum_{k=3}^{M+2} P_k(t) + R_{M+2}(t)
\]

and expand \( R_2(t) \) by the multinomial theorem to obtain

\[
\frac{1}{1 - f(t)} = \sum_{j=0}^{M} \left( \frac{-\sum_{k=3}^{M+2} P_k(t)}{P_2^{j+1}(t)} \right)^j + S_M(t). \tag{13}
\]

Here

\[
S_M(t) = \frac{R_2^{M+1}(t)}{P_2^{M+1}(t)(1 - f(t))} + \\
+ \sum_{j=1}^{M} \Sigma' C_{i_1, \ldots, i_m} \frac{R_{M+2}^{i_1}(t)P_3^{i_2}(t) \ldots P_{M+2}^{i_m}(t)}{P_2^{j+1}(t)}
\]

where \( \Sigma' \) is a finite sum with \( i_1 + i_3 + \ldots + i_{M+2} = j \) and \( i_1 \geq 1 \). We claim that \( S_M \) has \( N \) derivatives with respect to \( t_1 \) that are locally integrable. Granted this we can integrate by parts \( N \) times to obtain

\[
\left| \int e^{it \cdot x} \hat{\phi}(ct) \bar{\zeta}_Q(t) \psi(t) S_M(t) dt \right| \leq \frac{C}{x_1^N}. \tag{14}
\]
To see that $S_M(t)$ has $N$ locally integrable derivatives, note that the moment assumption on $p(x)$ implies that $f(t)$ is differentiable $N$ times with respect to $t_1$. Hence $S_M(t)$ is differentiable $N$ times if $t \neq 0$. Furthermore, in a neighborhood of the origin every quotient occurring in $S_M$ is bounded by a constant times $(|t_1| + |t'||^2)^{1/(M-1)}$ (see (10)). As each differentiation with respect to $t_1$ introduces at worst a multiplicative factor of size $(|t_1| + |t'||^2)^{-1}$ (see (11)), we get

$$\left| \frac{\partial^N S_M(t)}{\partial t_1^N} (t) \right| \leq \frac{C}{(|t_1| + |t'||^2)^{N/2}} \in L^1_{\text{loc}}(\mathbb{R}^d).$$

In view of (7), (13) and (14) we have

$$I(x) = \sum_{j=0}^{M} \sum_{i=3j}^{(M+2)j} \frac{1}{2\pi} \int e^{i\cdot x \cdot \tilde{\phi}(s)} \tilde{Q}_o(t) \psi(t) \frac{q_{l,j}(t)}{P_{l}^{j+1}(t)} dt + O(x_1^{-N}), \quad x_1 \to + \infty,$$

where $q_{l,j}(t)$ is a polynomial satisfying

$$q_{l,j}(\lambda^2 t_1, \lambda t') = \lambda^l q_{l,j}(t_1, t').$$

Write

$$\psi(t) \frac{q_{l,j}(t)}{P_{l}^{j+1}(t)} = \frac{q_{l,j}(t)}{P_{l}^{j+1}(t)} + (\psi(t) - 1) \frac{q_{l,j}(t)}{P_{l}^{j+1}(t)}.$$

Now sufficiently high order derivatives with respect to $t_1$ to

$$(\psi(t) - 1) q_{l,j}(t) P_{l}^{-j+1}(t)$$

are integrable functions, so $((\psi(t) - 1) q_{l,j}(t) P_{l}^{-j+1}(t))^\gamma$ is a function in $x_1 > 0$ and is $O(x_1^{-k})$, $x_1 \to + \infty$, for any $k$. Thus we get from (15) that

$$I(x) = \sum_{j=0}^{M} \sum_{i=3j}^{(M+2)j} \phi_x \star \varphi_{Q_o} \star Q_{l,j}(x) + O(x_1^{-N}), \quad x_1 \to + \infty,$$

where $\hat{Q}_{l,j} = q_{l,j} P_{l}^{-j+1}$. From (16) we see that

$$\frac{q_{l,j}(\lambda^2 t_1, \lambda t')}{P_{l}^{j+1}(\lambda^2 t_1, \lambda t')} = \lambda^{l-2(j+1)} \frac{q_{l,j}(t)}{P_{l}^{j+1}(t)}$$
which implies
\[ Q_{l,j}(\lambda^2 x_1, \lambda x') = \frac{1}{\lambda^{d-1} + l - 2j} Q_{l,j}(x). \]

By letting \( x_1 = 1 \), \( x' = 0 \) and \( y_1 = \lambda^2 \) we get
\[ Q_{l,j}(y_1, 0) = \frac{C_{l,j}}{y_1^{d/2 - j}}, \quad y_1 > 0. \]

Now if we define \( \omega_k \) by
\[ \frac{1}{(2\pi x_1)^p} \exp\left( -\frac{|x'|^2}{2x_1} \right) \frac{\omega_k(x)}{x_1^{k/2}} = \sum_{l - 2j = k} Q_{l,j}(x), \tag{18} \]
we have \( \omega_k(x_1, 0) = c_k \) which agrees with (1). To see that \( \omega_k(x) \) has the representation (1) also when \( x' \neq 0 \), we first observe that the Fourier transform of
\[ w(x) = \begin{cases} \frac{1}{(2\pi x_1)^p} \exp\left( -\frac{|x'|^2}{2x_1} \right), & x_1 > 0 \\ 0, & x_1 \leq 0 \end{cases} \]
is
\[ \hat{w}(t) = P_2^{-1}(t), \]
see [1]. From this we obtain
\[ \left( \frac{\partial^x}{\partial x^{x_1}} x_1^\lambda w(x) \right) (t) = C_{x_1} \frac{t^x}{P_2^x(t)}, \]
where the derivatives and the Fourier transform are interpreted in the sense of distributions. As \( x_1^\lambda w(x) \in C_0^\infty \) if \( x \neq 0 \), we see that \( \frac{\partial^x}{\partial x^{x_1}} x_1^\lambda w(x) \) for \( x \neq 0 \) is a function obtained from \( x_1^\lambda w(x) \) by (ordinary) differentiation. From this the representation (1) follows easily by induction. We also see that the terms \( \phi_x \ast \chi_{Qr} \ast Q_{l,j} \) in (17) with \( l - 2j > M \) is \( O(x_1^{-N}) \), \( x_1 \to +\infty \). Thus (9) follows from (17). This completes the proof of (3), as it is an immediate consequence of (5), (6), (7) and (9).
To prove (4) we observe that if \( y \in \text{supp} \phi_\varepsilon \) then

\[ Q_1 - \varepsilon + x - y \subseteq Q + x \subseteq Q_1 + \varepsilon + x - y, \]

which implies

\[ \phi_\varepsilon \ast \chi_{Q_1 - \varepsilon} \ast v(x) \leq v(Q + x) = \int v(Q + x)\phi_\varepsilon(y)dy \leq \phi_\varepsilon \ast \chi_{Q_1 + \varepsilon} \ast v(x) \]

(19)

as \( v \) is a nonnegative measure. Furthermore,

\[
\left| \phi_\varepsilon \ast \chi_{Q_1 - \varepsilon} \ast \Omega_M(x) - \Omega_M(Q + x) \right| \leq \int \left| \Omega_M(Q_1 \pm \varepsilon + x - y) - \Omega_M(Q + x) \right| \phi_\varepsilon(y)dy.
\]

As \( y \in \text{supp} \phi_\varepsilon \) implies that the symmetric difference between \( Q_1 \pm \varepsilon + x - y \) and \( Q + x \) is included in \( \{ x; 1 - 2\varepsilon \leq |x_1| \leq 1 + 2\varepsilon \} \) and \( \Omega_M \) has a bounded density, we get

\[
\left| \phi_\varepsilon \ast \chi_{Q_1 - \varepsilon} \ast \Omega_M(x) - \Omega_M(Q + x) \right| \leq C\varepsilon.
\]

(20)

This completes the proof of the theorem as (19) and (20) implies (4).

**Remark 4:** It can be seen from the proof that the theorem is true for any measure that satisfies \( \lim_{|t| \to \infty} \inf |1 - f(t)| > 0 \). Also \( Q \) can be an arbitrary parallelepiped and the estimate is uniform for \( Q \) in bounded sets.


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