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AN ASYMPTOTIC SERIES EXPANSION OF THE MULTIDIMENSIONAL RENEWAL MEASURE

Hasse Carlsson[1] and Stephen Wainger[2]

1. Introduction and main theorem

Let $p(x)dx$ be an absolutely continuous probability distribution on Euclidean d-dimensional space with non-vanishing mean vector μ. As usual we define the renewal measure v by the formula

$$v(E) = \sum_{n=0}^{\infty} \int_E p^{n*}(x)\,dx$$

for any Borel set E. Then it is well known that, if say Q is the unit cube,

$$v(Q + \lambda\mu) \sim \frac{C}{\lambda^{\rho}}, \qquad \lambda \to +\infty,$$

where $\rho = \frac{1}{2}(d-1)$. See [1], [2], [4], and [6]. We are concerned here with the error

$$E(\lambda) = v(Q + \lambda\mu) - \frac{C}{\lambda^{\rho}}.$$

In one dimension the decay of $E(\lambda)$ is to a large extent independent of the distribution $p(x)$, provided $p(x)$ has sufficiently many moments. For example if $\int |x|^k p(x)\,dx < +\infty$, $k = 2, 3, 4, \ldots$, then

$$E(\lambda) = o(\lambda^{-k+1}), \qquad \lambda \to +\infty.$$

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(See however Stone [7] where it is shown that

$$E(\lambda) = R(\lambda) + o(\lambda^{-k}), \qquad \lambda \to +\infty,$$

where $R(\lambda)$ does depend on $p(x)$.) In contrast to this, in more than one dimension it is not true in general that

$$E(\lambda) = o(\lambda^{-(\rho+1)}), \qquad \lambda \to +\infty,$$

no matter how many moments are assumed finite. In fact $E(\lambda)$ has an asymptotic series expansion which is very much dependent on the complete structure of $p(x)$.

One may get a feeling for this phenomena by considering the following special example. Suppose dP is a singular measure in the plane supported on the line $x = 1$ and smooth on that line. So for a C_0^∞ function f,

$$dP(f) = \int f(1, y)g(y)\,dy, \quad \int g(y)\,dy = 1$$

with $g \in C_0^\infty$. Suppose $\int yg(y)\,dy = 0$. Then dP^{*n} will be supported on the line $x = n$ and its distribution on this line is governed by the local central limit theorem which is very distributional dependent. It follows obviously that $v(Q + n)$ is also very dependent on the distribution of g.

To state our theorem we need some notation. Let π be the hyperplane through the origin perpendicular to μ. For x in $\mathbb{R}^d$ let x_1 be the projection of x on μ and let x' be the projection on π. We let $x_2, \ldots, x_d$ denote the standard coordinates in π of x'. Then we consider the covariance matrix

$$B = (E[X_i X_j])_{i,j=2,\ldots,d}$$

where $X = (X_1, \ldots, X_d)$ is a random vector with distribution $p(x)\,dx$. By ω_j we mean an expression of the form

$$Q_0(x') + \frac{Q_1(x')}{x_1^{1/2}} + \ldots + \frac{Q_{n_j}(x')}{x_1^{n_j/2}} \tag{1}$$

where each Q_k is a homogeneous polynomial of degree k in x' whose coefficients are determined by the moments of $p(x)$.

We shall prove the following result.

THEOREM: *If $p(x)$ has a sufficient number of finite moments then*

$$E_k(x) = v(Q + x) - \int_{Q+x} h_k(y)dy = o\left(\frac{1}{x_1^{\rho + k/2}}\right), \quad x_1 \to +\infty, \tag{2}$$

where

$$h_k(x) = \frac{|\mu|^{\rho - 1}}{(\det B)^{1/2}(2\pi x_1)^\rho} \exp\left(-\frac{|\mu| B^{-1}(x', x')}{2x_1}\right) \cdot$$
$$\cdot \left(1 + \frac{\omega_1(x)}{x_1^{1/2}} + \frac{\omega_2(x)}{x_1} + \ldots + \frac{\omega_k(x)}{x_1^{k/2}}\right).$$

The estimate is uniform in x'.

REMARK 1: ω_j is bounded in any paraboloid $|x'|^2 \leq Cx_1$ and thus the behavior of $v(Q + x)$ in such a paraboloid is determined by (2) up to "order" $\rho + \dfrac{k}{2}$.

REMARK 2: The explicit form of (1) in terms of the moments of $p(x)$ is rather complicated. For instance in $\mathbb{R}^2$ we have

$$\omega_1(x) = \frac{1}{2}\frac{x_2}{x_1^{1/2}}\left(\mu_{0,3}\frac{\mu_1^2}{\mu_2^2} - \mu_{1,1}\frac{\mu_1}{\mu_2}\right) +$$
$$+ \frac{x_2^3}{x_1^{3/2}}\left(\frac{1}{2}\mu_{1,1}\frac{\mu_1^2}{\mu_2^2} - \frac{1}{6}\mu_{0,3}\frac{\mu_1^3}{\mu_2^3}\right)$$

where $\mu_i = E[X_i^i]$ and $\mu_{i,j} = E[X_1^i X_2^j]$.

REMARK 3: We will prove (2) if

$$E[|X_1|^{\max(1,\rho) + \frac{1}{2}(k+3)}] < +\infty$$

and

$$E[|X'|^{k+5}] < +\infty.$$

In fact one can prove the sharper result

$$E_k(x) = o\left(\frac{1}{x_1^{\rho + \frac{1}{2}(k+\alpha)}}\right), \qquad x_1 \to +\infty,$$

for any $\alpha < 1$ if one assumes that

$$E[|X_1|^{\max(1,\rho)+\frac{1}{2}(k+\alpha)+\varepsilon}] < +\infty$$

and

$$E[|X'|^{2+k+\alpha+\varepsilon}] < +\infty$$

for some $\varepsilon > 0$. For the stronger result one needs to use more delicate techniques such as those developed in [1].

In proving the theorem we need the "correct" method of expanding $f(t) = \hat{p}(t)$. We write

$$f(t) = \sum_{j=0}^{m} p_j(t) + E_m(t),$$

where $p_j(\lambda^2 t_1, \lambda t') = \lambda^j p_j(t_1, t')$, for $\lambda > 0$. This method of counting degrees has been employed in several complex variables and operators on nilpotent groups. See [3] and [5].

2. Proof of the theorem

Let $\phi(x)$ be C^∞ and have support in the unit cube $Q = \{x; |x_i| \leq 1\}$ and put $\phi_\varepsilon(x) = \varepsilon^{-d}\phi(x/\varepsilon)$. Let χ_E denote the indicator function of the set E, and put $Q_r = \{x; |x_i| \leq r\}$. Let Ω_k be the measure with density h_k, let N denote the integer $[\rho + \frac{1}{2}k] + 1$ and put $M = 2(N - \rho)$. We shall prove

$$|\phi_\varepsilon * \chi_{Q_r} * (v - \Omega_M)(x)| \leq C \frac{\log^d \frac{1}{\varepsilon}}{x_1^N}, \tag{3}$$

where C can be chosen uniformly for r bounded, and

$$\begin{aligned} \phi_\varepsilon * \chi_{Q_{1-\varepsilon}} * (v - \Omega_M)(x) - C\varepsilon \leq (v - \Omega_M)(Q + x) \leq \\ \leq \phi_\varepsilon * \chi_{Q_{1+\varepsilon}} * (v - \Omega_M)(x) + C\varepsilon. \end{aligned} \tag{4}$$

(2) follows easily from (3) and (4) by putting $\varepsilon = x_1^{-m}$ with m large enough. (As $N \geq \rho + \frac{1}{2}(k + 1)$ and $M \geq k + 1$, we obtain a sharper result than (2). This is possible as we assume a stronger moment condition than necessary; compare Remark 3.)

We assume without loss of generality that $|\mu| = 1$ and that B is the identity matrix.

Let us turn to the estimate (3). Set $f(t) = \int e^{-it\cdot x}p(x)\,dx$. As observed in [1], $\hat{v} = (1-f)^{-1} \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. So

$$\phi_\varepsilon * \chi_{Q_r} * v(x) = \frac{1}{(2\pi)^d}\int e^{it\cdot x}\frac{\hat{\phi}(\varepsilon t)\hat{\chi}_{Q_r}(t)}{1-f(t)}\,dt. \tag{5}$$

Let $\psi(t)$ be C_0^∞ and 1 near the origin. Then, as $(1-f(t))^{-1}$ is bounded if t is bounded away from the origin and χ_{Q_r} and its derivatives are bounded by a constant times $\prod_{i=1}^d |t_i|^{-1}$, $|t| \to +\infty$, N integrations by parts gives

$$\left|\int e^{it\cdot x}\frac{\hat{\phi}(\varepsilon t)\hat{\chi}_{Q_r}(t)}{1-f(t)}(1-\psi(t))\,dt\right| \leq \frac{C\log^d\frac{1}{\varepsilon}}{x_1^N}. \tag{6}$$

It remains to consider

$$I(x) = \frac{1}{(2\pi)^d}\int e^{it\cdot x}\frac{\hat{\phi}(\varepsilon t)\hat{\chi}_{Q_r}(t)\psi(t)}{1-f(t)}\,dt. \tag{7}$$

The general idea is to use the Taylor expansion of f at the origin to prove that

$$I(x) = \phi_\varepsilon * \chi_{Q_r} * \Omega_M(x) + O(x_1^{-N}), \qquad x_1 \to +\infty. \tag{8}$$

By expanding $e^{-it\cdot x}$ in a Taylor series and integrating term by term, we get

$$f(t) = 1 - \sum_{j=2}^{n} P_j(t) + R_n(t), \tag{9}$$

where $P_j(t) = P_j(t_1, t')$ is a polynomial homogeneous in the sense that

$$P_j(\lambda^2 t_1, \lambda t') = \lambda^j P_j(t).$$

The coefficients of P_j are determined by the moments of $p(x)$. From the Taylor expansion of $e^{-it\cdot x}$ and the moment condition imposed on $p(x)$, we get if $n \leq M+2$ that

$$\begin{aligned} R_n(t) &= \int\left(e^{-it\cdot x} - \sum_{\|\alpha\|\leqq n}\frac{(-it\cdot x)^\alpha}{\alpha!}\right)p(x)\,dx = \\ &= O((|t_1| + |t'|^2)^{\frac{1}{2}(n+1)}), \qquad t \to 0, \end{aligned} \tag{10}$$

where $\|\alpha\| = 2\alpha_1 + \alpha_2 + \ldots + \alpha_d$. Furthermore, if $l \leq n/2$

$$\frac{\partial^l R_n}{\partial t_1^l}(t) = \int (-ix_1)^l \left(e^{-it\cdot x} - \sum_{\|\alpha\| \leq n-2l} \frac{(-it\cdot x)^\alpha}{\alpha!} \right) p(x)\,dx = $$
$$= O((|t_1| + |t'|^2)^{\frac{1}{2}(n+1)-l}), \qquad t \to 0, \tag{11}$$

and $\partial^l R_n / \partial t_1^l$ is bounded if $n/2 < l \leq N$.

Write

$$\frac{1}{1-f(t)} = \frac{1}{P_2(t)} + \frac{1}{1-f(t)} - \frac{1}{P_2(t)} =$$
$$= \frac{1}{P_2(t)} + \frac{R_2(t)}{P_2(t)(1-f(t))}.$$

By iteration, we find

$$\frac{1}{1-f(t)} = \sum_{j=0}^{M} \frac{R_2^j(t)}{P_2^{j+1}(t)} + \frac{R_2^{M+1}(t)}{P_2^{M+1}(t)(1-f(t))}. \tag{12}$$

We write

$$R_2(t) = -\sum_{k=3}^{M+2} P_k(t) + R_{M+2}(t)$$

and expand $R_2(t)$ by the multinomial theorem to obtain

$$\frac{1}{1-f(t)} = \sum_{j=0}^{M} \frac{\left(-\sum_{k=3}^{M+2} P_k(t) \right)^j}{P_2^{j+1}(t)} + S_M(t). \tag{13}$$

Here

$$S_M(t) = \frac{R_2^{M+1}(t)}{P_2^{M+1}(t)(1-f(t))} +$$
$$+ \sum_{j=1}^{M} \Sigma' C_{i_1,\ldots,i_m} \frac{R_{M+2}^{i_1}(t) P_3^{i_3}(t) \ldots P_{M+2}^{i_{M+2}}(t)}{P_2^{j+1}(t)}$$

where Σ' is a finite sum with $i_1 + i_3 + \ldots + i_{M+2} = j$ and $i_1 \geq 1$. We claim that S_M has N derivatives with respect to t_1 that are locally integrable. Granted this we can integrate by parts N times to obtain

$$\left| \int e^{it\cdot x} \hat{\phi}(\varepsilon t) \hat{\chi}_{Q_r}(t) \psi(t) S_M(t)\,dt \right| \leq \frac{C}{x_1^N}. \tag{14}$$

To see that $S_M(t)$ has N locally integrable derivatives, note that the moment assumption on $p(x)$ implies that $f(t)$ is differentiable N times with respect to t_1. Hence $S_M(t)$ is differentiable N times if $t \neq 0$. Furthermore, in a neighborhood of the origin every quotient occurring in S_M is bounded by a constant times $(|t_1| + |t'|^2)^{\frac{1}{2}(M-1)}$ (see (10)). As each differentiation with respect to t_1 introduces at worst a multiplicative factor of size $(|t_1| + |t'|^2)^{-1}$ (see (11)), we get

$$\left| \frac{\partial^N S_M(t)}{\partial t_1^N}(t) \right| \leq \frac{C}{(|t_1| + |t'|^2)^{\rho + 1/2}} \in L^1_{\text{loc}}(\mathbb{R}^d).$$

In view of (7), (13) and (14) we have

$$I(x) = \sum_{j=0}^{M} \sum_{l=3j}^{(M+2)j} \frac{1}{(2\pi)^d} \int e^{it \cdot x} \hat{\phi}(\varepsilon t) \hat{\chi}_{Q_r}(t) \psi(t) \frac{q_{l,j}(t)}{P_2^{j+1}(t)} \, dt + O(x_1^{-N}),$$
$$x_1 \to +\infty, \qquad (15)$$

where $q_{l,j}(t)$ is a polynomial satisfying

$$q_{l,j}(\lambda^2 t_1, \lambda t') = \lambda^l q_{l,j}(t_1, t'). \qquad (16)$$

Write

$$\psi(t) \frac{q_{l,j}(t)}{P_2^{j+1}(t)} = \frac{q_{l,j}(t)}{P_2^{j+1}(t)} + (\psi(t) - 1) \frac{q_{l,j}(t)}{P_2^{j+1}(t)}.$$

Now sufficiently high order derivatives with respect to t_1 to

$$(\psi(t) - 1) q_{l,j}(t) P_2^{-(j+1)}(t)$$

are integrable functions, so $\{(\psi(t) - 1) q_{l,j}(t) P_2^{-(j+1)}(t)\}^{\vee}$ is a function in $x_1 > 0$ and is $O(x_1^{-k})$, $x_1 \to +\infty$, for any k. Thus we get from (15) that

$$I(x) = \sum_{j=0}^{M} \sum_{l=3j}^{(M+2)j} \phi_\varepsilon * \chi_{Q_r} * Q_{l,j}(x) + O(x_1^{-N}), \qquad x_1 \to +\infty, \qquad (17)$$

where $\hat{Q}_{l,j} = q_{l,j} P_2^{-(j+1)}$.

From (16) we see that

$$\frac{q_{l,j}(\lambda^2 t_1, \lambda t')}{P_2^{j+1}(\lambda^2 t_1, \lambda t')} = \lambda^{l - 2(j+1)} \frac{q_{l,j}(t)}{P_2^{j+1}(t)}$$

which implies

$$Q_{l,j}(\lambda^2 x_1, \lambda x') = \frac{1}{\lambda^{d-1+l-2j}} Q_{l,j}(x).$$

By letting $x_1 = 1$, $x' = 0$ and $y_1 = \lambda^2$ we get

$$Q_{l,j}(y_1, 0) = \frac{C_{l,j}}{y_1^{\rho+l/2-j}}, \qquad y_1 > 0.$$

Now if we define ω_k by

$$\frac{1}{(2\pi x_1)^\rho} \exp\left(-\frac{|x'|^2}{2x_1}\right) \frac{\omega_k(x)}{x_1^{k/2}} = \sum_{l-2j=k} Q_{l,j}(x), \tag{18}$$

we have $\omega_k(x_1, 0) = c_k$ which agrees with (1). To see that $\omega_k(x)$ has the representation (1) also when $x' \neq 0$, we first observe that the Fourier transform of

$$w(x) = \begin{cases} \dfrac{1}{(2\pi x_1)^\rho} \exp\left(-\dfrac{|x'|^2}{2x_1}\right), & x_1 > 0 \\ 0, & x_1 \leq 0 \end{cases}$$

is

$$\hat{w}(t) = P_2^{-1}(t),$$

see [1]. From this we obtain

$$\left(\frac{\partial^\alpha}{\partial x^\alpha} x_1^p w(x)\right)^\wedge (t) = C_{\alpha,p} \frac{t^\alpha}{P_2^p(t)},$$

where the derivatives and the Fourier transform are interpreted in the sense of distributions. As $x_1^p w(x) \in C_0^\infty$ if $x \neq 0$, we see that $\dfrac{\partial^\alpha}{\partial x^\alpha} x_1^p w(x)$ for $x \neq 0$ is a function obtained from $x_1^p w(x)$ by (ordinary) differentiation. From this the representation (1) follows easily by induction. We also see that the terms $\phi_\varepsilon * \chi_{Q_r} * Q_{l,j}$ in (17) with $l - 2j > M$ is $O(x_1^{-N})$, $x_1 \to +\infty$. Thus (9) follows from (17). This completes the proof of (3), as it is an immediate consequence of (5), (6), (7) and (9).

To prove (4) we observe that if $y \in \operatorname{supp} \phi_\varepsilon$ then

$$Q_{1-\varepsilon} + x - y \subseteq Q + x \subseteq Q_{1+\varepsilon} + x - y,$$

which implies

$$\phi_\varepsilon * \chi_{Q_{1-\varepsilon}} * v(x) \leq v(Q + x) = \int v(Q + x)\phi_\varepsilon(y)dy \leq \phi_\varepsilon * \chi_{Q_{1+\varepsilon}} * v(x) \tag{19}$$

as v is a nonnegative measure. Furthermore,

$$\begin{aligned}&|\phi_\varepsilon * \chi_{Q_{1\pm\varepsilon}} * \Omega_M(x) - \Omega_M(Q + x)| \leq \\ &\quad \leq \int |\Omega_M(Q_{1\pm\varepsilon} + x - y) - \Omega_M(Q + x)|\,\phi_\varepsilon(y)dy.\end{aligned}$$

As $y \in \operatorname{supp} \phi_\varepsilon$ implies that the symmetric difference between $Q_{1\pm\varepsilon} + x - - y$ and $Q + x$ is included in $\{x; 1 - 2\varepsilon \leq |x_i| \leq 1 + 2\varepsilon\}$ and Ω_M has a bounded density, we get

$$|\phi_\varepsilon * \chi_{Q_{1\pm\varepsilon}} * \Omega_M(x) - \Omega_M(Q + x)| \leq C\varepsilon. \tag{20}$$

This completes the proof of the theorem as (19) and (20) implies (4).

Remark 4: It can be seen from the proof that the theorem is true for any measure that satisfies $\lim_{|t|\to\infty} \inf|1 - f(t)| > 0$. Also Q can be an arbitrary parallelepiped and the estimate is uniform for Q in bounded sets.

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