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AN INVERSION FORMULA FOR WEIGHTED ORBITAL INTEGRALS

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§1. Introduction

Let G be a reductive Lie group satisfying Harish–Chandra's basic assumptions. Let A be the split component of a parabolic subgroup of G and T a Cartan subgroup of G with $A \subseteq T$. Write $T = T_I T_R$ where T_I is compact and T_R is split. Then for $h \in T'$, the set of regular elements of T, and $f \in C_c^\infty(G)$, Arthur defines in [1c] a weighted integral of f over the orbit of h by

$$r_f^A(h) = \int_{T_R \backslash G} f(x^{-1}hx)v_A(x)d\dot{x} \tag{1.1}$$

where v_A is a certain weight function corresponding to A defined on $C_G(A)\backslash G$ and $d\dot{x}$ is a G-invariant measure on the quotient. When $A = \{1\}$, $r_f^{\{1\}}(h)$ is the ordinary orbital integral.

Arthur proves that the distributions $r^A(h): f \to r_f^A(h)$, $f \in C_c^\infty(G)$, are tempered, that is, extend continuously to $f \in \mathscr{C}(G)$, the Schwartz space of G, and have many properties analogous to those of ordinary orbital integrals. Such weighted orbital integrals occur in the Selberg trace formula for the case of non-compact quotient, and thus it is important to compute their Fourier transforms as tempered distributions [see 1 a, d, 5].

In the case that f is a matrix coefficient for a discrete series representation of class ω and with character θ_ω, Arthur proves that

$$r_f^A(h) = \varepsilon(T, A)(-1)^p \langle \theta_\omega, f \rangle \theta_\omega(h) \tag{1.2}$$

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where p is the dimension of A and $\varepsilon(T, A)$ is 1 if $A = T_R$ and is 0 otherwise. This gives the Fourier inversion formula for $r^A(h)$ restricted to the space ${}^\circ\mathscr{C}(G)$ of cusp forms on G. It also shows that the weighted orbital integrals, like ordinary orbital integrals, have important connections with the harmonic analysis on G.

In the case that $A = \{1\}$, (1.2) is a well-known theorem of Harish–Chandra. In order to motivate the results of this paper, it is useful to review other results of Harish–Chandra on orbital integrals. Thus let $P = MA_1N$ be a cuspidal parabolic subgroup of G. For ω an equivalence class of discrete series representations of M and $\nu \in \mathscr{F}$, the real dual of the Lie algebra of A_1, let $\theta_{\omega,\nu}$ be the corresponding unitary character induced from P. Let $W(\omega) = \{s \in N_G(A_1)/C_G(A_1),\ \ s\omega = \omega\}$. For $\alpha \in C_c^\infty(\mathscr{F})$, let $f = \varphi_\alpha$ be a wave packet corresponding to ω. Then Harish–Chandra proves that if $h \in T'$ where T is a Cartan subgroup of G with $\dim T_R \geq \dim A_1$, then

$$r_f(h) = \varepsilon(T, A_1)[W(\omega)]^{-1} \int_{\mathscr{F}} \langle \theta_{\omega,\nu}, f \rangle \theta_{\omega,\nu}(h)\, d\nu. \tag{1.3}$$

Here $\varepsilon(T, A_1)$ is 1 if $\dim T_R = \dim A_1$ and is 0 otherwise [2c].
Let $\mathscr{C}_{A_1}(G)$ denote the subspace of $\mathscr{C}(G)$ spanned by wave packets coming from some parabolic P with split component A_1. Then we see from (1.3) that for $f \in \mathscr{C}_{A_1}(G)$ and $h \in T'$, if $\dim A_1 < \dim T_R$, $r_f(h) = 0$, while if $\dim T_R = \dim A_1$, $r_f(h)$ is possibly non-zero but still is given by a simple formula. When $\dim A_1 > \dim T_R$, the formula for $r_f(h)$ becomes much more complicated (see [3b]).

Returning to the case of weighted orbital integrals, if $A = T_R$, then the distribution $r^A(h)$, $h \in T'$, is non-trivial on the space of cusp forms. Thus in this case we expect that for $f \in \mathscr{C}_{A_1}(G)$, $\dim A_1 > 0$, the Fourier inversion formula for $r_f^A(h)$ will be complicated. However, if $A \subsetneq T_R$, so that $r_f^A(h) = 0$ for all $f \in {}^\circ\mathscr{C}(G)$, it is reasonable to expect that $r_f^A(h)$, $f \in \mathscr{C}_{A_1}(G)$, may be given by a relatively simple formula for A_1 of sufficiently small dimension. This is indeed the case.

Thus let $P = MA_1N$ be a cuspidal parabolic subgroup of G with $\dim A_1 \leq \dim T_R - \dim A$. Let ω be an equivalence class of discrete series representations of M. For $\alpha \in C_c^\infty(\mathscr{F})$, let $f = \varphi_\alpha$ be a wave packet corresponding to ω. We will define a "weighted character" $\theta_{\omega,\nu}^A$ on T' so that

$$r_f^A(h) = \varepsilon(T, A, A_1)(-1)^p[W(\omega)]^{-1} \int_{\mathscr{F}} \langle \theta_{\omega,\nu}, f \rangle \theta_{\omega,\nu}^A(h)\, d\nu \tag{1.4}$$

where $\varepsilon(T, A, A_1)$ is 1 if $\dim T_R = \dim A + \dim A_1$ and is 0 otherwise.

Note that (1.4) shows that restricted to $\mathscr{C}_{A_1}(G)$, $\dim A_1 \leq \dim T_R - \dim A$, $r^A(h)$ is invariant as a distribution. However, $r^A(h)$ is not an invariant distribution on $\mathscr{C}(G)$. Thus when $f \in \mathscr{C}_{A_1}(G)$, $\dim A_1 > \dim T_R - \dim A$, we can expect the problem of computing $r_f^A(h)$ to become much more difficult.

In section §2 we review the basic definitions and results of Arthur on weighted orbital integrals and of Harish–Chandra on wave packets which will be needed to prove (1.4). In section §3 we define the "weighted characters" $\theta_{\omega,\nu}^A$ which appear in (1.4) and show that they retain many of the basic properties of the ordinary characters $\theta_{\omega,\nu}$. In section §4 we study distribution-valued functions on T' of the type that occur in (1.4), and in §5 we give the proof of (1.4).

§2. Background material

Let G be a reductive Lie group with Lie algebra $\mathfrak{g}$. Let K be a maximal compact subgroup of G and θ the corresponding Cartan involution. Let B be a real symmetric bilinear form on $\mathfrak{g}$. We will assume that (G, K, θ, B) satisfy the general assumptions of Harish–Chandra in [2b] and that Haar measures are normalized as in [2b].

Subgroups of G will be denoted by capital letters and the associated subalgebras by the corresponding lower case German letters. The complexification of any Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ will be denoted $\mathfrak{h}_{\mathbf{C}}$. All Cartan subgroups T of G will be assumed to be θ-stable. We will write T' for the set of regular elements of T and decompose $T = T_I T_R$ where $T_I = T \cap K$ and T_R is a vector subgroup of T with Lie algebra $\mathfrak{t}_R$ contained in the -1 eigenspace for θ. We will write $N_G(T)$ for the normalizer of T in G, T_0 for the center of T, and $W(G, T)$ for $N_G(T)/T_0$. $\Phi = \Phi(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$ will denote the set of roots of $\mathfrak{g}_{\mathbf{C}}$ with respect to $\mathfrak{t}_{\mathbf{C}}$, Φ_R and Φ_I the subsets of Φ taking real and pure imaginary values on $\mathfrak{t}$ respectively. Φ_C denotes the complement in Φ of $\Phi_R \cup \Phi_I$. For each $\gamma \in \Phi_C$ there is $\gamma^\sigma \in \Phi_C$ such that for all $H \in \mathfrak{t}$, $\gamma^\sigma(H) = \overline{\gamma(H)}$. The real dual of $\mathfrak{t}$ will be denoted by $\mathfrak{t}^*$, the complex dual by $\mathfrak{t}_{\mathbf{C}}^*$. We will identify elements of $\mathfrak{t}_{\mathbf{C}}$ and $\mathfrak{t}_{\mathbf{C}}^*$ via the bilinear form B. $W = W(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$ denotes the Weyl group corresponding to Φ. For any $\beta \in \Phi$, $s_\beta \in W$ denotes the reflection corresponding to β and ξ_β the character of T corresponding to β.

For the convenience of the reader we will review some definitions and lemmas of Arthur. The reader is referred to [1c] for details. Let A be a special vector subgroup of G, that is a split component of a parabolic subgroup of G as defined in [1c, §1] and $\mathscr{Y}$ an A-orthogonal set. Corresponding to $\mathscr{Y}$, Arthur defines a weight function $v(x : \mathscr{Y})$, $x \in G$, which is left-invariant by $C_G(A)$, the centralizer in G of A. Let $\mathscr{G}$ denote the universal envelopping algebra of $\mathfrak{g}_{\mathbf{C}}$, and for any $X \in \mathscr{G}$, let $c_0(X)$ denote the constant term of X. Let $\mathscr{G}_A$ be the set of elements in $\mathscr{G}$ invariant under the adjoint action of A. For $X \in \mathscr{G}_A$ we will write D_X for the right-invariant differential operator associated to X.

Let T be a Cartan subgroup of G with $A \subseteq T$. Then for all $h \in T'$, $f \in C_c^\infty(G)$, and $X \in \mathscr{G}_A$, Arthur defines

$$\langle r(h:\mathscr{Y}:X), f\rangle = \int_{T_R\backslash G} f(x^{-1}hx) D_X v(x:\mathscr{Y})\, d\dot{x} \tag{2.1}$$

where $d\dot{x}$ is a G-invariant measure on the quotient.

Let $\mathfrak{z}$ denote the centralizer of $\mathfrak{t}_R$ in $\mathfrak{g}$ and $Z(\mathfrak{t})$ the centralizer of $\mathfrak{z}$ in K. Let Φ_I^+ be a set of positive roots for $(\mathfrak{z}, \mathfrak{t})$. Let:

$$\Delta_+(h) = |\det(1 - Ad(h^{-1}))_{\mathfrak{g}/\mathfrak{z}}|^{1/2}, \qquad h \in T;$$

$$\Delta_I(H) = \prod_{\beta\in\Phi_I^+} (e^{\beta(H)/2} - e^{-\beta(H)/2}), \qquad H \in \mathfrak{t};$$

$$\tilde{\Delta}(\zeta, H) = \Delta_I(H)\Delta_+(\zeta \exp H), \qquad \zeta \in Z(\mathfrak{t}), H \in \mathfrak{t}.$$

For $\zeta \in Z(\mathfrak{t})$, $H \in \mathfrak{t}'(\zeta) = \{H \in \mathfrak{t} : \zeta \exp H \in T'\}$, and $f \in C_c^\infty(G)$, define

$$\begin{aligned} R_f^{T,A}(\zeta, H:\mathscr{Y}:X) = \langle R(\zeta, H:\mathscr{Y}:X), f\rangle = \\ = \tilde{\Delta}(\zeta, H)\langle r(\zeta \exp H:\mathscr{Y}:X), f\rangle. \end{aligned} \tag{2.2}$$

LEMMA 2.3 (Arthur): *Let $\zeta \in Z(\mathfrak{t})$. For each $H \in \mathfrak{t}'(\zeta)$ the distribution $R(\zeta, H:\mathscr{Y}:X)$ is tempered. For every $f \in \mathscr{C}(G)$, the function $R_f(\zeta, H:\mathscr{Y}:X)$ is infinitely differentiable for $H \in \mathfrak{t}'(\zeta)$.*

Let $\mathscr{Z}$ denote the center of $\mathscr{G}$, $S(\mathfrak{t}_{\mathbf{C}})$ the symmetric algebra on $\mathfrak{t}_{\mathbf{C}}$, $I(\mathfrak{t}_{\mathbf{C}})$ the set of Weyl group invariants in $S(\mathfrak{t}_{\mathbf{C}})$, and $\gamma = \gamma_{\mathfrak{g}/\mathfrak{t}}$ the Harish–Chandra isomorphism from $\mathscr{Z}$ onto $I(\mathfrak{t}_{\mathbf{C}})$. Arthur defines ideals $\mathscr{G}_A(0) \subseteq \mathscr{G}_A(1) \subseteq \ldots$ of $\mathscr{G}_A$ so that $\bigcup_{r\geq 0} \mathscr{G}_A(r) = \mathscr{G}_A$, $\mathscr{G}_A(r)\mathscr{G}_A(r') \subseteq \mathscr{G}_A(r + r')$, $r, r' \geq 0$, $D_X v(x:\mathscr{Y}) = 0$ if $X \in \mathscr{G}_A(p+1)$, $p = \dim A$, and $c_0(X) = 0$ if $X \in \mathscr{G}_A(1)$.

LEMMA 2.4 (Arthur): *For any $z \in \mathscr{Z}$ there are elements $\{X_i : 1 \leq i \leq r\}$ in $\mathscr{G}_A(1)$ and differential operators $\{\partial_i : 1 \leq i \leq r\}$ on $\mathfrak{t}'(\zeta)$ so that for every $\zeta \in Z(\mathfrak{t})$, $H \in \mathfrak{t}'(\zeta)$, $X \in \mathscr{G}_A$, and $f \in \mathscr{C}(G)$,*

$$R_{zf}(\zeta, H:\mathscr{Y}:X) - R_f(\zeta, H; \partial\gamma(z):\mathscr{Y}:X) = \sum_{i=1}^{r} R_f(\zeta, H; \partial_i:\mathscr{Y}:XX_i).$$

Fix $\zeta \in Z(\mathfrak{t})$ and $\beta \in \Phi_R(\zeta) = \{\beta \in \Phi_R : \xi_\beta(\zeta) = 1\}$. Let $\mathfrak{t}_\beta^0 = \{H \in \mathfrak{t} : \beta(H) = 0\}$, $\mathfrak{a}_\beta = \mathfrak{a} \cap \mathfrak{t}_\beta^0$, and $A_\beta = \exp(\mathfrak{a}_\beta)$. Let $H'_\beta \in \mathfrak{t}$ be dual to $2\beta/\langle\beta,\beta\rangle$ and let X'_β and $Y'_\beta \in \mathfrak{g}$ be root vectors for β satisfying $[X'_\beta, Y'_\beta] = H'_\beta$, $Y'_\beta = -\theta X'_\beta$. Then $\mathfrak{t}_\beta = \mathfrak{t}_\beta^0 + \mathbf{R}(X'_\beta - Y'_\beta)$ is a Cartan subalgebra of $\mathfrak{g}$ and we denote the corresponding Cartan subgroup by T_β. Let $\Lambda = \exp(-\pi i/4 \operatorname{ad}(X'_\beta + Y'_\beta))$ be the associated Cayley transform. Let $n_\beta(A)$ denote the cosine of the angle in $\mathfrak{t}_R$ between β and $\mathfrak{a}$. If $H \in \mathfrak{t}'(\zeta)$, set $\tau_\beta(H) = n_\beta(A)\|H'_\beta\| \log|e^{\beta(H)/2} - e^{-\beta(H)/2}|$, and define

$$S_f^\beta(\zeta, H : \mathscr{Y} : X) = R_f(\zeta, H : \mathscr{Y} : X) + \tau_\beta(H) R_f^{T, A_\beta}(\zeta, H : \mathscr{Y}_\beta : X) \tag{2.5}$$

where $\mathscr{Y}_\beta$ is an A_β-orthogonal set depending on $\mathscr{Y}$. Let $\mathfrak{t}_\beta^0(\zeta) = \{H \in \mathfrak{t}_\beta^0 : \xi_\alpha(\zeta \exp H) \neq 1$ for any $\alpha \in \Phi$, $\alpha \neq \pm\beta\}$. For $H_0 \in \mathfrak{t}_\beta^0(\zeta)$, write $S(H_0)^\pm = \lim_{t \to 0^\pm} S(H_0 + tH'_\beta)$.

LEMMA 2.6 (Arthur): *Let* $u \in S(\mathfrak{t}_C)$, $f \in \mathscr{C}(G)$. *Then for* $H_0 \in \mathfrak{t}_\beta^0(\zeta)$, $S_f^\beta(\zeta, H_0; \partial u : \mathscr{Y} : X)^+ - S_f^\beta(\zeta, H_0; \partial u : \mathscr{Y} : X)^- = n_\beta(A) \lim_{\theta \to 0} R_f^{T_\beta, A_\beta}(\zeta, H_0 + \theta(X'_\beta - Y'_\beta); \partial\Lambda(s_\beta u - u) : \mathscr{Y}_\beta : X)$ *where the limits all exist uniformly for* H_0 *in compacta of* $\mathfrak{t}_\beta^0(\zeta)$.

LEMMA 2.7 (Arthur): *Let* $u \in S(\mathfrak{t}_C)$, $f \in \mathscr{C}(G)$. *Then for* $H_0 \in \mathfrak{t}_\beta^0(\zeta)$,

$$\begin{aligned}
&\lim_{\theta \to 0^+} R_f^{T_\beta, A_\beta}(\zeta, H_0 + \theta(X'_\beta - Y'_\beta); \partial\Lambda u : \mathscr{Y}_\beta : X) - \\
&\lim_{\theta \to 0^-} R_f^{T_\beta, A_\beta}(\zeta, H_0 + \theta(X'_\beta - Y'_\beta); \partial\Lambda u : \mathscr{Y}_\beta : X) = \\
&-\pi i \|H'_\beta\| \lim_{t \to 0} R_f^{T, A_\beta}(\zeta, H_0 + tH'_\beta; \partial u : \mathscr{Y}_\beta : X)
\end{aligned}$$

where the limits all exist uniformly for H_0 *in compacta of* $\mathfrak{t}_\beta^0(\zeta)$.

Lemma 2.7 gives boundary conditions for R_f across any hyperplane determined by a singular imaginary root. It follows easily from the proof of 2.6 (Theorem 6.1 of [1c]) and facts about ordinary orbital integrals that if β is a compact root of $(\mathfrak{g}, \mathfrak{t})$ and $H_0 \in \mathfrak{t}$ satisfies $e^{\beta(H_0)} = 1$, $\xi_\alpha(\zeta \exp H_0) \neq 1$ for any $\alpha \in \Phi$, $\alpha \neq \pm\beta$, then $R_f(\zeta, H : \mathscr{Y} : X)$ extends to a smooth function around $H = H_0$.

For $H \in \mathfrak{t}'(\zeta)$, let $m(H) = \min\{|1 - \xi_\alpha(\zeta \exp H)^{-1}| : \alpha \in \Phi$ and $\alpha|_\mathfrak{a} \neq 0\}$. Let $L(\zeta \exp H) = |\log m(H)|$.

LEMMA 2.8: *Given any* $u \in S(\mathfrak{t}_C)$ *there is a continuous seminorm* ν *on*

$\mathscr{C}(G)$ so that for all $f \in \mathscr{C}(G)$ and $H \in \mathfrak{t}'(\zeta)$,

$$|R_f(\zeta, H; \partial u : \mathscr{Y} : X)| \leq \nu(f)(1 + L(\zeta \exp H))^p.$$

PROOF: In the case that $u = 1$, this result is a special case of Corollary 7.4 of [1c]. For general $u \in S(\mathfrak{t}_{\mathbf{C}})$ it can be obtained by using the argument of Arthur in Lemma 8.1 of [1c]. □

We now turn to results of Harish–Chandra on wave packets which can be found in [2c, §20]. Let $P = MA_1N$ be a cuspidal parabolic subgroup of G. Let $\varepsilon_2(M)$ denote the set of equivalence classes of irreducible unitary square-integrable representations of M. Let $\mathscr{C}_\omega(M)$ denote the closed subspace of $\mathscr{C}(M)$ spanned by K_M-finite matrix coefficients of ω where $K_M = K \cap M$. For any $\nu \in \mathscr{F} = \mathfrak{a}_1^*$, let $\pi_{\omega,\nu}$ be the tempered unitary representation of G induced from $\omega \otimes e^{i\nu} \otimes 1$ on P. Let $\theta_{\omega,\nu}$ and θ_ω denote the characters of $\pi_{\omega,\nu}$ and ω considered as functions on G' and M' respectively. For $f \in \mathscr{C}(G)$, write

$$\langle \theta_{\omega,\nu}, f \rangle = \int_G f(x)\overline{\theta_{\omega,\nu}(x)}\, dx.$$

For $\omega \in \varepsilon_2(M)$, $\psi \in \mathscr{C}_\omega(M)$, $\alpha \in C_c^\infty(\mathscr{F})$, and $x \in G$, define

$$\varphi_\alpha(x) = \int_{\mathscr{F}} \alpha(\nu) E(P : \psi : \nu : x)\mu(\omega : \nu)\, d\nu \tag{2.9}$$

where $E(P : \psi : \nu)$ is the Eisenstein integral defined in [2b] and $\mu(\omega : \nu)\, d\nu$ is the Plancherel measure corresponding to $\pi_{\omega,\nu}$, $\nu \in \mathscr{F}$. Then $\varphi_\alpha \in \mathscr{C}(G)$ is called a wave packet for $\omega \in \varepsilon_2(M)$, and $\alpha \to \varphi_\alpha$ is a continuous mapping from $C_c^\infty(\mathscr{F})$ into $\mathscr{C}(G)$. Extend $\mathfrak{a}_1$ to a Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_I + \mathfrak{a}_1$ of $\mathfrak{g}$ with $\mathfrak{h}_I \subseteq \mathfrak{m}$. Let $\lambda \in i\mathfrak{h}_I^*$ correspond to the infinitesimal character of ω. For $q \in I(\mathfrak{h}_{\mathbf{C}})$, let $p(q)$ be the polynomial function on $\mathscr{F}$ given by $p(q : \nu) = q(\lambda + i\nu)$, $\nu \in \mathscr{F}$. Then if $q \in I(\mathfrak{h}_{\mathbf{C}})$ and $z = \gamma_{\mathfrak{g}/\mathfrak{h}}^{-1}(q) \in \mathscr{Z}$, then

$$z\varphi_\alpha = \varphi_{p(q)\alpha}. \tag{2.10}$$

Finally, for ω and ψ fixed as above, there is a constant c so that for all $\alpha \in C_c^\infty(\mathscr{F})$, $\nu \in \mathscr{F}$,

$$\langle \theta_{\omega,\nu}, \varphi_\alpha \rangle = c \sum_{s \in W(\omega)} \alpha(s\nu). \tag{2.11}$$

§3. Weighted characters

Let A, B be subspaces of a Euclidean vector space with $\dim A = \dim B = m$. Let $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_m\}$ be orthonormal bases of B and A respectively. Define $c(B, A) = |\det X|$ where X is the $m \times m$ matrix with entries $x_{ij} = \langle v_i, w_j \rangle$. Then $c(B, A) = c(A, B)$ is independent of the choices of orthonormal bases and is equal to the volume of a unit cube in B projected onto A.

For any vector $v \neq 0$, let $n_v(A)$ be the cosine of the angle between v and A and A_v, B_v be the subspaces of A, B respectively which are orthogonal to v.

LEMMA 3.1: *If $v \in B$, $v \neq 0$, then $c(B, A) = n_v(A)c(B_v, A_v)$.*

PROOF: Pick an orthonormal basis for B so that $v_1 = v/\|v\|$. Let v_A be the projection of v onto A. If $v_A = 0$, then $n_v(A) = 0$ and $\langle v_1, w \rangle = 0$ for all $w \in A$ so that $c(A, B) = 0$. Assume $v_A \neq 0$. Choose an orthonormal basis for A with $w_1 = v_A/\|v_A\|$. Then $\langle v_1, w_1 \rangle = n_v(A)$ and $\langle v_1, w_j \rangle = 0$ for $j \geq 2$. Thus $c(A, B) = n_v(A)|\det X^*|$ where $X^* = (\langle v_i, w_j \rangle)$, $2 \leq i$, $j \leq m$ is the matrix corresponding to A_v and B_v. □

We will now use the constants $c(B, A)$ to define the weighted characters which appears in (1.4).

Let $P = MA_1N$ be a cuspidal parabolic subgroup of G. Write $L = MA_1$. Let $\omega \in \varepsilon_2(M)$ and $\nu \in \mathscr{F} = \mathfrak{a}_1^*$. For T a Cartan subgroup of G, let $H_1, \ldots, H_k$ be a complete set of representatives for distinct L-conjugacy classes of Cartan subgroups of L for which $H_i = x_i T x_i^{-1}$, $x_i \in G$, $1 \leq i \leq k$. For $h \in T'$, write $h_i = x_i h x_i^{-1} \in H_i'$. Then

$$\theta_{\omega,\nu}(h) = \Delta_+^G(h)^{-1} \sum_{i=1}^{k} \sum_{w \in W(L, H_i)\backslash W(G, H_i)} \Delta_+^L(wh_i)(\theta_\omega \otimes e^{i\nu})(wh_i) \qquad (3.2)$$

where Δ_+^G, Δ_+^L are the functions Δ_+ defined on T and H_i, $1 \leq i \leq k$, in §2 when T and H_i are considered as Cartan subgroups of G and L respectively. Note that if no conjugate of T lies in L, then $\theta_{\omega,\nu} = 0$ on T'.

Now let A be a special vector subgroup of G with $A \subseteq T_R$ and $\dim T_R = \dim A + \dim A_1$. For $h \in T'$ define $x_1, \ldots, x_k$ and $h_1, \ldots, h_k$ as above. For $1 \leq i \leq k$ and $w \in W(G, H_i)$, $\operatorname{ad} x_i^{-1} w^{-1} A_1 \subseteq T_R$ and is independent of the representative of the coset $W(L, H_i)w$ chosen. Define B_w to be the orthogonal complement in T_R of $\operatorname{ad} x_i^{-1} w^{-1} A_1$. (We consider T_R as a Euclidean vector space via the exponential map isomorphism with $\mathfrak{t}_R$.)

Then B_w is a subspace of T_R with $\dim B_w = \dim A$. Define

$$\theta^A_{\omega,\nu}(h) = \Delta^G_+(h)^{-1} \sum_{i=1}^{k} \sum_{w \in W(L,H_i)\backslash W(G,H_i)} c(B_w, A)\Delta^L_+(wh_i)(\theta_\omega \otimes e^{i\nu})(wh_i). \tag{3.3}$$

Note that $\theta^A_{\omega,\nu}$ is not an invariant function on G'. It is easy to check from the definition that in fact, if $h \in T'$ and $y \in G$, then

$$\theta^A_{\omega,\nu}(h) = \theta^{yAy^{-1}}_{\omega,\nu}(yhy^{-1}). \tag{3.4}$$

For $\zeta \in Z(\mathfrak{t})$ and $H \in \mathfrak{t}'(\zeta)$, define

$$\tilde{\Phi}_{\omega,\nu}(\zeta, H) = \tilde{\Delta}(\zeta, H)\theta_{\omega,\nu}(\zeta \exp H) \text{ and}$$
$$\tilde{\Phi}^A_{\omega,\nu}(\zeta, H) = \tilde{\Delta}(\zeta, H)\theta^A_{\omega,\nu}(\zeta \exp H).$$

Assume for simplicity that $T \subseteq L$. (Because of (3.4) this leads to no loss of generality.) Let $\mathfrak{t}_M = \mathfrak{t} \cap \mathfrak{m}$ and write $\tilde{\Phi}_\omega(\zeta, H) = \Delta_I(H)\Delta^L_+(\zeta \exp H)\theta_\omega(\zeta \exp H)$ for $H \in \mathfrak{t}'_M(\zeta)$. Let $\lambda \in \mathfrak{t}^*_{M,\mathbf{C}}$ correspond to the infinitesimal character of ω. That is, λ is a regular element of $\mathfrak{t}^*_{M,\mathbf{C}}$ so that $\tilde{\Phi}_\omega(\zeta, H; \partial q) = q(\lambda)\tilde{\Phi}_\omega(\zeta, H)$ for all $q \in I(\mathfrak{t}_{M,\mathbf{C}})$ and $H \in \mathfrak{t}'_M(\zeta)$. Fix $\zeta \in Z(\mathfrak{t})$ and let $\Omega(\zeta) = \{H \in \mathfrak{t} : \beta(H) \neq 0 \text{ for all } \beta \in \Phi_R(\zeta)\}$.

Lemma 3.5: *For any connected component F of $\Omega(\zeta)$ there are constants $c_s(F)$, $s \in W = W(\mathfrak{g}_\mathbf{C}, \mathfrak{t}_\mathbf{C})$, so that for all $H \in F$,*

$$\tilde{\Phi}_{\omega,\nu}(\zeta, H) = \sum_{s \in W} c_s(F)\exp(s(\lambda + i\nu)(H))$$

and

$$\tilde{\Phi}^A_{\omega,\nu}(\zeta, H) = \sum_{s \in W} c_s(F)c(B_s, A)\exp(s(\lambda + i\nu)(H)).$$

Here $c_s(F) = 0$ unless $sA_1 \subseteq T_R$, and in this case B_s is the orthogonal complement in T_R of sA_1. Further, if $\beta \in \Phi_R(\zeta)$ and F, $s_\beta F$ are adjacent chambers of $\Omega(\zeta)$, then $c_s(F) = c_s(s_\beta F)$ unless $\beta|_{s\mathfrak{a}_1} = 0$.

Proof: Fix $H \in F$ and let $h = \zeta \exp H$. Define $x_1, \ldots, x_k$ as in (3.2). Let W_i be a set of representatives for the cosets $W(L, H_i)\backslash W(G, H_i)$, $1 \leq i \leq k$. Then using (3.2) and (3.3),

$$\tilde{\Phi}_{\omega,\nu}(\zeta, H) = \sum_{i=1}^{k} \sum_{w \in W_i} \varepsilon(w)(\tilde{\Phi}_\omega \otimes e^{i\nu})(w\zeta_i, wH_i)$$

and

$$\tilde{\Phi}^A_{\omega,\nu}(\zeta, H) = \sum_{i=1}^{k} \sum_{w\in W_i} \varepsilon(w)c(B_w, A)(\tilde{\Phi}_\omega \otimes e^{i\nu})(w\zeta_i, wH_i)$$

where $\zeta_i = x_i\zeta x_i^{-1}$, $H_i = \operatorname{ad} x_i(H)$, $1 \le i \le k$, and $\varepsilon(w) = \Delta_I(H)^{-1}\Delta_I(wH_i) = \pm 1$ depends on the choices of positive systems of imaginary roots which have been made, but not on H. Fix $1 \le i \le k$, and write $\mathfrak{h}$ for the Lie algebra of H_i. Write $W_M = W(\mathfrak{m}_{\mathbf{C}}, \mathfrak{t}_{M,\mathbf{C}})$. Then there is $y \in M_{\mathbf{C}}$ so that $\operatorname{Ad} y(\mathfrak{t}_{\mathbf{C}}) = \mathfrak{h}_{\mathbf{C}}$. Using the theory of characters on M, for every $w \in W_i$ and $\sigma \in W_M$ there are uniquely determined constants $c_\sigma(i, w)$ depending only on the component of $\mathfrak{t}'(i, w) = \{H \in \mathfrak{t} : \beta(H) \neq 0$ for all $\beta \in \Phi_R(\zeta)$ such that $\beta | \operatorname{Ad} x_i^{-1} w^{-1} \mathfrak{a}_1 \neq 0\}$ containing F so that

$$(\tilde{\Phi}_\omega \otimes e^{i\nu})(w\zeta_i, wH_i) = \sum_{\sigma\in W_M} c_\sigma(i, w) \exp(s(i, w)\sigma(\lambda + i\nu)(H))$$

where $s(i, w) \in W$ represents the action of $\operatorname{Ad} x_i^{-1} w^{-1} \operatorname{Ad} y$ on $\mathfrak{t}_{\mathbf{C}}$ and so satisfies $s(i, w)A_1 = \operatorname{Ad} x_i^{-1} w^{-1} A_1 \subseteq T_R$. Note that for all $\sigma \in W_M$, $s(i, w)\sigma A_1 = s(i, w)A_1$, so that $B_w = B_{s(i,w)\sigma}$ for all $\sigma \in W_M$. Further, one can check that any $s \in W$ can be written in at most one way as $s = s(i, w)\sigma$, $1 \le i \le k$, $w \in W_i$, $\sigma \in W_M$. Thus we can write $\tilde{\Phi}_{\omega,\nu}$ and $\tilde{\Phi}^A_{\omega,\nu}$ as claimed in the lemma where $c_s(F) = 0$ if s is not of the form $s = s(i, w)\sigma$ for some $1 \le i \le k$, $w \in W_i$, and $\sigma \in W_M$, and if $s = s(i, w)\sigma$, then $c_s(F) = \varepsilon(w)c_\sigma(i, w)$. If $\beta \in \Phi_R(\zeta)$, F, $s_\beta F$ are adjacent chambers of $\Omega(\zeta)$, and $\beta|_{s\mathfrak{a}_1} \neq 0$, $s = s(i, w)\sigma$, then F and $s_\beta F$ lie in the same component of $\mathfrak{t}'(i, w)$ so that $c_s(F) = c_s(s_\beta F)$. □

An immediate consequence of (3.5) is

$$\tilde{\Phi}^A_{\omega,\nu}(\zeta, H; \partial q) = q(\lambda + i\nu)\tilde{\Phi}^A_{\omega,\nu}(\zeta, H) \text{ for all } H \in \mathfrak{t}'(\zeta),\ q \in I(\mathfrak{t}_{\mathbf{C}}). \tag{3.6}$$

LEMMA 3.7: *Given any $u \in S(\mathfrak{t}_{\mathbf{C}})$, there exist constants c, r so that $|\tilde{\Phi}^A_{\omega,\nu}(\zeta, H; \partial u)| \le c(1 + \|H_R\|)^r$ for all $H = H_I + H_R \in \mathfrak{t}'(\zeta)$.*

PROOF: By results of Harish-Chandra [2b], such an estimate is valid for $\tilde{\Phi}_{\omega,\nu}(\zeta, H)$. This implies that $c_s(F) = 0$ for any $s \in W$ for which $\operatorname{Re}(s(\lambda + i\nu))(H) > 0$ for any $H \in F$. Using (3.5), then, the estimate holds for $\tilde{\Phi}^A_{\omega,\nu}(\zeta, H)$. □

Now fix $\beta \in \Phi_R(\zeta)$, and use the notation of (2.6). Let Ω_0 be a relatively compact open subset of $\mathfrak{t}^0_\beta(\zeta)$.

LEMMA 3.8: *For all* $u \in S(\mathfrak{t}_{\mathbf{C}})$, $H_0 \in \Omega_0$,

$$\tilde{\Phi}^A_{\omega,\nu}(\zeta, H_0; \partial u)^+ - \tilde{\Phi}^A_{\omega,\nu}(\zeta, H_0; \partial u)^- =$$
$$= n_\beta(A) \lim_{\theta \to 0} \tilde{\Phi}^{A_\beta}_{\omega,\nu}(\zeta, H_0 + \theta(X'_\beta - Y'_\beta); \partial A(u - s_\beta u))$$

where the limits exist uniformly for $H_0 \in \Omega_0$.

PROOF: Let F and $s_\beta F$ be components of $\Omega(\zeta)$ with $H_0 \in \bar{F} \cap s_\beta \bar{F}$ and $H_0 + tH'_\beta \in F$ for $t > 0$ and sufficiently small. Write $c_s^+ = c_s(F)$, $c_s^- = c_s(s_\beta F)$, $s \in W$. Using (3.5),

$$\tilde{\Phi}^A_{\omega,\nu}(\zeta, H_0; \partial u)^\pm = \sum_{s \in W_0} [c_s^\pm c(B_s, A) u(s(\lambda + i\nu)) +$$
$$+ c^\pm_{s_\beta s} c(B_{s_\beta s}, A) s_\beta u(s(\lambda + i\nu))] \exp(s(\lambda + i\nu)(H_0))$$

where W_0 is a set of coset representatives for $\{I, s_\beta\}\backslash W$. If $\beta|_{\mathfrak{a}_1} = 0$ so that $T_\beta \subseteq L$, we can use (3.5) directly to obtain a similar expression for $\tilde{\Phi}^{A_\beta}_{\omega,\nu}$ on T'_β. However, for the general case we must combine (3.5) and (3.4) to see that there are constants d_s, $s \in W$, such that for all H in an open subset of $\mathfrak{t}_\beta$ containing H_0, $\tilde{\Phi}^{A_\beta}_{\omega,\nu}(\zeta, H) = \sum_{s \in W} d_s c(\tilde{B}_s, A_\beta) \exp(A s(\lambda + i\nu)(H))$ where $d_s = 0$ unless $sA_1 \subseteq (T_\beta)_R$ and in that case $\tilde{B}_s$ is the orthogonal complement in $(T_\beta)_R$ of sA_1. (If no conjugate of T_β is contained in L, then of course $d_s = 0$ for all $s \in W$.) From work of Hirai [4] and the observations of Arthur in [1c, Thm. 9.1] it is known that $\tilde{\Phi}_{\omega,\nu}$ is continuous at H_0 and that

$$\tilde{\Phi}_{\omega,\nu}(\zeta, H_0; \partial H'_\beta)^+ - \tilde{\Phi}_{\omega,\nu}(\zeta, H_0; \partial H'_\beta)^- =$$
$$= 2 \lim_{\theta \to 0} \tilde{\Phi}_{\omega,\nu}(\zeta, H_0 + \theta(X'_\beta - Y'_\beta); \partial A H'_\beta).$$

Thus we see that for all $s \in W_0$,

(i) $c_s^+ + c^+_{s_\beta s} - c_s^- - c^-_{s_\beta s} = 0$ and that

(ii) $c_s^+ - c^+_{s_\beta s} - c_s^- + c^-_{s_\beta s} = 2(d_s - d_{s_\beta s})$.

By considering separately the cases that $s_\beta u = u$ and $s_\beta u = -u$, $u \in S(\mathfrak{t}_{\mathbf{C}})$, it will be enough to prove that for all $s \in W_0$

(iii) $(c_s^+ - c_s^-) c(B_s, A) + (c^+_{s_\beta s} - c^-_{s_\beta s}) c(B_{s_\beta s}, A) = 0$ and that

(iv) $(c_s^+ - c_s^-) c(B_s, A) - (c^+_{s_\beta s} - c^-_{s_\beta s}) c(B_{s_\beta s}, A) =$
$= 2n_\beta(A)(d_s - d_{s_\beta s}) c(\tilde{B}_s, A_\beta)$.

Suppose first that $sA_1 \not\subseteq T_R$. Then also $s_\beta s A_1 \not\subseteq T_R$ as $s_\beta T_R = T_R$, so that $c_s^\pm = c^\pm_{s_\beta s} = d_s = d_{s_\beta s} = 0$. If $sA_1 \subseteq T_R$ and $\beta|_{s\mathfrak{a}_1} \neq 0$, then

$sA_1 \not\subseteq (T_\beta)_R$, and again the same is true of $s_\beta s$, so that $c_s^+ = c_s^-$, $c_{s_\beta s}^+ = c_{s_\beta s}^-$, and $d_s = d_{s_\beta s} = 0$. Finally, suppose that $sA_1 \subseteq T_R$ and $\beta|_{sa_1} = 0$. Then $s_\beta s A_1 = sA_1$ so that $B_s = B_{s_\beta s}$. Also $\exp(H'_\beta) \in B_s$ so that using (3.1), $c(B_s, A) = n_\beta(A) c((B_s)_\beta, A_\beta)$. But since $\beta|_{sa_1} = 0$, $(B_s)_\beta = \tilde{B}_s$. Thus (iii) and (iv) are satisfied in all cases. □

§4. Distribution-valued functions on $\mathfrak{t}$

In [1c], to prove (1.2) Arthur shows that for a fixed matrix coefficient f of the discrete series representation π and for fixed $\zeta \in Z(\mathfrak{t})$,

$$\psi(H) = \tilde{\Delta}(\zeta, H)[r_f^A(\zeta \exp H) - \varepsilon(T, A)(-1)^p \langle \theta_\pi, f \rangle \theta_\pi(\zeta \exp H)]$$

is a smooth function of $H \in \mathfrak{t}'(\zeta)$ which is an eigenfunction of ∂q for all $q \in I(\mathfrak{t}_{\mathbf{C}})$, extends to a continuously differentiable function in a neighborhood of any $H_0 \in \mathfrak{t}_\beta^0(\zeta)$, $\beta \in \Phi_R(\zeta)$, and is of moderate growth. Then using techniques of Harish–Chandra he shows that any such function must be zero.

In our situation, in order to obtain a differential equation for ∂q, $q \in I(\mathfrak{t}_{\mathbf{C}})$, we must consider functions ψ not only of $H \in \mathfrak{t}'(\zeta)$, but also of the $\alpha \in C_c^\infty(\mathscr{F})$ which are used to form the wave packets $f = \varphi_\alpha$. In this section we will give sufficient conditions on such a function $\psi(H:\alpha)$ to guarantee that $\psi = 0$. Then in §5 we will prove (1.4) by showing that

$$\psi(H:\alpha) = \tilde{\Delta}(\zeta, H)\Bigg[r_{\varphi_\alpha}^A(\zeta \exp H) - \varepsilon(T, A, A_1)(-1)^p [W(\omega)]^{-1} \int_{\mathscr{F}} \langle \theta_{\omega,\nu}, \varphi_\alpha \rangle \theta_{\omega,\nu}^A(\zeta \exp H)\, d\nu \Bigg]$$

satisfies these conditions.

For simplicity we assume that $T \subseteq L$. We also assume that T is not a fundamental Cartan subgroup of L. Fix $\zeta \in Z(\mathfrak{t})$ and $\lambda \in \mathfrak{t}_{M,\mathbf{C}}^*$ corresponding to the infinitesimal character of some $\omega \in \varepsilon_2(M)$. Let V be a subspace of $\mathscr{F}$ and $U \subseteq V$ any open subset of V. For $q \in I(\mathfrak{t}_{\mathbf{C}})$, let $p(q)$ be the polynomial on U given by $p(q:\nu) = q(\lambda + i\nu)$, $\nu \in U$. For $v \in S(\mathfrak{t}_{\mathbf{C}})$ and $s \in W$, let $p_s(v)$ be the polynomial on U given by $p_s(v:\nu) = v(s(\lambda + i\nu))$, $\nu \in U$. Let $\rho_I = \frac{1}{2}\sum_{\beta \in \Phi_I^+} \beta$.

Define $E(U)$ to be the complex vector space consisting of all functions ψ on $\mathfrak{t}'(\zeta) \times C_c^\infty(U)$ satisfying:

(4.1) for each $\alpha \in C_c^\infty(U)$, $\psi(H:\alpha) = e^{\rho_I(H)} f_\alpha(\zeta \exp H)$ where f_α is a smooth function on T';

(4.2) for each $q \in I(\mathfrak{t}_{\mathbf{C}})$. $\psi(H; \partial q : \alpha) = \psi(H : p(q)\alpha)$ for all $H \in \mathfrak{t}'(\zeta)$, $\alpha \in C_c^\infty(U)$;

(4.3) for each $\beta \in \Phi_R(\zeta)$, $H_0 \in \mathfrak{t}_\beta^0(\zeta)$, $\psi(H : \alpha)$ extends to a smooth function in a neighborhood of H_0 for all $\alpha \in C_c^\infty(U)$;

(4.4) for all $\alpha \in C_c^\infty(U)$, $\psi(H : \alpha)$ extends to a C^∞ function on $\Omega_I = \mathfrak{t}_I + \{H \in \mathfrak{t}_R : \beta(H) \neq 0$ for any $\beta \in \Phi$ with $\beta|_{\mathfrak{t}_R} \neq 0\}$;

(4.5) for each fixed $H \in \mathfrak{t}'(\zeta)$, $u \in S(\mathfrak{t}_{\mathbf{C}})$, $\alpha \mapsto \psi(H; \partial u : \alpha)$ defines a distribution on U. Further, for any $u \in S(\mathfrak{t}_{\mathbf{C}})$ there is a continuous seminorm μ on $C_c^\infty(U)$ and a constant r so that

$|\psi(H; \partial u : \alpha)| \leq \mu(\alpha)(1 + L(\zeta \exp H))^p(1 + \|H_R\|)^r$ for all $H = H_I + H_R \in \mathfrak{t}'(\zeta)$, $\alpha \in C_c^\infty(U)$.

We are of course primarily interested in showing that $E(\mathscr{F}) = \{0\}$. However, in order to do this, it is necessary to use the various spaces $E(U)$ defined above. Note that when $U \subseteq U'$ are open subsets of V, then for $\psi \in E(U')$, the restriction of ψ to $\mathfrak{t}'(\zeta) \times C_c^\infty(U)$ is an element of $E(U)$. For S a subalgebra of $S(\mathfrak{t}_{\mathbf{C}})$, $s \in W$, and $U \subseteq V \subseteq \mathscr{F}$ as above, define $E(U : s : S) = \{\psi \in E(U) : \psi(H; \partial v : \alpha) = \psi(H : p_s(v)\alpha)$ for all $v \in S$, $H \in \mathfrak{t}'(\zeta)$, $\alpha \in C_c^\infty(U)\}$. When $S = S(\mathfrak{t}_{\mathbf{C}})$, we write $E(U : s : S(\mathfrak{t}_{\mathbf{C}})) = E(U : s)$.

LEMMA 4.6: *Let U be an open subset of a subspace V of $\mathscr{F}$. Let $\psi \in E(U : s)$ for some $s \in W$ and let Ω be a convex open subset of $\mathfrak{t}$ on which ψ extends to a smooth function and such that $L(\zeta \exp H)$ is bounded on compact subsets of Ω. Then there is a fixed distribution T on U so that $\psi(H : \alpha) = T(\exp(p_s(H))\alpha)$ for all $H \in \Omega$, $\alpha \in C_c^\infty(U)$.*

PROOF: Let P be a fixed point in Ω and fix $\alpha \in C_c^\infty(U)$. For any $H_0 \in \Omega$, let $H = H_0 - P$. Then since Ω is convex, $P + tH \in \Omega$, $0 \leq t \leq 1$, and using Taylor's theorem, for any $q > 0$ there is a $0 < \tau < 1$ with

$$\psi(H_0 : \alpha) = \sum_{r=0}^{q-1} \frac{\psi(P; \partial H^r : \alpha)}{r!} + \frac{\psi(P + \tau H; \partial H^q : \alpha)}{q!}$$
$$= \psi\left(P : \left(\sum_{r=0}^{q-1} \frac{p_s(H)^r}{r!}\right)\alpha\right) + \psi(P + \tau H : p_s(H)^q/q!\alpha).$$

Using (4.5) and the assumption that $L(\zeta\, exp\, H)$ is bounded on compact subsets of Ω, there is a constant C so that for all $q \geq 0$, $|\psi(P + \tau H : p_s(H)^q\alpha)|/q! \leq C\mu(\alpha p_s(H)^q/q!)$ where μ is a continuous seminorm on $C_c^\infty(U)$. But as q goes to ∞, $\alpha p_s(H)^q/q!$ converges to zero and $\alpha \sum_{r=0}^{q-1} p_s(H)^r/r!$ converges to $\alpha \exp p_s(H)$ in $C_c^\infty(U)$ so that $\psi(H_0 : \alpha)$

$= \psi(P:\exp(p_s(H))\alpha)$. Thus if we let $T(\alpha) = \psi(P:\exp(-p_s(P))\alpha)$, we see that $\psi(H_0:\alpha) = T(\exp(p_s(H_0))\alpha)$. $\square$

As before, U is an open subset of a subspace V of $\mathscr{F}$. Via duality, we think of V also as a subspace of $\mathfrak{t}_R$.

LEMMA 4.7: *Let $s \in W$ with $sV \subseteq \mathfrak{t}_R$. Then $E(U:s) = \{0\}$.*

PROOF: Since $\lambda \in \mathfrak{t}^*_{M,\mathbf{C}}$ corresponds to the infinitesimal character of some $\omega \in \varepsilon_2(M)$, we know that $\langle \lambda, \beta \rangle \neq 0$ for every $\beta \in \Phi_M = \{\beta \in \Phi : \beta|_{\mathfrak{a}_1} = 0\}$. Suppose that $s\lambda|_{\mathfrak{a}_1} = 0$. Then using [3a] $s\lambda$ is also regular with respect to Φ_M. We have assumed that T is not a fundamental Cartan subgroup of L and that P is a cuspidal parabolic subgroup of G. Thus Φ_M contains real roots so that $s\lambda|_{\mathfrak{t}_{M,R}} \neq 0$. Since $\mathfrak{t}_R = \mathfrak{a}_1 \oplus \mathfrak{t}_{M,R}$, we see in any case that $s\lambda|_{\mathfrak{t}_R} \neq 0$. But considered as an element of $\mathfrak{t}^*_{\mathbf{C}}$, $\lambda \in i\mathfrak{t}^*_I + \mathfrak{t}^*_R = \{\mu \in \mathfrak{t}^*_{\mathbf{C}} : \mu|_{\mathfrak{t}_I}$ takes pure imaginary values and $\mu|_{\mathfrak{t}_R}$ takes real values$\}$. Since this real subspace of $\mathfrak{t}^*_{\mathbf{C}}$ is stable under W, $s\lambda$ takes real values on $\mathfrak{t}_R$.

Let $\mathfrak{t}_2$ be the orthogonal complement of sV in $\mathfrak{t}_R$. For all $H \in \mathfrak{t}_2$, $p_s(H:v) = s(\lambda + iv)(H) = s\lambda(H)$ is independent of $v \in V$. Since $s\lambda|_{\mathfrak{t}_R} \neq 0$ and $s\lambda|_{sV} = 0$, we can choose $H_2 \in \mathfrak{t}_2$ with $s\lambda(H_2) \neq 0$ and $\beta(H_2) \neq 0$ for every $\beta \in \Phi_R(\zeta)$ for which $\beta|_{\mathfrak{t}_2} \neq 0$. Let $\mathfrak{t}_1 = \{H \in \mathfrak{t} : \langle H, H_2 \rangle = 0\}$. Fix $H_1 \in \mathfrak{t}_1$ so that for $t \in \mathbf{R}$, $H_1 + tH_2 \in \mathfrak{t}'(\zeta)$ for all but finitely many values of t, and so that $H_1 + tH_2 \in \bigcup_{\beta \in \Phi_R(\zeta)} \mathfrak{t}^0_\beta(\zeta)$ whenever $H_1 + tH_2 \notin \mathfrak{t}'(\zeta)$.

Fix $\beta \in \Phi_R(\zeta)$ and $t_0 \in \mathbf{R}$ with $H_0 = H_1 + t_0 H_2 \in \mathfrak{t}^0_\beta(\zeta)$. Let $\psi \in E(U:s)$. By (4.6) there are distributions $T^\pm$ so that for $t > 0$ and small enough that $H_0 \pm tH_2 \in \mathfrak{t}'(\zeta)$; $\psi(H_0 \pm tH_2 : \alpha) = T^\pm(\alpha \exp p_s(H_0 \pm tH_2)) = c^\pm(\alpha)\exp(t_0 \pm t)(s\lambda(H_2))$ where for all $\alpha \in C^\infty_c(U)$, $c^\pm(\alpha) = T^\pm(\alpha \exp p_s(H_1))$. By the continuity of ψ at H_0, $c^+(\alpha) = c^-(\alpha)$ for all $\alpha \in C^\infty_c(U)$.

Since we can do this for any value of t_0 with $H_1 + t_0 H_2 \notin \mathfrak{t}'(\zeta)$, we see that for any $\alpha \in C^\infty_c(U)$ there is a constant $c(\alpha)$ so that $\psi(H_1 + tH_2 : \alpha) = c(\alpha) e^{ts\lambda(H_2)}$ for all $t \in \mathbf{R}$. But since $s\lambda(H_2)$ is real-valued and non-zero, this contradicts the growth condition on ψ unless $c(\alpha) = 0$. For fixed H_2 as above, the set of points $H_1 + tH_2$ with $t \in \mathbf{R}$ and $H_1 \in \mathfrak{t}_1$ satisfying the above hypotheses is dense in $\mathfrak{t}'(\zeta)$. Thus $\psi(H:\alpha) = 0$ for all $H \in \mathfrak{t}'(\zeta)$, $\alpha \in C^\infty_c(U)$. $\square$

For V a subspace of $\mathscr{F}$, let $\Phi_V = \{\beta \in \Phi : \langle \beta, v \rangle = 0$ for all $v \in V\}$. Let $V' = \{v \in V : \langle \beta, v \rangle \neq 0$ for all $\beta \in \Phi \backslash \Phi_V\}$.

LEMMA 4.8: *Let $s \in W$ with $sV \nsubseteq \mathfrak{t}_R$. Then for all $v \in V'$, $sv|_{\mathfrak{t}_I} \neq 0$.*

PROOF: Fix $s \in W$ and assume that $sv|_{t_I} = 0$ for some $v \in V'$. Then $v, sv \in t_R^* = \{\mu \in t^* : \mu|_{t_I} = 0\}$. Let $W_1 = \{s_1 \in W : s_1 t_R = t_R\}$. Then for every $\beta \in \Phi_R \cup \Phi_I, s_\beta \in W_1$. Also if $\gamma \in \Phi_c$ with $\langle \gamma, \gamma^\sigma \rangle = 0$, then $s_\gamma s_\gamma^\sigma \in W_1$. As in [3a] it is easy to see that there is $s_1 \in W_1$ so that $v, s_1 sv$ are separated only by hyperplanes corresponding to roots $\gamma \in \Phi_c$ with $\langle \gamma, \gamma^\sigma \rangle > 0$. Thus there are $\gamma_1, \dots, \gamma_k \in \Phi_c$ so that $\langle \gamma_i, \gamma_i^\sigma \rangle > 0$, $1 \le i \le k$, and $v = s_{\gamma_1} \dots s_{\gamma_k} s_1 sv$. But since $v \in V'$, this implies that $s_{\gamma_1} \dots s_{\gamma_k} s_1 sV = V$ and $sV = s_1^{-1} s_{\gamma_k} \dots s_{\gamma_1} V$. For $1 \le i \le k$, $\langle \gamma_i, \gamma_i^\sigma \rangle > 0$ implies that $\gamma_i - \gamma_i^\sigma = \beta_i \in \Phi_I$ so that $\gamma_i = \gamma_R + \beta_i/2$ for some $\gamma_R \in t_R^*$. Thus $s_{\gamma_k} \dots s_{\gamma_1} V \subseteq V + \sum_{i=1}^k \mathbf{R}\gamma_i \subseteq t_R + \sum_{i=1} \mathbf{R}\beta_i$. But for $s_1 \in W_1$, $s_1^{-1}(t_R + \sum_{i=1}^k \mathbf{R}\beta_i) \subseteq t_R + \sum_{\beta \in \Phi_I} \mathbf{R}\beta$. Thus $sV \subseteq t_R + \sum_{\beta \in \Phi_I} \mathbf{R}\beta$. Since $sV \nsubseteq t_R$, there is $\beta \in \Phi_I$ with $\beta|_{sV} \neq 0$. Thus $s^{-1}\beta \in \Phi \backslash \Phi_V$ so that $\langle \beta, sv \rangle \neq 0$ since $v \in V'$. This contradicts the assumption that $sv|_{t_I} = 0$. □

LEMMA 4.9: *Suppose $s \in W$ with $sV \nsubseteq t_R$. Then for any open subset U of V', $E(U:s) = \{0\}$.*

PROOF: Define Ω_I as in (4.4). Clearly $L(\zeta \exp H)$ is bounded for H in compact subsets of Ω_I since for $\beta \in \Phi$ with $\beta|_{\mathfrak{a}} \neq 0$, $\xi_\beta(\zeta \exp H) \neq 1$ for all $H \in \Omega_I$. Let F be a connected component of Ω_I. Then by (4.6), for $\psi \in E(U:s)$ there is a distribution T on U so that $\psi(H:\alpha) = T(\alpha \exp p_s(H))$ for all $H \in F$, $\alpha \in C_c^\infty(U)$. By (4.1) $\psi(H:\alpha) = e^{\rho_I(H)} f_\alpha(\zeta \exp H)$ for some smooth function f_α defined on T'. Let $H_0 \in L = \{H \in t_I : \exp(H/2) = 1\}$. Then for all $H \in F$, $H + H_0 \in F$ and $\psi(H:\alpha) = \psi(H + H_0:\alpha)$ so that $T(\exp p_s(H)(1 - \exp p_s(H_0))\alpha) = 0$. Since $\exp(-p_s(H)) \in C^\infty(U)$, this implies that for all $\alpha \in C_c^\infty(U)$ and $H_0 \in L$, $T((1 - \exp p_s(H_0))\alpha) = 0$.

Because $s\lambda$ takes pure imaginary values on t_I and siv, $v \in U$, takes real values on t_I, $\exp(p_s(H_0)) = 1$ only if $sv(H_0) = 0$. Fix $v_0 \in U$. Since $v_0 \in V'$, $sv_0|_{t_I} \neq 0$ by (4.8). Since L spans t_I, there is $H_0 \in L$ with $sv_0(H_0) \neq 0$. Let $U_0 = \{v \in U : sv(H_0) \neq 0\}$. Then U_0 is an open neighborhood of v_0 in U and $(1 - \exp p_s(H_0))^{-1} \in C^\infty(U_0)$, so that for any $\alpha \in C_c^\infty(U)$ with support contained in U_0, $T(\alpha) = 0$. Thus v_0 is not in the support of T and since $v_0 \in U$ was arbitrary, $T = 0$. □

For a subspace V of $\mathscr{F}$ and $s \in W$, let
$W(sV) = \{w \in W : ws(\lambda + iv) = s(\lambda + iv)$ for all $v \in V\} = \{w \in W : ws\lambda = s\lambda$ and $wsv = sv$ for all $v \in V\}$ and $S(sV) = \{v \in S(t_{\mathbf{C}}) : wv = v$ for all $w \in W(sV)\}$.

LEMMA 4.10: *For U any open subset of V' and for all $s \in W$, $E(U:s:\mathrm{S(sV)}) = \{0\}$.*

Proof: We will show that $E(U:s:S(sv)) \subseteq E(U:s)$ which is zero by (4.7) and (4.9).

Let $\mathfrak{t}_0 = \{H \in \mathfrak{t}_{\mathbf{C}} : s\lambda(H) = 0$ and $sv(H) = 0$ for all $v \in V\}$. Clearly for $v \in S(\mathfrak{t}_0)$, if v has no constant term, then $p_s(v:v) = v(s(\lambda + iv)) = 0$ for all $v \in U$ so that $p_s(v) = 0$ as an element of $C^\infty(U)$. Let $\mathfrak{t}_1 = \mathbf{C}(s\lambda) \oplus \mathbf{C}(sV)$. Then $\mathfrak{t}_1$ is the orthogonal complement in $\mathfrak{t}_{\mathbf{C}}$ of $\mathfrak{t}_0$ so that $S(\mathfrak{t}_{\mathbf{C}}) = S(\mathfrak{t}_0) \otimes S(\mathfrak{t}_1)$ and $S(\mathfrak{t}_1) \subseteq S(sV)$. Thus to show that $E(U:s:S(sV)) \subseteq E(U:s)$ it is enough to show that for all $v \in S(\mathfrak{t}_0)$, $\psi \in E(U:s:S(sV))$,

$$\psi(H, \partial v:\alpha) = \psi(H:p_s(v)\alpha) \text{ for all } H \in \mathfrak{t}'(\zeta), \alpha \in C_c^\infty(U). \tag{*}$$

Let $I(\mathfrak{t}_0) = \{v \in S(\mathfrak{t}_0) : wv = v$ for all $w \in W(sV)\}$. $W(sV)$ is the pointwise stabilizer in W of $\mathfrak{t}_1$ so that $W(sV)$ is generated by reflections in roots which vanish on $\mathfrak{t}_1$, that is roots lying in $\mathfrak{t}_0$. Thus using a standard argument of Harish–Chandra [2a], there are $u_1, \ldots, u_k \in S(\mathfrak{t}_0)$, u_i homogeneous of degree d_i, $1 \le i \le k$, so that each $v \in S(\mathfrak{t}_0)$ can be written as $v = \sum_{i=1}^k u_i q_i$ for some $q_1, \ldots, q_k \in I(\mathfrak{t}_0)$.

Let $d = \max\{d_1, \ldots, d_k\}$. Suppose $v \in S(\mathfrak{t}_0)$ is homogeneous of degree $\ell > d$. Then each q_i is homogeneous of degree $\ell - d_i > 0$ so that $p_s(q_i) = 0$, $1 \le i \le k$, and $p_s(v) = \sum_{i=1}^k p_s(u_i)p_s(q_i) = 0$. Thus for any $\psi \in E(U:s:S(sV))$, $\alpha \in C_c^\infty(U)$, and $H \in \mathfrak{t}'(\zeta)$, $\psi(H; \partial v:\alpha) = \sum_{i=1}^k \psi(H; \partial u_i : p_s(q_i)\alpha) = 0 = \psi(H:p_s(v)\alpha)$.

Now let $v_0 \in S(\mathfrak{t}_0)$ be homogeneous of degree k, $1 \le k \le d$, and assume inductively that for $v \in S(\mathfrak{t}_0)$ homogeneous of degree greater than k and $\psi \in E(U:s:S(sV))$, property (*) holds. For $\psi \in E(U:s:S(sV))$, define $v_0\psi$ by $v_0\psi(H:\alpha) = \psi(H; \partial v_0 : \alpha)$ for $H \in \mathfrak{t}'(\zeta)$, $\alpha \in C_c^\infty(U)$. Clearly $v_0\psi \in E(U:s:S(sV))$. Further, if $v \in S(\mathfrak{t}_0)$ is homogeneous of degree ≥ 1, then $v_0\psi(H; \partial v:\alpha) = \psi(H; \partial(vv_0):\alpha) = \psi(H:p_s(vv_0)\alpha) = 0 = v_0\psi(H:p_s(v)\alpha)$ by the induction hypothesis. If $v \in S(\mathfrak{t}_0)$ is constant, $v \in I(\mathfrak{t}_0)$ so that also in this case v and $v_0\psi$ satisfy (*). Thus $v_0\psi \in E(U:s)$ so that $v_0\psi = 0$; that is, $\psi(H; \partial v_0 : \alpha) = 0 = \psi(H:p_s(v_0)\alpha)$ for all $H \in \mathfrak{t}'(\zeta)$, $\alpha \in C_c^\infty(U)$. Thus for any $v_0 \in S(\mathfrak{t}_0)$, v_0 homogeneous of degree ≥ 1, and $\psi \in E(U:s:S(sV))$, ψ and v_0 satisfy (*). Again, since (*) always holds for terms of degree 0, we are done. □

Let U be an open subset of $\mathscr{F}$. For $\psi \in E(U)$ and $u \in S(\mathfrak{t}_{\mathbf{C}})$, define $u\psi(H:\alpha) = \psi(H; \partial u:\alpha)$. For each $f \in C^\infty(U)$, $\psi \in E(U)$, define $f\psi(H:\alpha) = \psi(H:f\alpha)$. Clearly the above give algebra actions of $S(\mathfrak{t}_{\mathbf{C}})$ and $C^\infty(U)$ on $E(U)$ which commute. Thus $Y(U) = C^\infty(U) \otimes S(\mathfrak{t}_{\mathbf{C}})$ acts on $E(U)$. For $y = \sum_{i=1}^k f_i \otimes u_i \in Y(U)$ and $s \in W$, define $sy = \sum_{i=1}^k f_i \otimes su_i$ and define $p_s(y)$ to be the C^∞ function on U given by $p_s(y:v) = \sum_{i=1}^k f_i(v)p_s(u_i:v)$, $v \in U$. Define $Y_0(U) = \{y \in Y(U) : p_s(y) = 0$ for all $s \in W\}$, $Y^I(U) = C^\infty(U) \otimes I(\mathfrak{t}_{\mathbf{C}})$, and $Y_0^I(U) = Y_0(U) \cap Y^I(U)$.

LEMMA 4.11: $Y_0(U) = \{\sum_{i=1}^k v_i y_i : v_i \in S(\mathfrak{t}_{\mathbf{C}}),\ y_i \in Y_0^I(U)\}$.

PROOF: We know from [2a] that there are homogeneous elements $u_1, \ldots, u_w \in S(\mathfrak{t}_{\mathbf{C}})$, $w = [W]$, so that each $u \in S(\mathfrak{t}_{\mathbf{C}})$ can be written uniquely as $u = \sum_{i=1}^w u_i q_i$ where $q_i \in I(\mathfrak{t}_{\mathbf{C}})$, $1 \le i \le w$. Write $W = \{s_1, \ldots, s_w\}$ and $p_{s_i} = p_i$, $1 \le i \le w$. Fix $v \in U \cap \mathscr{F}'$. Then $\{s(\lambda + iv) : s \in W\}$ is a set of w distinct points in $\mathfrak{t}_{\mathbf{C}}^*$. Thus there are polynomials $v_1, \ldots, v_w \in S(\mathfrak{t}_{\mathbf{C}})$ so that $p_i(v_j : v) = v_j(s_i(\lambda + iv)) = \delta_{ij}$, $1 \le i,\ j \le w$. For $1 \le j \le w$, write $v_j = \sum_{k=1}^w q_{kj} u_k$ where $q_{kj} \in I(\mathfrak{t}_{\mathbf{C}})$, $1 \le k \le w$. Then for $1 \le i,\ j \le w$, $\delta_{ij} = p_i(v_j : v) = \sum_{k=1}^w p(q_{kj} : v) p_i(u_k : v)$. Thus if A_v is the $w \times w$ matrix with entries $a_{ij}(v) = p_i(u_j : v)$, $1 \le i, j \le w$, we see that A_v is invertible so that $\det A_v \ne 0$.

Now let $y = \sum_{i=1}^k f_i \otimes v_i$ denote an arbitrary element of $Y_0(U)$. For $1 \le i \le k$, write $v_i = \sum_{j=1}^w q_{ij} u_j$ where $q_{ij} \in I(\mathfrak{t}_{\mathbf{C}})$ and u_j are as above, $1 \le j \le w$. Then we can write $y = \sum_{j=1}^w u_j y_j$ where $y_j = \sum_{i=1}^k f_i \otimes q_{ij} \in Y^I(U)$ for $1 \le j \le w$. Since $y \in Y_0(U)$, for all $1 \le i \le w$, $v \in U$, $p_i(y : v) = \sum_{j=1}^w p(y_j : v) p_i(u_j : v) = 0$. Now since for each $v \in U \cap \mathscr{F}'$ the matrix A_v is non-singular, this implies that for $v \in U \cap \mathscr{F}'$, $p(y_j : v) = 0$, $1 \le j \le w$. But $U \cap \mathscr{F}'$ is dense in U so that $p(y_j : v) = 0$, $1 \le j \le w$, for all $v \in U$. □

LEMMA 4.12: *For all* $y \in Y_0(U)$, $\psi \in E(U)$, $y\psi = 0$.

PROOF: By (4.11) it is enough to show that $y\psi = 0$ for all $y \in Y_0^I(U)$. Write $y = \sum_{i=1}^k f_i \otimes q_i$ where $q_i \in I(\mathfrak{t}_{\mathbf{C}})$, $1 \le i \le k$. Then for all $H \in \mathfrak{t}'(\zeta)$, $\alpha \in C_c^\infty(U)$, $y\psi(H : \alpha) = \sum_{i=1}^k \psi(H; \partial q_i : f_i \alpha) = \sum_{i=1}^k \psi(H : p(q_i) f_i \alpha) = \psi(H : p(y)\alpha) = 0$. □

Let $v_0 \in \mathscr{F}$. Let $\Phi_0 = \{\beta \in \Phi : \langle \beta, v_0 \rangle = 0\}$, $V = \{v \in \mathscr{F} : \langle \beta, v \rangle = 0$ for all $\beta \in \Phi_0\}$. Then $v_0 \in V'$. For $s \in W$, define $W(sV)$ and $S(sV)$ as in (4.10). Note that for $s \in W$, $v \in S(sV)$, sV and $p_s(v)$ depend only on the coset of s in $W/W(V)$.

LEMMA 4.13: *There is a neighborhood* U *of* v_0 *in* $\mathscr{F}$ *so that* $E(U) = \sum_{s \in W/W(V)} E(U : s : S(sV))$.

PROOF: Let $s_0 = 1, s_1, \ldots, s_k$ be a set of representatives for the cosets $W(V)\backslash W$. For $0 \le i \le k$, write p_i for p_{s_i}. Let $H_0 \in \mathfrak{t}_{\mathbf{C}}$ be dual to $\lambda + iv_0$. Then for $1 \le i \le k$, since $v_0 \in V'$, $p_i(H_0 : v_0) \ne p_0(H_0 : v_0)$. Let U be a neighborhood of v_0 in $\mathscr{F}$ for which $p_i(H_0 : v) \ne p_0(H_0 : v)$ for all $v \in U$, $1 \le i \le k$. Then

$$y_1 = \prod_{i=1}^k \frac{H_0 - p_i(H_0)}{p_0(H_0) - p_i(H_0)} \in Y(U)$$

and $wy_1 = y_1$ for all $w \in W(V)$. For any $w \in W(V)$,

$$p_{ws_i}(y_1) = p_{s_i}(w^{-1}y_1) = p_i(y_1) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{if } 1 \le i \le k. \end{cases}$$

Thus

$$p_s(y_1) = \begin{cases} 1 & \text{if } s \in W(V) \\ 0 & \text{if } s \notin W(V). \end{cases}$$

Now for any s, $t \in W$,

$$p_s(ty_1) = p_{t^{-1}s}(y_1) = \begin{cases} 1 & \text{if } s \in tW(V) \\ 0 & \text{if } s \notin tW(V). \end{cases}$$

Thus for

$$y = \sum_{s \in W/W(V)} sy_1 \in Y^I(U),\ p(y) = \sum_{s \in W/W(V)} p_1(sy) = 1$$

so that $y\psi = \psi$ for all $\psi \in E(U)$. But for $s \in W$ and $v \in S(sV)$, $(v - p_s(v))(sy_1) \in Y_0(U)$ so that for all $\psi \in E(U)$, $sy_1\psi \in E(U:s:S(sV))$. Thus $\psi = \sum_{s \in W/W(V)} sy_1\psi$ gives the required decomposition. $\square$

Note that (4.13) and (4.10) do not combine to imply that every $v_0 \in \mathscr{F}$ has a neighborhood U in $\mathscr{F}$ so that $E(U) = \{0\}$. This is because in the statement of (4.10) the set U is an open subset of V', not of $\mathscr{F}$, and unless $v_0 \in \mathscr{F}'$, V is a proper subspace of $\mathscr{F}$.

Suppose $v_0 \in \mathscr{F}'$. Then $\Phi_0 = \emptyset$, $V = \mathscr{F}$, and for all $s \in W$, $W(sV) = \{1\}$, $S(sV) = S(\mathfrak{t}_{\mathbf{C}})$. In the proof of (4.13) we could have picked the neighborhood U of v_0 in $\mathscr{F}$ small enough so that $U \subseteq \mathscr{F}'$. Thus using (4.7) and (4.9), $E(U) = \sum_{s \in W} E(U:s) = \{0\}$. This shows that for any $\psi \in E(\mathscr{F}')$ and $\alpha \in C_c^\infty(\mathscr{F}')$ with support contained in U, $\psi(H:\alpha) = 0$ for all $H \in \mathfrak{t}'(\zeta)$. That is, v_0 is not in the support of the distribution $\psi(H)$ for all $H \in \mathfrak{t}'(\zeta)$. But $v_0 \in \mathscr{F}'$ was arbitrary so that $\psi(H) = 0$ for all $H \in \mathfrak{t}'(\zeta)$. Thus $E(\mathscr{F}') = \{0\}$.

That is, for all $\psi \in E(\mathscr{F})$ and $H \in \mathfrak{t}'(\zeta)$, the support of the distribution $\psi(H)$ is contained in the singular set $\mathscr{F}^s = \{v \in \mathscr{F} : v \notin \mathscr{F}'\}$ which is a finite union of hyperplanes $V_\beta = \{v \in \mathscr{F} : \langle \beta, v \rangle = 0\}$ for some $\beta \in \Phi$, $\beta|_{\mathfrak{a}_1} \neq 0$. For U an open subset of $\mathscr{F}$, V a subspace of $\mathscr{F}$, $s \in W$, and S a subalgebra of $S(\mathfrak{t}_{\mathbf{C}})$, write $E(U : U \cap V : s : S) = \{\psi \in E(U:s:S)$: for all $H \in \mathfrak{t}'(\zeta)$, $\operatorname{supp} \psi(H) \subseteq U \cap V\}$. We will also write $E(U : U \cap V)$ for the analogous subset of $E(U)$. We have seen above that $E(\mathscr{F}) = E(\mathscr{F} : \mathscr{F}^s) = \sum_{\beta \in \Phi_1} E(\mathscr{F} : V_\beta)$, $\Phi_1 = \{\beta \in \Phi : \beta|_{\mathfrak{a}_1} \neq 0\}$.

We now need to recall a classical theorem about distributions on $\mathbf{R}^n$ which are supported on a subspace. Let U be an open subset of $\mathbf{R}^n$, V a subspace with $U \cap V \neq \emptyset$. We identify V with $\mathbf{R}^k \times \{0\}$ for some $0 \leq k \leq n$. For $\varphi \in C_c^\infty(U)$, let $\bar{\varphi} \in C_c^\infty(U \cap V)$ denote the restriction of φ to $U \cap V$. For any distribution T on $U \cap V$ there is a distribution $\bar{T}$ on U given by $\bar{T}(\varphi) = T(\bar{\varphi})$, $\varphi \in C_c^\infty(U)$. Clearly if D is any differential operator on U and T is any distribution on $U \cap V$, then $(D\bar{T})(\varphi) = T(\overline{D^*\varphi})$ gives a distribution on U supported on $U \cap V$. Let $Q = \{(q_1, \ldots, q_{n-k}) : q_i \in \mathbf{N},\ 1 \leq i \leq n-k\}$. For $q \in Q$ a multi-index, let D^q denote the corresponding differential operator on U with respect to the $\mathbf{R}^{n-k}$ variables transverse to $V = \mathbf{R}^k \times \{0\}$.

THEOREM 4.14 [6]: *Let T be a distribution on U, supported on $U \cap V$. Then for every $q \in Q$ there is a unique distribution T_q on $U \cap V$ so that $T = \sum_q D^q \bar{T}_q$. Further, the sum is locally finite.*

Now suppose U is an open subset of $\mathscr{F}$, V is a subspace of $\mathscr{F}$ which has non-trivial intersection with U, and $\psi \in E(U : U \cap V)$. Then using (4.14), for each $H \in \mathfrak{t}'(\zeta)$ and $q \in Q$ there is a unique distribution $\psi_q(H)$ on $U \cap V$ so that $\psi(H) = \sum_q D^q \overline{\psi_q(H)}$. For each $H \in \mathfrak{t}'(\zeta)$, the sum is locally finite. But in fact, using the full strength of (4.5), if Ω is a relatively compact open subset of U, there is an $N \geq 0$ so that for every $H \in \mathfrak{t}'(\zeta)$ and $\alpha \in C_c^\infty(\Omega)$,

$$\psi(H : \alpha) = \sum_{|q| \leq N} (-1)^{|q|} \psi_q(H : \overline{D^q \alpha}). \tag{4.15}$$

Further, since for any $\beta \in C_c^\infty(U \cap V)$ and $q \in Q$ we can find $\alpha \in C_c^\infty(U)$ with $\overline{D^q \alpha} = \beta$, $\overline{D^{q'} \alpha} = 0$, $q' \neq q$, it is easy to see that each ψ_q must satisfy conditions (4.1), (4.3), (4.4), and (4.5) as a function on $\mathfrak{t}'(\zeta) \times C_c^\infty(U \cap V)$.

LEMMA 4.16: *Suppose U is an open subset of $\mathscr{F}$ and V is a subspace of $\mathscr{F}$ so that $U \cap V \subseteq V'$. Then $E(U : U \cap V : s : S(sV)) = \{0\}$ for all $s \in W$.*

PROOF: Suppose $\psi \in E(U : U \cap V : s : S(sV))$. Assume $\psi \neq 0$. We will show this produces a contradiction. Fix $v_0 \in U \cap V$ such that v_0 is in the support of $\psi(H)$ for some $H \in \mathfrak{t}'(\zeta)$. Let Ω be a relatively compact neighborhood of v_0 in U. For $H \in \mathfrak{t}'(\zeta)$ and $\alpha \in C_c^\infty(\Omega)$ decompose $\psi(H : \alpha)$ as in (4.15) where N is chosen as small as possible. Then there is a $q \in Q$ so that $|q| = N$ and $\psi_q(H)$ is non-trivial on $C_c^\infty(\Omega \cap V)$ for some $H \in \mathfrak{t}'(\zeta)$. Let $s \in W$ and $v \in S(sV)$. For $\beta \in C_c^\infty(\Omega \cap V)$ choose $\alpha \in C_c^\infty(\Omega)$ so that $\overline{D^q \alpha}$

$= \beta. \overline{D^{q'}\alpha} = 0$, $q' \neq q$. Then since $\psi \in E(U:s:S(sV))$, for any $v \in S(sV)$,

$$\psi_q(H;\partial v:\beta) = (-1)^{|q|}\psi(H:\partial v:\alpha) = (-1)^{|q|}\psi(H:p_s(v)\alpha) = \\ (-1)^{|q|} \sum_{|q'| \leq N} (-1)^{|q'|}\psi_{q'}(H:\overline{D^{q'}(p_s(v)\alpha)}) = \psi_q(H:p_s(v)\beta).$$

Thus the restriction of ψ_q to $\mathfrak{t}'(\zeta) \times C_c^\infty(\Omega \cap V)$ is an element of $E(\Omega \cap V:s:S(sV))$. Thus $\psi_q(H:\beta) = 0$ for all $\beta \in C_c^\infty(\Omega \cap V)$, $H \in \mathfrak{t}'(\zeta)$ by (4.10). This contradicts the assumption that ψ_q is non-trivial on $C_c^\infty(\Omega \cap V)$. □

For V a subspace of $\mathscr{F}$, define Φ_V and V' as before and let $\mathscr{F}'_V = \{v \in \mathscr{F} : \langle \beta, v\rangle \neq 0 \text{ for } \beta \in \Phi \backslash \Phi_V\}$. Then $\mathscr{F}'_V$ is an open subset of $\mathscr{F}$ and $\mathscr{F}'_V \cap V = V'$. When $V = \mathscr{F}$, $\mathscr{F}'_V = \mathscr{F}'$ and when $V = \{0\}$, $\mathscr{F}'_V = \mathscr{F}$.

THEOREM 4.17: $E(\mathscr{F}) = \{0\}$.

We will show that $E(\mathscr{F}) = \{0\}$ by using downward induction on $\dim V$ to prove that $E(\mathscr{F}'_V) = \{0\}$ for all V. We have already established this for $V = \mathscr{F}$. The statement for $V = \{0\}$ will give the theorem.

Let V be a subspace of $\mathscr{F}$ with $\dim V < \dim \mathscr{F}$. *We can assume inductively that for subspaces V_1 with* $\dim V_1 > \dim V$, $E(\mathscr{F}'_{V_1}) = \{0\}$. *Thus* $E(\mathscr{F}'_V) = E(\mathscr{F}'_V : V')$. *For $v_0 \in V'$ there is a neighborhood U of v_0 in $\mathscr{F}'_V$ so that* $E(U:U \cap V) = \sum_{s \in W/W(V)} E(U:U \cap V:s:S(sV))$ *by* (4.13). *But since* $U \cap V \subseteq V'$, *by* (4.16), $E(U:U \cap V:s:S(sV)) = \{0\}$ *for all* $s \in W$. *Thus* $E(U:U \cap V) = E(U) = \{0\}$. *Thus for any* $\psi \in E(\mathscr{F}'_V)$, $H \in \mathfrak{t}'(\zeta)$, $v_0 \notin \operatorname{supp} \psi(H)$. *Since this is true for all $v_0 \in V'$ and* $\operatorname{supp} \psi(H) \subseteq V'$, $\psi = 0$.

§5. Proof of the main theorem

Let $P = MA_1N$ be a cuspidal parabolic subgroup of G. For $\omega \in \varepsilon_2(M)$ and $v \in \mathscr{F} = \mathfrak{a}_1^*$, let $\theta_{\omega,v}$ be the character of the corresponding induced representation and let

$$\varphi_\alpha(x) = \int_{\mathscr{F}} \alpha(v)E(P:\psi:v:x)\mu(\omega:v)\,dv, \; x \in G,$$

be a wave packet where ψ is a K_M-finite matrix coefficient for ω on M and $\alpha \in C_c^\infty(\mathscr{F})$. Let $W(\omega) = \{s \in N_G(A_1)/MA_1 : s\omega = \omega\}$.

Let A be a special vector subgroup of G of dimension p and $\mathscr{Y}$ be an A-orthogonal set. Let $X \in \mathscr{G}_A$ and let $c_0(X)$ be its constant term. Let $T = T_I T_R$ be a θ-stable Cartan subgroup of G which contains A and satisfies $\dim T_R \geq \dim A + \dim A_1$. Let $\varepsilon(T, A, A_1)$ be 1 if $\dim T_R = \dim A + \dim A_1$ and be zero otherwise. When $\varepsilon(T, A, A_1) = 1$ let $\theta^A_{\omega,\nu}$ be the weighted character defined in (3.3). Let $f = \varphi_\alpha$, $\alpha \in C_c^\infty(\mathscr{F})$.

THEOREM 5.1: *For any* $h \in T'$,

$$r_f(h:\mathscr{Y}:X) = \varepsilon(T, A, A_1)c_0(X)(-1)^p[W(\omega)]^{-1}\int_{\mathscr{F}} \langle \theta_{\omega,\nu}, f\rangle \theta^A_{\omega,\nu}(h)\,d\nu.$$

PROOF: For the first part of the proof we will repeat the argument used by Arthur in Theorem 9.1 of [1c].

Suppose $p = 0$. Then $\mathscr{G}_A = \mathscr{G}$ and for all $X \in \mathscr{G}$, $r_f(h:\mathscr{Y}:X) = c_0(X)\int_{T_R\backslash G} f(x^{-1}hx)d\dot{x}$ and the result follows by results of Harish–Chandra in §24 of [2c] summarized here as (1.3).

Let $p > 0$ and assume inductively that the theorem is true for any $\tilde{T}$ and $\tilde{A}$ with $\dim \tilde{A} < p$. Fix $X \in \mathscr{G}_A(r)$, $r \geq 0$. If $r > p$ then $D_X v(x:\mathscr{Y}) = 0$ and $c_0(X) = 0$. Thus we can assume inductively that the theorem is true for $X \in \mathscr{G}_A(r')$, $r' > r$.

Fix $\zeta \in Z(\mathfrak{t})$. For $H \in \mathfrak{t}'(\zeta)$, $\alpha \in C_c^\infty(\mathscr{F})$, write

$$\psi(H:\alpha) = \tilde{\Delta}(\zeta, H)\Big\{r_{\varphi_\alpha}(\zeta \exp H:\mathscr{Y}:X) - c_0(X)\varepsilon(T, A, A_1)(-1)^p[W(\omega)]^{-1}\int_{\mathscr{F}} \langle \theta_{\omega,\nu}, \varphi_\alpha\rangle \theta^A_{\omega,\nu}(\zeta \exp H)\,d\nu\Big\}.$$

We must show that $\psi \in E(\mathscr{F})$ so that $\psi(H:\alpha) = 0$.

We may as well assume that $T \subseteq L$. Since $p = \dim A > 0$, T is not fundamental. We know that for each $\alpha \in C_c^\infty(\mathscr{F})$, $r_{\varphi_\alpha}(h:\mathscr{Y}:X)$ and $\theta^A_{\omega,\nu}(h)$ are smooth functions of $h \in T'$. Further, for $h = \zeta \exp H \in T'$, $h \to e^{-\rho_I(H)}\tilde{\Delta}(\zeta, H) = \prod_{\beta \in \Phi_I^+}(1 - \xi_{-\beta}(h))\Delta_+(h)$ is a smooth function on T'. Thus ψ is a function on $\mathfrak{t}'(\zeta) \times C_c^\infty(\mathscr{F})$ satisfying (4.1).

Let $z \in \mathscr{Z}$ and let $q = \gamma(z) \in I(\mathfrak{t}_{\mathbf{C}})$. Then by (3.6), $\tilde{\Phi}^A_{\omega,\nu}(\zeta, H; \partial q) = q(\lambda + i\nu)\tilde{\Phi}^A_{\omega,\nu}(\zeta, H)$ where $\lambda \in \mathfrak{t}^*_{M,\mathbf{C}}$ corresponds to $\omega \in \varepsilon_2(M)$. Also $\langle \theta_{\omega,\nu}, zf\rangle = q(\lambda + i\nu)\langle \theta_{\omega,\nu}, f\rangle$. Further, because of the induction hypothesis on r and (2.4), $R_f(\zeta, H; \partial q:\mathscr{Y}:X) = R_{zf}(\zeta, H:\mathscr{Y}:X)$ since for all $X_i \in \mathscr{G}_A(1)$, $XX_i \in \mathscr{G}_A(r + 1)$ so that by the induction hypothesis $R_f(\zeta, H:\mathscr{Y}:XX_i) = 0$. Now since $f = \varphi_\alpha$, $zf = \varphi_{p(q)\alpha}$ by (2.10). Combining the above observations we see that ψ satisfies (4.2).

Fix $\beta \in \Phi_R(\zeta)$. Then

$$S^\beta_f(\zeta, H:\mathscr{Y}:X) = R_f(\zeta, H:\mathscr{Y}:X) + \tau_\beta(H)R_f^{T, A_\beta}(\zeta, H:\mathscr{Y}:X).$$

If $n_\beta(A) = 0$, then $\tau_\beta(H) = 0$. If $n_\beta(A) \neq 0$, then $\dim A_\beta < p$ so by the induction hypothesis the theorem holds for R_f^{T,A_β}. But $\dim T_R \geq \dim A + \dim A_1 > \dim A_\beta + \dim A_1$ so that $\varepsilon(T:A_\beta:A_1) = 0$. Thus in any case $S_f^\beta(\zeta, H:\mathscr{Y}:X) = R_f(\zeta, H:\mathscr{Y}:X)$, so that using (2.6), for any $H_0 \in \mathfrak{t}_\beta^0(\zeta)$, $u \in S(\mathfrak{t}_{\mathbf{C}})$, $R_f(\zeta, H_0; \partial u:\mathscr{Y}:X)^+ - R_f(\zeta, H_0; \partial u:\mathscr{Y}:X)^- = -n_\beta(A)\lim_{\theta\to 0} R_f^{T_\beta,A_\beta}(\zeta, H_0 + \theta(X'_\beta - Y'_\beta); \partial(A(u - s_\beta u)):\mathscr{Y}_\beta:X)$ where the limits exist uniformly for H_0 in compacta of $\mathfrak{t}_\beta^0(\zeta)$. Again, either $n_\beta(A) = 0$ or else the theorem can be applied to $R_f^{T_\beta,A_\beta}$. Combining this with (3.8) we see that for any $H_0 \in \mathfrak{t}_\beta^0(\zeta)$, $u \in S(\mathfrak{t}_{\mathbf{C}})$, $\psi(H_0; \partial u:\alpha)^+ = \psi(H_0; \partial u:\alpha)^-$ where the limits exist uniformly on compacta of $\mathfrak{t}_\beta^0(\zeta)$. Thus we see that ψ satisfies (4.3).

Using (2.8) and the fact that $\alpha \to \varphi_\alpha$ is a continuous mapping of $C_c^\infty(\mathscr{F})$ into $\mathscr{C}(G)$ we see that $R\varphi_\alpha$ satisfies the growth condition (4.5). Using (2.11) and (3.5), for any $u \in S(\mathfrak{t}_{\mathbf{C}})$, $\int_{\mathscr{F}} \langle \theta_{\omega,\nu}, \varphi_\alpha \rangle \tilde{\Phi}_{\omega,\nu}^A(\zeta, H; \partial u)\,d\nu$ is a finite sum of terms of the form

$$I(H:\alpha) = c\, c_s(F) c(B_s:A) e^{s\lambda(H)} \int_{\mathscr{F}} \alpha(t\nu) p_s(u:\nu) e^{is\nu(H)}\,d\nu$$

where $t \in W(\omega)$, $s \in \{w \in W: wA_1 \subseteq T_R\}$, F is the connected component of $\Omega(\zeta)$ containing H, and $p_s(u)$ is the polynomial on $\mathscr{F}$ given by $p_s(u:\nu) = u(s(\lambda + i\nu))$. Since $sA_1 \subseteq T_R$, $s\nu(H)$ is real for all $H \in \mathfrak{t}$. Further, by (3.7), $c_s(F) = 0$ unless $\operatorname{Re} s\lambda(H) < 0$ for all $H \in \mathscr{F}$. Thus there are a constant C and a continuous seminorm μ on $C_c^\infty(\mathscr{F})$ so that

$$|I(H:\alpha)| \leq C \int_{\mathscr{F}} |\alpha(t\nu) p_s(u:\nu)|\,d\nu \leq C\mu(\alpha) \text{ for all } \alpha \in C_c^\infty(\mathscr{F}),$$

$H \in \mathfrak{t}'(\zeta)$. Thus ψ satisfies (4.5).

To finish the theorem we must know that for every $\alpha \in C_c^\infty(\mathscr{F})$, $H \to \psi(H:\alpha)$ extends to a C^∞ function on $\Omega_I = \mathfrak{t}_I + \{H \in \mathfrak{t}_R : \beta(H) \neq 0$ for any $\beta \in \Phi$ with $\beta|_{\mathfrak{t}_R} \neq 0\}$. Note that for $H \in \Omega_I$ and $\beta \in \Phi$, if $\xi_\beta(\zeta \exp H) = 1$, then $\beta \in \Phi_I$ and $\beta(H) \in 2\pi i\mathbf{Z}$. Because of (3.5), it is enough to show that R_f, $f = \varphi_\alpha$, extends smoothly to Ω_I. To prove this we need another induction.

Suppose that T is a Cartan subgroup of G with $\dim T_R$ maximal. Then every imaginary root of $(\mathfrak{g}, \mathfrak{t})$ is compact so that, using the remarks following (2.7), ψ extends to a C^∞ function about any semi-regular point in Ω_I. Since ψ and all its derivatives are bounded in a neighborhood of any singular point of Ω_I, it follows from the usual argument that ψ extends to a C^∞ function on Ω_I. Thus in this case $\psi \in E(\mathscr{F}) = \{0\}$ and the theorem is proved.

Assume now that T is a Cartan subgroup of G with $\dim \mathfrak{t}_R = k$ not maximal, and assume that the theorem is true for Cartan subgroups $\tilde{T}$ of G with $\dim \tilde{\mathfrak{t}}_R > k$. For all such $\tilde{T}$ with $A \subseteq \tilde{T}_R$, $R_f^{\tilde{T},A} = 0$ since $\dim \tilde{T}_R > \dim T_R \geq \dim A + \dim A_1$.

Let β be any singular imaginary root of $(\mathfrak{g}, \mathfrak{t})$. Then using (2.7), the jump of any derivative of $R_f^{T,A}$ across the hyperplane $\beta(H) = 0$ is a multiple of $R_f^{\tilde{T},A}$ for a Cartan subgroup $\tilde{T}$ of G with $\dim \tilde{T}_R = \dim T_R + 1$. Thus by the induction hypothesis the jump is zero. The formula for the jump of $R_f(\zeta, H : X : \mathscr{Y})$ across a hyperplane of the form $\beta(H) = 2\pi i n$ can also be obtained by (2.7) by using a possibly different ζ, so again we see that R_f extends smoothly to a neighborhood of any semi-regular point of Ω_I, and hence to Ω_I. Thus ψ satisfies (4.4) and is zero.

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