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VECTOR BUNDLES ON THE CONE OVER A CURVE

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Let $X \subset \mathbb{P}^n_k$ be a projectively normal curve over an algebraically closed field $k$, and let $C(X) \subset \mathbb{A}^{n+1}$ be the affine cone over $X$. The problem studied in this paper is to determine whether $K_0(C(X)) = \mathbb{Z}$, where $K_0$ denotes the Grothendieck group of vector bundles on $C(X)$ (see [2] for definitions). This is an important special case of a question raised by Murthy, as to whether $K_0(A) = \mathbb{Z}$ for any normal graded ring $A = \bigoplus_{n \geq 0} A_n$, finitely generated over $A_0 = k$, where $k$ is a field (see Bass [1]). Spencer Bloch recently showed that $K_0(A) \neq \mathbb{Z}$ for $A = \mathbb{C}[X, Y, Z]/(Z^2 - X^3 - Y^7)$ giving a counterexample to Murthy's question. However, one still suspected that the result would be true for cones. Partial positive results were known (see Varley [3]).

It turns out that the problem has a very different flavour in characteristic 0 than in positive characteristics. First consider the case of characteristic $p > 0$. We have

**Theorem 1:** Let $A = \bigoplus_{n \geq 0} A_n$ be a normal graded ring, finitely generated over $A_0 = k$, where $k$ is algebraically closed of characteristic $p > 0$. Suppose that $A$ is Cohen–Macaulay, and that the vertex (corresponding to the ideal $\bigoplus_{n > 0} A_n$) is the only singularity of $\text{Spec } A$. Then $A_0(\text{Spec } A) = 0$.

Here $A_0(\text{Spec } A)$ denotes the subgroup of $K_0(A)$ generated by the classes of smooth points of $\text{Spec } A$. Now $K_0(A)$ is generated by the class of the trivial line bundle, and classes of sub-varieties not meeting the singular locus (see [3]). Since $\text{Pic } A = (0)$ for a normal graded ring $A$, we deduce (see §1)
COROLLARY (1.3): Let $A$ be as in Theorem 1. Suppose that $\dim A = 2$. Then $K_0(A) = \mathbb{Z}$.

This answers Murthy's question affirmatively in the two dimensional case, and thus includes the result on cones over curves.

Next, we have a partial positive result in characteristic 0.

THEOREM 2: Let $X \subset \mathbb{P}^n_k$ be a projectively normal curve, where $k$ is an algebraically closed field of characteristic 0. Assume that $X$ is not contained in a hyperplane, and that $\deg X \leq 2n - 1$. Then $A_0(C(X)) = 0$, where $C(X) \subseteq \mathbb{A}^{n+1}$ is the affine cone over $X$.

As a consequence, we obtain

THEOREM 2': Let $X/k$ be a curve of genus $g$, and $D$ a divisor on $X$ such that $\deg D \geq 2g + 1$. Then $A_0(C(X)) = 0$, where $C(X)$ is the cone over $X$ in the embedding $X \subset \mathbb{P}^n$, and satisfying $\deg X \leq 2n - 1$.

Using the cancellation theorem of Murthy and Swan, we can formulate the above theorems as follows.

THEOREM: Let $k$ be an algebraically closed field, and let $A = \bigoplus_{n \geq 0} A_n$ be a finitely generated graded $k$-algebra with $A_0 = k$. Then every projective module over $A$ is free, in each of the following cases:

i) $\text{char } k = p > 0$, and $A$ is normal of dimension 2.

ii) $\text{char } k = 0$, and $\text{Spec } A$ is the cone over a projectively normal curve $X$ properly contained in $\mathbb{P}^n$, and satisfying $\deg X \leq 2n - 1$.

iii) $\text{char } k = 0$, and $A = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD))$ where $X/k$ is a smooth curve of genus $g$, and $D$ a divisor on $X$ satisfying $\deg D \geq 2g + 1$.

Finally, we construct an infinite family of examples of cones over $\mathbb{C}$ which admit non-trivial vector bundles. Let $L$ denote the field of algebraic numbers.

THEOREM 3: Let $X \subset \mathbb{P}^n_L$ be a projectively normal curve such that $H^1(X, \mathcal{O}_X(1)) \neq 0$. Then if $C(X_\mathbb{C})$ denotes the cone over the corresponding complex curve, we have $K_0(C(X_\mathbb{C})) \neq \mathbb{Z}$. (In fact, a slight modification of our argument will show that $K_0(C(X_\mathbb{C}))$ is uncountable).

One remarkable fact about theorem 3 is the following. For a curve $X \subset \mathbb{P}^n_\mathbb{C}$, let $Y \subset \mathbb{P}^{n+1}_\mathbb{C}$ denote the projective cone over $X$. Let $Z \subset Y$ be the blow up of $Y$ at the vertex. Then $Y \cong \mathbb{P}(\mathcal{O}_X \oplus \mathcal{O}_X(1))$. The Leray spectral sequence applied to the map $\pi$ yields an exact sequence

$$0 \to H^1(Y, \mathcal{O}_Y) \to H^1(Z, \mathcal{O}_Z) \to \Gamma(Y, R^1\pi_*\mathcal{O}_Z) \to$$

$$\to H^2(Y, \mathcal{O}_Y) \to H^2(Z, \mathcal{O}_Z) \to 0.$$
Since $Z$ is a ruled surface, $p_g(Z) = 0$, and $q = g(X)$, the genus of $X$. (In fact, all elements of $H^1(Z, \mathcal{O}_Z)$ are pulled back from $H^1(X, \mathcal{O}_X)$.) Now $R^1\pi_*\mathcal{O}_Z$ is a torsion sheaf on $Y$ supported only at the vertex of the cone. From the formal function theorem (see [7]), $\Gamma(R^1\pi_*\mathcal{O}_Z)$ has a filtration whose associated graded module is $\bigoplus_{m \geq 0} H^1(E, \mathcal{O}_Z/m^{m+1})$ where $E$ is the exceptional set, and $I$ is its sheaf of ideals on $Z$. Now $E$ is a section of the fibration $Z \to X$, hence $E \cong X$. One easily checks that $I/I^2 \cong \mathcal{O}_X(1)$, and thus $I^m/I^{m+1} \cong \mathcal{O}_X(m)$. Also, the map $H^1(Z, \mathcal{O}_Z) \to \Gamma(R^1\pi_*\mathcal{O}_Z)$ maps the former isomorphically onto $H^1(E, \mathcal{O}_E)$. Hence $h^2(Y, \mathcal{O}_Y)$ vanishes precisely when $h^1(X, \mathcal{O}_X(1)) = 0$. Thus, the curves $X \subset \mathbb{P}^r_C$ with $H^1(X, \mathcal{O}_X(1)) \cong 0$ correspond precisely to the cones $Y$ with “geometric genus” (i.e. $h^2(\mathcal{O}) > 0$). Hence, Theorem 3 may be regarded as an analogue for cones of a famous result of Mumford on the infinite dimensionality of the Chow group of zero cycles on a surface with $p_g > 0$ (see [5]). In fact, one might conjecture that at least for cones, $A_0(C(X)) = 0 \iff p_g(Y) = 0$ (where $p_g$ stands for $h^2(\mathcal{O})$); this is the analogue of a conjecture of Bloch for smooth surfaces with $p_g = 0$ (see [13], ch. 1 for motivation and further references for that conjecture).

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§1) Results in characteristic $p > 0$

In this section we prove theorem 1. In this paper, the Chow group of zero cycles will always be the subgroup of the Grothendieck group $K_0$ generated by the classes of smooth points. In particular, it is not the same (in general) as the Chow group of Fulton [6] when the variety we are dealing with is singular. The proof of the theorem is based on two lemmas:

**Lemma (1.1):** Let $Y$ be an affine normal variety with isolated singularities over an algebraically closed field (of arbitrary characteristic). If $U \subset Y$ is an open (dense) set, then $A_0(Y)$ is generated by the classes of smooth points of $U$. $A_0(Y)$ is a divisible group.
PROOF: If $Y$ is a curve, the result holds from the theory of Jacobians. In general, if $P \in Y$ is a smooth point, we can find a curve $C \subset Y$ such that $P \in C$, $C \cap U \neq \emptyset$, and $C$ misses the singular locus of $Y$; we may take $C$ to be smooth. Then there is a natural map $\text{Pic} C \to A_0(Y)$, and the class of $P$ is in the image of this map. Hence the result follows from the previous case.

**Lemma (1.2):** Let $A = \bigoplus A_n$ be a graded normal ring of dimension 1, where $A_0 = k$, and $A$ is finitely generated over $A_0$. Then $A \cong k[t]$, where $t$ is homogeneous (perhaps of degree $d > 1$).

**Proof:** It is amusing to give two proofs. First, an algebraic one. Let $M = \bigoplus A_n$. Then $A_M$ is a P.I.D. as $A$ is normal. Since $MA_M$ is generated by one element, but also has a set of homogeneous generators, it is generated by one homogeneous element (Nakayama's Lemma). Let $MA_M = fA_M$, with $f \in A$ homogeneous, and let $g \in A$ be any homogeneous element of positive degree. Since $g \in f \cdot A_M$, $g = u \cdot f^n = \frac{u_1}{u_2} \cdot f^n$, where $u_1, u_2 \in A - M$. Comparing homogeneous terms of lowest degree on both sides of $u_2g = u_1f^n$, we see that we may assume $u_1, u_2$ to be homogeneous. Since $u_i \notin M$, $u_i \in A_0 = k$. Thus $A = k[f]$ (since every element of $A$ is a finite sum of homogeneous elements).

The second proof is geometric – since Spec $A$ is an affine curve over $k$ with a non-trivial $\mathbb{G}_m$-action, it is a rational curve. Since it is normal, and has no units (because $A$ is graded) apart from $k^*$, it must be $\mathbb{A}^1_k$. The group of automorphisms of $\mathbb{A}^1_k$ fixing a point is $\mathbb{G}_m$; hence the grading on the coordinate ring of $\mathbb{A}^1_k$ induced from $A$ must be the usual one.

**Proof of Theorem 1:** We first give a simple proof in the case when $A$ is the homogeneous coordinate ring of a plane curve.

Let $X = \text{Proj} A \subset \mathbb{P}_k^2$ be a smooth plane curve, and let $C(X) \subset \mathbb{A}^3$ be the cone over $X$ (so that $C(X) = \text{Spec} A$). Let $0 \in C(X)$ be the vertex, and let $\pi: C(X) - \{0\} \to X$ be the projection. Let $P \in C(X)$ be a smooth point and $\pi(P) = \bar{P}$. Choose a line $l \subset \mathbb{P}_k^2$ such that $l \cap X = \{P_1, \ldots, P_n\}$, where $P_1 = \bar{P}$, and $n = \deg X$, and the $P_i$ are distinct. Then $\pi^{-1}(l) \cup \{0\} = S_1 \cap C(X)$, where $S_1 \subset \mathbb{A}^3$ is a plane (the cone over the line $l$). Thus $S_1 \cap C(X) = l_1 \cup \ldots \cup l_n$, where $\pi(l_i - \{0\}) = P_i$ and the $l_i$ are lines on $S_1$ which concur at 0. We can choose coordinates $x$ and $y$ on $S_1$ so that $l_1$ is the $y$-axis, $P \in l_1$ is the point $(0, 1)$, and the lines $l_2, \ldots, l_n$ respectively have slopes $\lambda_2, \ldots, \lambda_n$. Thus

$$S_1 \cap C(X) = \text{Spec} k[x, y]/(x \prod_{i=2}^n (y - \lambda_i x))$$
Consider the function \( f = 1 - (y - \lambda_2 x)^r \prod_{i=3}^n (y - \lambda_i x) \) (where \( r > 0 \) will be chosen in a moment). Then \( f \) is identically equal to 1 on \( I_2 \cup \ldots \cup I_n \). On \( I_1, x = 0 \), so that \( f|_{I_1} = 1 - y^{r+n-2} \). Choose \( r > 0 \) to be the smallest integer such that \( r + n - 2 = p^r \), where \( p = \text{char} \ k \). Then \( f|_{I_1} = (1 - y)^p \). Thus, the zero cycle \( (P) \) represents a \( p^r \)-torsion element of \( \text{Pic}(S_1 \cap C(X)) \), and hence a \( p^r \)-torsion element of \( A_0(C(X)) \). Since \( A_0(C(X)) \) is generated by the classes of smooth points \( P \in C(X) \), we have \( p^r \cdot A_0(C(X)) = 0 \). Now the divisibility of \( A_0(C(X)) \) forces \( A_0(C(X)) = 0 \).

Now we give the proof in the general case, based on the same idea. Let \( \mathcal{O}(1) \) denote the sheaf on \( \text{Proj} \ A \) associated to the graded module \( \bigoplus A_n \) (see [7], ch. II). Fix a large integer \( m > 0 \) such that \( \mathcal{O}(m) \) embeds in \( \text{Proj} \ A \) is some projective space, and let \( \text{dim} \ \text{Proj} \ A = r \) (so that \( \text{dim} \ A = r + 1 \)). Let \( 0 \in \text{Spec} \ A \) be the vertex, and let \( \pi: \text{Spec} \ A - 0 \to \text{Proj} \ A \) be the projection. Let \( U \subset \text{Spec} \ A - 0 \) be the inverse image of the locus of smooth points of \( \text{Proj} \ A \), and let \( P \in U \). Let \( \pi(P) = \bar{P} \). Choose \( r \) general hyperplane sections of \( \text{Proj} \ A \) through \( \bar{P} \), so that the intersection of all of them with \( \text{Proj} \ A \) consists of \( d = \text{deg}(\text{Proj} \ A) \) points \( P_1 = \bar{P}, P_2, \ldots, P_d \), where all the \( P_i \) are smooth. Let \( Y = \pi^{-1}\{\{P_1, \ldots, P_d\} \cup \{0\} \) so that \( Y = \text{Spec}(A/(f_1, \ldots, f_r)) \) where \( f_i \) is homogeneous of degree \( m \) for each \( i \). Since \( A \) is Cohen-Macaulay, and height \( (f_1, \ldots, f_r) = r \), the ring \( B = A/(f_1, \ldots, f_r) \) is a reduced graded ring of dimension 1. The minimal primes \( \mathcal{P}_1, \ldots, \mathcal{P}_d \) of \( B \) satisfy \( \text{Spec}(B/(\mathcal{P}_i) = \pi^{-1}(P_i) - 0 \). Let \( \bar{B} \) denote the normalisation of \( B \) (i.e. its integral closure in its total quotient ring), and let \( I \subset B \) be the conductor of \( B \to \bar{B} \) (see [12] for the definition). We have the well known exact sequence (where \( * \) denotes the unit group)

\[
0 \to B^* \to \bar{B}^* \to (\bar{B}/I\bar{B})^*/(B/I B)^* \to \text{Pic} \ B \to \text{Pic} \ \bar{B} \to 0
\]

From lemma (1.2), \( \bar{B} \cong \bigoplus \limits_{i=1}^d k[t_i] \), so that \( \text{Pic} \ \bar{B} = 0 \). Thus we have a surjection \( (\bar{B}/I\bar{B})^*/(\text{Image} \ \bar{B}^*) \to \text{Pic} \ B \). Since \( \bar{B} \cong \bigoplus \limits_{i=1}^d k[t_i] \), \( \bar{B}^* \cong \bigoplus \limits_{i=1}^d k^* \), also \( \bar{B}/I\bar{B} \cong \bigoplus \limits_{i=1}^d k[t_i]/(t_i^{n_i}) \) for some exponents \( n_i > 0 \). Now \( \left( \frac{k[t_i]}{(t_i^n)} \right)^* \)

\[
= k^*R_n,
\]

where \( R_n \) is \( p^r \)-torsion for any \( v \) such that \( p^r \geq n \) (this boils down to the identity

\[
(1 + a_1 t + a_2 t^2 + \ldots + a_d t^d)^{p^r} = 1 + a_1^{p^r} t^{p^r} + \ldots + a_d^{p^r} t^{d \cdot p^r}.
\]

Thus if \( p^r \geq \sup n_i \), then \( p^r \cdot (\text{Pic} \ B) = 0 \). If we can find a value of \( v > 0 \)
such that \( v \) is independent of the choice of the initial point \( P \in U \), and
the hyperplane sections \( f_1, \ldots, f_r \), then as in the case of plane curves we
can conclude that \( p^* \cdot A_0(\text{Spec } A) = 0 \); hence \( A_0(\text{Spec } A) = 0 \) by divisibility.
The rest of the proof will consist in showing that by shrinking \( U \) to some non-empty open subset \( V \),
and for any choices of \( f_1, \ldots, f_r \) corresponding to any given point \( P \in V \), the resulting exponent \( v \) is bounded
by some preassigned number depending only on \( A \).

Factor the inclusion \( B \subset B\) as \( B \subset \bigoplus_{i=1}^d B/\mathcal{P}_i \) and
\[
\bigoplus_{i=1}^d B/\mathcal{P}_i \subset \bigoplus_{i=1}^d (B/\mathcal{P}_i) \cong \bigoplus_{i=1}^d k[r].
\]

If \( J_1 \) and \( J_2 \) are the respective conductors, then \( J_1 J_2 B \subset IB = I \). Thus if
\( J_1 B = \bigoplus t^{n_i} k[r] \), and \( J_2 B = \bigoplus t^{n_2} k[r] \), it is enough to separately bound
the exponents \( n_{i_1} \) and \( n_{i_2} \) for all \( i \).

Let \( A = k[\phi_1, \ldots, \phi_s] \) where \( \phi_i \in A \) is homogeneous of degree \( m_i \) for
each \( i \). For any homogeneous prime \( \mathcal{P}_o \) of \( A \) of height \( r \) (corresponding
to a point \( \mathcal{P}_o \in \text{Proj } A \)), \( A/\mathcal{P}_o \cong k[r] \), where \( t \) has degree \( e \) (say) in the
grading induced from \( A \). Thus \( A/\mathcal{P}_o \subset k[u] \), where \( u^e = t \), and \( u \) has
degree 1. Clearly \( \phi_i \) is mapped to an element of degree \( m_i \) in \( k[u] \), which
is homogeneous i.e. \( \phi_i \mapsto x_i u^{m_i} \) for some \( x_i \in k \). So \( A/\mathcal{P}_o \cong
\cong k[\alpha_1 u^{m_1}, \ldots, \alpha_s u^{m_s}] \subset k[u] \). Now \( \alpha_i = 0 \Leftrightarrow \phi_i \) (considered as a section
of \( \mathcal{O}(m_i) \)) vanishes at \( \mathcal{P}_o \in \text{Proj } A \). Hence deleting the finite set of zeroes
of the sections \( \phi_1, \ldots, \phi_s \in \Gamma(\text{Proj } A, \bigoplus \mathcal{O}(m)) \), and correspondingly shrinking
our open set \( U \subset \text{Spec } A - \{0\} \) to a smaller open set \( V \), we may
assume that none of the \( \alpha_i \) vanish when \( \mathcal{P}_o \) is one of the primes we
construct by taking hyperplane sections of \( \text{Proj } A \). But if \( \alpha_1, \ldots, \alpha_s \) are
non-zero, then \( k[\alpha_1 u^{m_1}, \ldots, \alpha_s u^{m_s}] = k[u^{m_1}, u^{m_2}, \ldots, u^{m_s}] \) i.e. all the \( B/\mathcal{P}_i \)
are isomorphic. Since \( m_1, \ldots, m_s \) depends only on \( A \), this bounds the
exponents \( n_{i_2} \) for \( J_2 \).

We claim that if \( J = \bigcap_i (\mathcal{P}_i + \bigcap_{j \neq i} \mathcal{P}_j) \), then \( J \subset J_1 \). By definition of
the conductor, \( J_1 \) is the largest ideal in \( B \) which is a \( \bigoplus B/\mathcal{P}_r \)-module.

Hence, to verify the claim, it is enough to prove the following –

given \( a_1, \ldots, a_d \in J \), there exists \( a \in B \) such that \( a - a_i \in \mathcal{P}_i \) (i.e.
\( a \mapsto (\bar{a}_1, \ldots, \bar{a}_d) \) under \( B \subset \bigoplus B/\mathcal{P}_i \). But if \( a_i = b_i + c_i \) with \( b_i \in \mathcal{P}_i \),
\( c_i \in \bigcap_{j \neq i} \mathcal{P}_j \), then \( a = \sum_{i=1}^d c_i \) works, since \( a - a_i = \sum_{j \neq i} c_j - b_i \in \mathcal{P}_i \) as
desired. Now suppose \( f \in J \) satisfies \( f = (\beta_1 t^{r_1}, \ldots, \beta_d t^{r_d}) \) where
\( \beta_1, \ldots, \beta_d \neq 0 \). Then clearly \( v_i \geq n_{i_2} \) for all \( i \). So if we can suitably bound \( v_i \),
we will be done. In fact, by symmetry it suffices to find \( f \) with suitably bounded \( v_1 \), and \( \beta_1 \neq 0 \).
Consider the homomorphisms $A \to B \to B/\mathcal{P}_i$, where we identify $B/\mathcal{P}_i$ with $k[u^m, \ldots, u^m_s] \subset k[u]$. Then for each $i \neq 1$, we can find $\mu, \nu$ such that $1 \leq \mu, \nu \leq s$, and $m_i^\mu \cdot m_i^\nu - m_i^\mu \cdot m_i^\nu \neq 0$ (i.e. if two points of Proj $A$, namely $P_i$ and $P_s$, are distinct, then they have distinct "weighted homogeneous" coordinates – recall that $\phi_i \mapsto u^m_i \cdot m_i^\mu$ under $A \to B \to B/\mathcal{P}_i$).

Let $\gamma \in A$ be the element defined by $\gamma = \gamma_{\mu, \nu} = m_i^\mu \cdot \phi_i^\mu - m_i^\mu \cdot \phi_i^\nu$. Then the image of $\gamma$ in $B/\mathcal{P}_i$ is

$$\alpha_i^{m_i^\mu} \cdot (\alpha_i^\mu \cdot u^m_i)^{m_i^\nu} - \alpha_i^{m_i^\nu} \cdot (\alpha_i^\nu \cdot u^m_i)^{m_i^\mu} = (\alpha_i^{m_i^\mu} \cdot \alpha_i^\mu - \alpha_i^{m_i^\nu} \cdot \alpha_i^\nu) \cdot u^m_i = 0$$

i.e. the image $\bar{\gamma}$ of $\gamma$ in $B$ actually lies in $\mathcal{P}_i$. The image of $\gamma$ in $B/\mathcal{P}_i$ is

$$[\alpha_i^{m_i^\mu} \cdot \alpha_i^\mu - \alpha_i^{m_i^\nu} \cdot \alpha_i^\nu] \cdot u^m_i = \delta_i \cdot u^i,$$

where $\delta_i \neq 0$, and $t_i$ is bounded by a number depending only on $A$. Since $\mathcal{P}_i + J = \mathcal{P}_i + \bigcap_{j \neq i} \mathcal{P}_j$, the element $\gamma_0 = \prod_{i=2} \bar{\gamma}_{i, i}$ is such that the image of $\gamma_0$ in $B/\mathcal{P}_i$ is actually in $J \cdot B/\mathcal{P}_i$, and hence in $J_1 \cdot B/\mathcal{P}_1$. But $\gamma_0 \cdot k[u] = u^{t_2} + \ldots + t_4 k[u]$, and $t_2 + \ldots + t_4$ is bounded by a number depending only on $A$.

This completes the proof of Theorem 1.

COROLLARY (1.3): Let $A$ be as in Theorem 1. Then if $\dim A = 2$, we have $K_0(A) = \mathbb{Z}$. Hence all vector bundles on Spec $A$ are trivial.

PROOF: By a remark of Murthy (see [1]) we know that $\text{Pic} A = (0)$. By the standard argument using the cancellation theorem of Bass, it suffices to prove that vector bundles of rank 2 represent trivial elements of $K_0(A)$ to prove that $K_0(A) = \mathbb{Z}$. Now we can find a section of a given vector bundle of rank 2 which has isolated zeroes at smooth points of Spec $A$. If $P$ is the projective $A$-module of global sections of the bundle, we have an exact sequence

$$0 \to L \to P^* \to I \to 0,$$

where $I \subset A$ is the ideal of zeroes of the chosen section, $P^*$ is the dual projective module. An argument using the determinant shows that $L \cong A \cdot P^* \in \text{Pic} A$ i.e. $L \cong A$. Next, $[A/I] \in K_0(A)$ gives an element of $A_0(\text{Spec} A)$ which is trivial by Theorem 1. Hence $[A] = [I]$ in $K_0(A)$. Putting these facts together gives $[P^*] \cong [A^{\otimes 2}]$ i.e. all vector bundles of rank 2 are stably trivial. This proves $K_0(A) = \mathbb{Z}$. Now the cancellation theorem of Murthy and Swan [4] proves that all vector bundles are trivial.

The argument needed to deduce the triviality of vector bundles from the vanishing of the Chow group works in all characteristics (see section 2 of this paper).

§2) Some positive results in characteristic 0

In this section we obtain partial positive results for cones over
smooth projective curves in characteristic zero. Our result is a slight improvement on results of Varley (see [3]). The proof is based on an idea of Ojanguren [8], who used it to prove the result for plane cubics.

**THEOREM 2:** Let $X \subset \mathbb{P}^n_k$ be a projectively normal curve over the algebraically closed field $k$ of characteristic 0. Assume that $X$ is not contained in a hyperplane, and has degree at most $2n - 1$. Then $A_0(C(X)) = 0$, where $C(X) \subset \mathbb{A}^{n+1}$ is the affine cone over $X$. Hence vector bundles on $C(X)$ are trivial. (See the proof of Corollary (1.3).)

**PROOF:** The triviality of vector bundles follows from the vanishing of the Chow group, using the triviality of line bundles (a remark of Murthy – see [1]) and the cancellation theorem of Murthy and Swan [4]. The proof of the vanishing of $A_0(C(X))$ is based on two lemmas.

Let $\deg X = d \leq 2n - 1$, and set $r = d - n$.

**LEMMA (2.1):** Assume that $P \in X$ is not a Weierstrass point. Let $H \subset \mathbb{P}^n$ be the osculating hyperplane to $X$ at $P$, so that the zero cycle $(H \cdot X) = n(P) + \sum_{i=1}^r (P_i)$ (where the $P_i \in X$ may not be distinct from each other or from $P$, in general). Then $\{P, P_1, \ldots, P_r\}$ span a $\mathbb{P}^r \subset \mathbb{P}^n$.

**PROOF:** Suppose that $\{P, P_1, \ldots, P_r\} \subset L \subset \mathbb{P}^n$, where $L$ is a linear space of dimension $r - 1$. Since the space $\tilde{L}$ of hyperplanes (in the dual projective space) which contain $L$ is a $\mathbb{P}^{n-r}$, we have

$$h^0(\mathcal{O}_X(D - P - \sum_{i=1}^r P_i)) \geq n - r + 1 \quad \text{(where } \mathcal{O}_X(D) = \mathcal{O}_X(1))$$

Choosing the representative $n(P) + \sum_{i=1}^r (P_i) \in |D|$, we have

$$h^0(\mathcal{O}_X(n - 1)P)) \geq n - r + 1.$$ 

Now $\deg X = d$, and $\dim |D| = n$. Since $n > d/2$, the divisor $D$ is non-special by Clifford's Theorem (see [7], ch. IV). Hence by the Riemann–Roch theorem, the genus $g$ of $X$ satisfies

$$g = \deg D - \dim |D| = d - n = r.$$ 

Since $n - 1 = g + (n - r - 1)$,

$$h^0(\mathcal{O}_X(n - 1)P)) \geq n - r + 1 \Rightarrow h^0(\mathcal{O}_X(gP)) \geq (n - r + 1) - (n - r - 1) = 2$$

i.e. $P \in X$ is a Weierstrass point.
**Lemma (2.2):** There is a non-empty open set $U \subset X$ with the following property – if $P \in U$, then there exist points $P_0, P_1, \ldots, P_{r-1}$ of $X$ such that (i) $P, P_0, \ldots, P_{r-1}$ span a $P^r \subset P^n$, and (ii) if $H$ is the osculating hyperplane to $P_0$, then $(H \cdot X) = n(P_0) + (P) + \sum_{i=1}^{r-1} P_i$.

**Proof:** The set $S$ in the dual projective space of hyperplanes which parametrizes osculating hyperplanes is birational to $X$. As $s$ ranges over $S$, the hyperplane sections $(H(s) \cdot X)$ have the form $(H(s) \cdot X) = nP(s) + \sum_{i=1}^{r-1} P_i(s)$ (through the individual $P_i(s)$ don't make sense, the zero cycle $\sum_{i=1}^{r-1} P_i(s)$ does). Then the lemma amounts to the claim that $\sum_{i=1}^{r-1} P_i(s)$ is not independent of $s$. Suppose that the lemma is false. Let $L \subset P^n$ be the $P^{r-1}$ spanned by $P_1, \ldots, P_r$ (for general $s \in S$, $P_1(s), \ldots, P_r(s)$ span a $P^{r-1}$, by Lemma (2.1)). Projection from $L$ to a suitable $P^{n-r}$ yields a curve $\overline{X} \subset P^{n-r}$ with the following property – if $P \in \overline{X}$ is general, then there exists a hyperplane $H_p \subset P^{n-r}$ such that the local intersection multiplicity $(H_p \cdot \overline{X})_p \geq n$ (choose $H_p$ to be the image of a suitable osculating hyperplane to $X$). But this is impossible – at a general point of a curve in $P^{n-r}$, the maximum local intersection multiplicity with a hyperplane is $n - r$. This contradiction finishes the proof of the lemma.

We now prove theorem 2. Let $0 \in C(X)$ be the vertex, and $\phi : C(X) - (0) \rightarrow X$ the projection. Let $P \in \phi^{-1}(U)$, where $U \subset X$ is the open set of lemma (2.2). Then if $\overline{P} = \phi(P)$, we can find $P_0, \ldots, P_{r-1} \in X$ and a hyperplane $H$ such that $(H \cdot X) = n(P_0) + (\overline{P}) + \sum_{i=1}^{r-1} P_i$. Then $\phi^{-1}(H) \cup \{0\} \cong \mathbb{A}^n \subset \mathbb{A}^{n+1}$ (by abuse of notation, let $\phi$ also denote the projection $\mathbb{A}^{n+1} - \{0\} \rightarrow \mathbb{P}^n$). Also, $(\phi^{-1}(H) \cup \{0\}) \cap C(X) = I_0 \cup I_1 \ldots \cup I_{r-1} \cup \bar{I}$, where $I_i$ and $\bar{I}$ are lines through $0$ in $\phi^{-1}(H) \cup \{0\}$. Since $P, P_0, \ldots, P_{r-1}$ can be chosen to span a $P^r \subset P^n$ by lemma (2.2), the lines $\bar{I}, I_0, \ldots, I_{r-1}$ span an $\mathbb{A}^{r+1} \subset \phi^{-1}(H) - \{0\}$, and satisfy $\phi(I_i - \{0\}) = P_i$, $\phi(\bar{I} - \{0\}) = \overline{P}$, and $P \in \bar{I} - \{0\}$. The lines $\bar{I}, I_1, \ldots, I_{r-1}$ occur with multiplicity 1 in the intersection $(\phi^{-1}(H) \cup \{0\}) \cap C(X)$, while $I_0$ occurs with multiplicity $n$.

There exists a unique linear subspace $L \cong \mathbb{A}^r$, with $L \subset \text{span} \{\bar{I}, I_0, \ldots, I_{r-1}\}$, such that $P \in L$, and $L \cap \text{span} \{I_0, \ldots, I_{r-1}\} = \phi$. (This is just the unique $\mathbb{A}^r$ through $P$ which is parallel to $\text{span} \{I_0, \ldots, I_{r-1}\} \cong \mathbb{A}^n$). If $f = 0$ is the equation of $L$ in the affine space $\mathbb{A}^{r+1} = \text{span} \{I_0, I_1, \ldots, I_{r-1}\}$, then the restriction of $f$ to the curve $Y = (\phi^{-1}(H) \cup \{0\}) \cap C(X)$ is a regular function on $Y$ whose divisor of zeroes is $(P)$. Thus $(P) = 0$ in Pic$^0(Y)$, and hence in $A_0(C(X))$. By lemma (1.1), this proves the result.

We easily deduce theorem 2' from theorem 2.
**THEOREM 2':** Let $X$ be a smooth curve of genus $g$ over an algebraically closed field of characteristic $0$. Let $D$ be a divisor on $X$ such that $\deg D \geq 2g + 1$. (Thus $D$ is very ample – see [7], ch. IV). Assume that $X$ is projectively normal in this embedding. Then $A_0(A) = 0$, where $A = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD))$.

**PROOF:** Since $\deg D \geq 2g + 1$, $D$ is non-special. Hence by the Riemann–Roch theorem, $n = \dim |D| = \deg D - g$. We claim that $\deg D \geq 2n - 1$ (so that theorem 2 applies). For

$$2n - 1 - \deg D = 2(\deg D - g) - 1 - \deg D = \deg D - (2g + 1) \geq 0.$$

**REMARK:** In fact, a result of Castelnuovo implies that for the range of degrees in theorem 2', $X$ will always be projectively normal. See [12], p. 52.

**§3) A counterexample in characteristic 0**

In this section we construct examples of cones over projectively normal complex curves which admit non-trivial vector bundles. Let $L$ denote the field of algebraic numbers.

**THEOREM 3:** Let $X \subset \mathbb{P}^n_L$ be a projectively normal curve such that $H^1(X, \mathcal{O}_X(1)) \neq 0$. Then $K_0(C(X_C)) \neq \mathbb{Z}$.

**COROLLARY (3.1):** Let $X$ be a non-hyperelliptic curve, defined over $L$. Then $K_0(A) \neq \mathbb{Z}$, where $A = \bigoplus_{n \geq 0} H^0(X_C, \omega_{X_C}^n)$. (The cone over the canonical embedding.)

**COROLLARY (3.2):** Let $X \subset \mathbb{P}^2_L$ be a smooth curve of degree at least 4. Then $C(X_C)$ admits non-trivial vector bundles.

This is in contrast to the situation in characteristic $p > 0$, and to the situation for analytic vector bundles (since any analytic vector bundle on a contractible Stein space is trivial). The method of proof is based on an idea of Spencer Bloch. He showed that $\mathbb{C}[x, y, z]/(z^7 - x^2 - y^3)$ provides a counterexample to the statement of theorem 1 in characteristic 0. Let me sketch his idea.

Let $X = \text{spec } \mathbb{C}[x, y, z]/(z^7 - x^2 - y^3)$. Then the origin is the only sin-
gular point of $X$. Let $\tilde{X}$ be a projective surface containing $X$ as an open subset, such that $\tilde{X}_{\text{sing}} = X_{\text{sing}} = \{0\}$, the origin. Let $\pi: \tilde{X} \to X$ be a resolution of the singularity. Then $\tilde{X}$ can be chosen so that $\pi^{-1}(\{0\})$ is a cuspidal rational curve $E$. Now $K_0(\tilde{X}) = \mathbb{Z} \oplus \text{Pic}(\tilde{X})$, and $SK_1(\tilde{X}) \cong \text{Pic}(\tilde{X}) \otimes \mathbb{C}^* \cong (\mathbb{C}^*)^{\otimes n}$ for some $n$, since $\tilde{X}$ is a rational surface. Also $SK_1(E) \cong \Omega^1_{\mathbb{C}/\mathbb{Z}}$, the module of Kahler differentials of $\mathbb{C}$ (see [7] for the definition, some properties and references). Since $\mathbb{C}$ has uncountable transcendence degree over $\mathbb{Q}$, and $\text{Hom}_{\mathbb{C}}(\Omega^1_{\mathbb{C}/\mathbb{Z}}, \mathbb{C}) = \text{(vector space of all derivations } \mathbb{C} \to \mathbb{C})$, $\Omega^1_{\mathbb{C}/\mathbb{Z}}$ is a $\mathbb{C}$-vector space of uncountable dimension.

Now one considers the diagram

$$
\begin{array}{cccccc}
K_1(\tilde{X}) & \xrightarrow{\alpha} & K_1(E) & \xrightarrow{\beta} & K_0(\tilde{X}, E) & \to \to K_0(\tilde{X}) \to K_0(E) \\
\uparrow & & \uparrow & & \pi^* & \uparrow \& \uparrow \\
K_1(X) & \xrightarrow{\alpha'} & K_1(\{0\}) & \xrightarrow{\beta'} & K_0(X, \{0\}) & \to \to K_0(X) \to K_0(\{0\})
\end{array}
$$

Here $K_0(\tilde{X}, E)$ and $K_0(\tilde{X}, P)$ are relative $K$-groups (we give the definitions below). Clearly $\alpha'$ is onto, as $K_1(\{0\}) = \mathbb{C}^*$; hence $\beta' = 0$. It turns out that points of $\tilde{X} - \{0\}$ admit cycle classes in $K_0(\tilde{X}, \{0\})$, and similarly for $K_0(\tilde{X}, E)$. Define $F_0K_0(\tilde{X}, E)$ to be the subgroup of $K_0(\tilde{X}, E)$ generated by classes of points of $\tilde{X} - E$, and similarly define $F_0K_0(X, \{0\})$. Evidently $\pi^*: F_0K_0(\tilde{X}, \{0\}) \to F_0K_0(\tilde{X}, E)$ as $\pi: \tilde{X} - E \cong X - \{0\}$. One main ingredient of the proof is a geometric description of $\beta|_{SK_1(E)}$. A class in $SK_1(E)$ is represented by finite sets of points of $E - E_{\text{sing}}$, together with non-zero elements of the residue fields of each of the points. If $P_1, \ldots, P_r \in E - E_{\text{sing}}$, and $\alpha_1, \ldots, \alpha_r \in \mathbb{C}^*$ (where we think of $\alpha_i \in \mathbb{C}(P_i)^*$), we choose a curve $C \subset \tilde{X}$ which meets $E$ transversally at $P_1, \ldots, P_r$. If $C$ meets $E$ at additional points $P_{r+1}, \ldots, P_s$, assume that these intersections are also transverse and the points $P_i \in E$ are all smooth. Let $\alpha_i \in \mathbb{C}(P_i)^*$ be set equal to 1 for $r + 1 \leq i \leq s$. Choose a rational function $f \in \mathbb{C}(C)^*$, such that $f(P_i) = \alpha_i$ ($1 \leq i \leq s$). Then the element $\beta((\{P_1, \alpha_1\}, \ldots, \{P_s, \alpha_s\}) = (\text{cycle class of the divisor of } f) \in F_0K_0(\tilde{X}, E)$. Once one has this, one can show that $F_0K_0(\tilde{X}, E) \neq 0$, and hence $F_0K_0(\tilde{X}, \{0\}) \neq 0$, as desired.

In our case, we have to work harder, because $SK_1$ of the ambient surface maps onto $SK_1$ of the exceptional set when we resolve the singularity of the cone. However, if we work with a multiple of the exceptional set, then obstructions to the triviality of vector bundles appear.

Let $X_L \subset \mathbb{P}^n_L$ be our given curve, and let $Y_L$ be the affine cone over $X_L$. We will make use of the following convention – unless "L" appears as a subscript on the symbol for a variety, we will be working over $\mathbb{C}$. Let $\tilde{Y}$ be the blow up of $Y$ at $P$. Then $\tilde{Y} \cong V(\mathcal{O}_X(-1))$, a ruled surface over $X$, and the exceptional set $\pi^{-1}(P) = E_0$ (where $P \in Y$ is the vertex) is a section of $\tilde{Y} \to X$ with normal bundle $\cong \mathcal{O}_X(-1)$. Let $E$ be the subscheme
2E_0; thus if I is the sheaf of ideals of E_0 on Y, then E is defined by the sheaf I^2. We write “2P” for the scheme spec A/M^2, where Y = spec A, and M is the maximal ideal of P.

For any scheme T, let \( \mathcal{P}(T) \) denote the category of locally free sheaves of finite rank; \( \mathcal{H}(T) \) will denote the category of coherent sheaves of finite homological dimension on T. If \( S \subseteq T \) is a subscheme, let \( \mathcal{H}(S, T) \) denote the category of \( \mathcal{O}_T \)-modules which are coherent, of finite homological dimension, and vanish on \( T - S \). If S is a single point \( x \in T \), we may write \( \mathcal{H}_x \) for \( \mathcal{H}(x, T) \).

Now we define the relative K-groups and cycle classes in them. Let \( i: Y \hookrightarrow X \) be a closed immersion. We have a natural map \( i^* : \mathcal{P}(X) \rightarrow \mathcal{P}(Y) \). For any exact category \( \mathcal{C} \), let \( BQ\mathcal{C} \) be the topological space (together with its natural base point) as defined by Quillen [9].

Then we have a natural map (of based spaces) \( BQ\mathcal{P}(X) \xrightarrow{i^*} BQ\mathcal{P}(Y) \).

Let \( F(i^*) \) denote the homotopy fiber of \( i^* \) (for a map \( (X, P) \rightarrow (Y, P') \) of based spaces, the homotopy fiber is the set of pairs \( (\omega, x) \) where \( x \in X \), \( \omega: [0, 1] \rightarrow Y \) is a path, with \( \omega(0) = P', \omega(1) = f(x) \). The base point is \( (\omega_0, P) \) where \( \omega_0: [0, 1] \rightarrow P' \). One of the basic properties of the homotopy fiber is that its homotopy groups fit into a long exact sequence with those of the domain and range. So if we set \( K_n(X, Y) = \pi_{n+1}(F(i^*), \ast) \), where \( \ast \in F(i^*) \) is the base point, then we have a long exact sequence

\[
\ldots \rightarrow K_d(X, Y) \rightarrow K_n(X) \rightarrow K_n(Y) \rightarrow K_{n-1}(X, Y) \rightarrow \ldots
\]

A general reference for the definitions and basic properties of higher K-groups is the fundamental paper [9] of Quillen. A summary of Quillen’s results, and some applications to questions in the theory of algebraic cycles, can be found in Bloch’s lecture notes [13].

Let \( Z \subseteq X - Y \) be a subscheme, closed in X, and of finite homological dimension. Then we claim that there is a natural cycle class \( [Z] \in K_0(X, Y) \). To construct it, we use the category \( \mathcal{H}_0(X) \subseteq \mathcal{H}(X) \), defined to be the full subcategory consisting of all coherent \( \mathcal{O}_X \)-modules \( \mathcal{F} \) satisfying \( \text{Tor}^i_\mathcal{O}(\mathcal{F}, \mathcal{O}_Y) = 0 \) for \( i > 0 \). Then the map \( i: Y \hookrightarrow X \) induces a functor \( i^* : \mathcal{H}_0(X) \rightarrow \mathcal{H}(Y) \). By Quillen’s resolution theorem [9], the maps \( BQ\mathcal{P}(X) \rightarrow BQ\mathcal{H}_0(X) \) and \( BQ\mathcal{P}(Y) \rightarrow BQ\mathcal{H}(Y) \) are homotopy equivalences. Hence the natural induced map between the homotopy fibers \( F(i^*) \) and \( F(i^*) \) is also a homotopy equivalence. The inclusion \( j: Z \hookrightarrow X \) induces a functor \( j_* : \mathcal{P}(Z) \rightarrow \mathcal{H}_0(X) \), since \( Z \cap Y = \emptyset \); and the composite functor \( i^* \circ j_* : \mathcal{P}(Z) \rightarrow \mathcal{H}(Y) \) is the 0-functor. Hence the induced map \( BQ\mathcal{P}(Z) \rightarrow BQ\mathcal{H}(Y) \) maps everything to the base point.
Hence we have an induced natural map $BQ\mathcal{P}(Z) \to F(i^*)$, and thus a map $K_0(Z) \to K_0(X, Y)$. The image of $[\mathcal{O}_Z] \in K_0(Z)$ under this map is the required cycle class; by construction, it maps to the usual cycle class in $K_0(X)$ under the natural map $K_0(X, Y) \to K_0(X)$.

The relative $K$-groups, and the cycle classes, enjoy the following naturality properties. If $i : Y \subset X$ and $i' : Y' \subset X'$, and $\pi : X \to X'$ is a morphism such that $\pi^{-1}(Y) = Y$, then we have a diagram

$$
\cdots \to K_n(X, Y) \to K_n(X) \to K_n(Y) \to K_{n-1}(X, Y) \to \cdots
$$

$$
\uparrow \quad \uparrow \quad \uparrow \quad \uparrow
$$

$$
\cdots \to K_n(X', Y') \to K_n(X') \to K_n(Y') \to K_{n-1}(X', Y') \to \cdots
$$

Further, let $Z' \subset X'$ be a subscheme of finite homological dimension, satisfying $Z' \cap Y' = \emptyset$, and $\operatorname{Tor}_i^{\mathcal{O}_{X'}}(\mathcal{O}_{Z'}, \mathcal{O}_{X'}) = 0$ for $i > 0$. Let $\mathcal{H}_1(X')$ denote the category of coherent $\mathcal{O}_{X'}$-modules $\mathcal{F}$ of finite homological dimension satisfying $\operatorname{Tor}_i^{\mathcal{O}_{X'}}(\mathcal{F}, \mathcal{O}_{Y'}) = 0$ for $i > 0$. Then consider the commutative square of categories (where $Z = \pi^{-1}(Z')$)

$$
\mathcal{P}(Z) \to \mathcal{H}_0(X)
$$

$$
\pi^* \uparrow \quad \uparrow \pi^*
$$

$$
\mathcal{P}(Z') \to \mathcal{H}_1(X')
$$

This gives the equation $\pi^*([Z']) = [Z]$ in $K_0(X, Y)$. Two cases where the hypothesis are satisfied are when $X - Y \to X' - Y'$ is an open immersion, and when the map $\pi$ is flat. In our applications, we only work with $K_0(X, Y)$ where $X - Y$ is smooth.

There is one technical point that we systematically ignore. When we say that a diagram of categories

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{F} & \mathcal{B} \\
H \downarrow & & \downarrow G \\
\mathcal{C} & \xrightarrow{K} & \mathcal{D}
\end{array}
$$

commutes, what we often mean is that the functors $G \circ F$ and $K \circ H$ are naturally equivalent. Thus, the corresponding diagram of classifying spaces

$$
\begin{array}{ccc}
B\mathcal{A} & \xrightarrow{B(F)} & B\mathcal{B} \\
B(H) \downarrow & & \downarrow B(G) \\
B\mathcal{C} & \xrightarrow{B(K)} & B\mathcal{D}
\end{array}
$$
only commutes up to homotopy. The induced map on homotopy fibers 
\( F(B(F)) \rightarrow F(B(K)) \) depends on the choice of this homotopy i.e. on the 
choice of the equivalence of functors \( G \circ F \cong K \circ H \). However, in all our 
situations, there is always one “natural” choice of the equivalence – for 
example, there is an obvious choice of a natural isomorphism 
\( (M \otimes_A N) \otimes_B P \cong M \otimes_A (N \otimes_B P) \); this is the kind of choice which has to 
be made consistently. More details will appear in my thesis. I wish to 
thank Professor Swan for pointing out that some care is needed here.

We need to make use of certain results from \( K \)-theory. We give them 
in a sequence of lemmas. Recall that \( L \) denotes the field of algebraic 
numbers.

**Lemma (3.2) (Van der Kallen [14]):** Let \( \mathcal{O} \) be a regular local ring con-
taining \( L \), and let \( \mathcal{O}[\tau]/(\tau^2) \) be the ring of dual numbers over \( \mathcal{O} \). Then 
\( K_2(\mathcal{O}[\tau]/(\tau^2)) \) fits into the exact sequence

\[
0 \rightarrow \Omega^1_{\mathcal{O}/L} \rightarrow K_2(\mathcal{O}[\tau]/(\tau^2)) \rightarrow K_2(\mathcal{O}) \rightarrow 0.
\]

The isomorphism \( \ker(K_2(\mathcal{O}[\tau]/(\tau^2)) \rightarrow K_2(\mathcal{O})) \cong \Omega^1_{\mathcal{O}/L} \) is given follows:
the kernel is generated by symbols of the form \( \{u, 1 + vt\} \) where \( u \in \mathcal{O}^*, \)
v \( \in \mathcal{O} \), and

\[
\{u, 1 + vt\} \rightarrow v \cdot \frac{du}{u} \in \Omega^1_{\mathcal{O}/L}.
\]

(Note that \( \Omega^1_{\mathcal{O}/L} = \Omega^1_{\mathcal{O}/L^2} \), since \( L/Q \) is separable algebraic.)

From now on, all differentials will be relative to \( L \) unless indicated
otherwise.

**Lemma (3.3) (Localisation sequence [11]):** Let \( U \hookrightarrow X \) be an open im-
mersion, where \( U \) is affine, and \( X - U \) is defined by an ideal sheaf which is 
locally principal and generated by a non zero-divisor. Let \( H \) be the 
category of coherent \( \mathcal{O}_X \)-modules which are 0 on \( U \) and have homological 
dimension 1 on \( X \). Then we have a localisation sequence

\[
\ldots \rightarrow K_{q+1}(U) \rightarrow K_q(H) \rightarrow K_q(X) \rightarrow K_q(U) \rightarrow \ldots.
\]

Now we come back to cones. Recall that \( E \subset \tilde{Y} \) is the non-reduced 
scheme \( "2E_0" \) where \( E_0 \) is the exceptional set. For any finite subscheme 
\( S \subset E \), the localisation sequence gives

\[
\ldots \rightarrow K_2(E) \rightarrow K_2(E - S) \rightarrow K_1(\mathcal{H}(S, E)) \rightarrow K_1(E) \rightarrow K_1(E - S) \rightarrow \ldots
\]
Taking limits over all such \( S \) (see Quillen [9], p. 96) we get

\[
\cdots \to K_2(E) \to K_2(F) \to \bigoplus_{x \in E} K_1(\mathcal{H}_x) \to K_1(E) \to K_1(F) \to \cdots
\]

where \( F \) is the local ring at the generic point of \( E \). Define \( SK_1(E) = \text{Ker}(K_1(E) \to K_1(F)) \). Then we have a presentation

\[
K_2(F) \to \bigoplus_{x \in E} K_1(\mathcal{H}_x) \to SK_1(E) \to 0.
\]

Since it is difficult to work with \( \mathcal{H}_x \), we wish to obtain another viewpoint on \( SK_1(E) \). To do this, replace \( E \) by \( \mathcal{O}_{x,E} \), for any closed point \( x \in E \), in the above argument. We obtain an exact sequence

\[
K_2(\mathcal{O}_{x,E}) \to K_2(F) \to K_1(\mathcal{H}_x) \to 0,
\]

because \( K_1(\mathcal{O}_{x,E}) \subseteq K_1(F) \) (since \( K_1(\text{local ring}) = \text{units} \)). Now let \( x \in E_0 \) be a smooth closed point; since any infinitesimal deformation of the regular local ring \( \mathcal{O}_{x,E_0} \) is trivial (as \( \mathcal{O}_{x,E_0} \) is essentially of finite type over \( \mathbb{C} \)), we see that \( \mathcal{O}_{x,E} \cong \mathcal{O}_{x,E_0}[t]/(t^2) \). Hence lemma (3.2) applies to give a diagram

\[
0 \to \Omega^1_{\mathcal{O}_{x,E_0}} \to K_2(\mathcal{O}_{x,E}) \to K_2(\mathcal{O}_{x,E_0}) \to 0
\]

\[
\downarrow \quad \downarrow \quad \downarrow
\]

\[
0 \to \Omega^1_{\mathcal{C}(E_0)} \to K_2(F) \to K_2(\mathcal{C}(E_0)) \to 0
\]

By a result of Dennis and Stein [10], \( K_2(\mathcal{O}_{x,E_0}) \subseteq K_2(\mathcal{C}(E_0)) \). Also, \( \Omega^1_{\mathcal{O}_{x,E_0}} \subseteq \Omega^1_{\mathcal{C}(E_0)} \), since \( \Omega^1_{\mathcal{O}_{x,E_0}} \) is a free \( \mathcal{O}_{x,E_0} \)-module, and the inclusion is just the localisation at the generic point. Hence \( K_2(\mathcal{O}_{x,E}) \subseteq K_2(F) \).

Let \( \eta \in E \) be the generic point; for any point \( x \in E \) let \( i_x : \{x\} \to E \). Then we have constructed an exact sequence of sheaves (for the Zariski topology)

\[
0 \to \mathcal{K}_{2,E} \to (i_\eta)_* K_2(F) \to \bigoplus_{x \in E \text{ closed}} (i_x)_* K_1(\mathcal{H}_x) \to 0,
\]

where \( \mathcal{K}_{2,E} \) is the sheaf associated to the presheaf \( U \mapsto K_2(\Gamma(U, \mathcal{O}_E)) \). Here \( K_2(F), K_1(\mathcal{H}_x) \) are regarded as constant sheaves supported on a subvariety.

Since \( (i_\eta)_* K_2(F), (i_x)_* K_1(\mathcal{H}_x) \) are flasque, they have no higher cohomology, and so we can use the above resolution of \( \mathcal{K}_{2,E} \) to compute its
cohomology. Hence, we obtain an isomorphism $SK_1(E) \cong H^1(E, \mathcal{K}_2, \mathcal{E})$, since both are presented as $\text{coker}(K_2(F) \to \bigoplus_{x \in E} K_1(\mathcal{K}_x))$.

Next, we go back to the identification of $\mathcal{O}_{x,E}$ with $\mathcal{O}_{x,E_0} \cdot (t)$. The map $\text{Ker}(K_2(\mathcal{O}_{x,E}) \to K_2(\mathcal{O}_{x,E_0}) \to \Omega^1_{\mathcal{E}_x, E_0}$ was given by $\{u, 1 + vt\} \mapsto v \frac{du}{u}$.

This is not quite canonical, as it involves the choice of $t$ generating $\text{Ker}(\mathcal{O}_{x,E} \to \mathcal{O}_{x,E_0})$. However, $v \frac{du}{u} \otimes t \in \Omega^1_{\mathcal{E}_x, E_0} \otimes \mathcal{E}_0 \cdot I/I^2$ is clearly canonical. Thus we obtain an exact sequence of sheaves

$$0 \to I/I^2 \otimes \mathcal{E}_0 \Omega^1_{\mathcal{E}_0} \to \mathcal{K}_2, E \to \mathcal{K}_2, E_0 \to 0.$$ 

In fact, this exact sequence splits naturally, using the fibration $Y \to X$ together with the isomorphism $X \cong E_0$ to split the inclusion $E_0 \subset E$. Hence, we have a naturally split exact sequence

$$0 \to H^1(E_0, \Omega^1_{\mathcal{E}_0} \otimes \mathcal{E}_0 \cdot I/I^2) \to SK_1(E) \to SK_1(E_0) \to 0.$$ 

Now $E_0 \cong X_L \times L \mathbb{C}$. Hence $\Omega^1_{\mathcal{E}_0} \cong (\Omega^1_{X_L} \otimes L \mathbb{C}) \oplus (\mathcal{O}_{X_L} \otimes L \mathcal{O}_L^1)$. This gives a corresponding splitting of $\Omega^1_{\mathcal{E}_0} \otimes \mathcal{E}_0 \cdot I/I^2$. Since $\Omega^1_{X_L} \otimes L \mathbb{C} = \Omega^1_{X/L},$ and $I/I^2 \cong \mathcal{O}_X(1)$, we have

$$H^1(E_0, (\Omega^1_{X_L} \otimes L \mathbb{C}) \otimes \mathcal{E}_0 \cdot I/I^2) = H^1(X, \mathcal{O}_X(1) \otimes \Omega^1_{X/L}) = 0$$

by Serre duality.

Again using $E_0 \cong X_L \times L \mathbb{C}$, and the Künneth formula, the other direct summand reduces to $H^1(X_L, \mathcal{O}_{X_L}(1)) \otimes \mathcal{O}_L^1$.

**Lemma (3.4):** $K_1(Y) \cong K_1(X)$, and the natural maps $K_1(Y) \to K_1(E)$ are injective. Further, $\text{coker}(K_1(Y) \to K_1(E)) \cong \text{Ker}(K_1(E) \to K_1(E_0))$.

**Proof:** Since $Y \to X$ is an $\mathbb{A}^1$-bundle, the first claim follows from [9], sec. 7, prop. (4.1). The remaining claims just exploit the fact that in $E_0 \subset E \subset Y \to X$, the composite $E_0 \to X$ is an isomorphism.

In particular, $H^1(X_L, \mathcal{O}_{X_L}(1)) \otimes \mathcal{O}_L^1 \subset K_0(Y, E)$. The next task is to imitate the geometric construction of the boundary map used by Spencer Bloch to show that at least some of these elements land in $F_0K_0(Y, E)$. 

From the diagram

\[
\begin{array}{cccc}
K_1(\tilde{Y}) & \longrightarrow & K_1(E) & \overset{\delta}{\longrightarrow} K_0(\tilde{Y},E) & \longrightarrow & K_0(\tilde{Y}) \\
\uparrow & & \pi^* & \pi^* & \uparrow \\
K_1(Y) & \longrightarrow & K_1(2P) & \overset{\psi}{\longrightarrow} K_0(Y,2P) & \phi & \longrightarrow K_0(Y)
\end{array}
\]

and the fact that \(SK_1(2P) = 0\), we claim that if \(\alpha \in H^1(X_L, \mathcal{O}_{X_L}(1) \otimes \Omega^1_L)\) is non-zero, and \(\delta \alpha = \pi^* \delta\), then \(\phi(\delta) \in K_0(Y)\) is also non-zero. For suppose \(\delta = \psi(\gamma)\). Then \(\pi^*(\gamma) - \alpha \in \text{Image}(K_1(\tilde{Y}) \to K_1(E))\), which maps isomorphically to \(K_1(E_0)\). But, by changing \(\gamma\) by an element of \(K_1(Y)\) (in fact, an element of \(\mathbb{C}^*\)) we may assume \(\gamma \mapsto 0\) in \(K_1(E)\). Clearly \(\pi^*(\gamma) \to 0\) in \(K_1(E_0)\), from

\[
\begin{align*}
K_1(E) & \to K_1(E_0) \\
\uparrow & \\
K_1(2P) & \to K_1(P)
\end{align*}
\]

Since \(\alpha \mapsto 0\) in \(K_1(E_0)\), \(\pi^*(\gamma) - \alpha \mapsto 0\) in \(K_1(E_0)\). But this forces \(\pi^*\gamma - \alpha = 0\) i.e. \(\pi^*(\gamma) = \alpha\). Since \(K_1(2P) \overset{\pi^*}{\longrightarrow} K_1(E) \to K_1(F)\) is injective (use the grading) while \(\alpha \in SK_1(E)\), this forces \(\alpha = 0\).

Now let \(C \subset \tilde{Y}\) be a smooth (possibly disconnected) affine closed curve. Then we claim there is a map between the sequence of \((C, C \cap E)\) and \((\tilde{Y}, E)\). Let \(j : C \hookrightarrow \tilde{Y}, j' : C \cap E \to E\). Then we have a diagram

\[
\begin{array}{ccc}
\mathcal{H}_0(\tilde{Y}) & \to & \mathcal{H}(E) \\
\mathcal{P}(C) & \to & \mathcal{P}(C \cap E)
\end{array}
\]

\((\mathcal{H}_0\) was defined when we introduced cycle classes).

This induces the maps between the sequences.

**Lemma (3.5):** Let \(C\) be a smooth affine curve, \(S \subset C\) a finite subscheme, \(\mathcal{O}_{S,C}\) the semilocal ring of \(S\) on \(C\). Then there is a commutative diagram (upto sign)

\[
\begin{array}{ccc}
\mathcal{O}^*_S & \overset{\eta}{\longrightarrow} & K_0(C,S) \\
\epsilon \uparrow & & \delta \\
\mathcal{O}^*_S \mathcal{O}_L & \overset{\epsilon}{\longrightarrow} & K_1(S)
\end{array}
\]
where $\partial : K_1(S) \to K_0(C, S)$ is the boundary map of the pair $(C, S)$, $\varepsilon : \mathcal{O}_{S, C}^* \to K_1(S)$ is the natural map on units, and $\eta$ sends $f \in \mathcal{O}_{S, C}^*$ to the cycle class of the divisor $(f)$ of $f$ on $C$.

PROOF: It clearly suffices to check that $\partial \circ \varepsilon(f) = \eta(f)$ for all $f \in \text{Image}(\mathcal{O}_C \to \mathcal{O}_{S, C} \cap \mathcal{O}_{S, C}^*)$. Such an $f$ can be regarded as a morphism $C \to \mathbb{A}^1$, and $[(f)] \in K_0(C, S)$ is just $f^*([0])$, where $[0] \in K_0(\mathbb{A}^1, f(S))$, and $f(S) \subset \mathbb{A}^1 - \{0\}$. So we are reduced to checking the claim in the case when $C \cong \mathbb{A}^1, S \subset \mathbb{A}^1 - \{0\}$, and $f = t$, the standard function on $\mathbb{A}^1$. The image of $t$ in $K_1(S)$ is a unit. If $\mathbb{A}^1 = \text{Spec } k[t], S = \text{Spec } k[t]/I$, then we have a diagram of rings

$$
\begin{array}{ccc}
k[t] & \to & k[t, t^{-1}] \\
\downarrow & & \downarrow \\
k[t] & \to & k[t]/I
\end{array}
$$

This induces a map between the localisation sequence for $(k[t], k[t, t^{-1}])$ and the exact sequence of the pair $(\mathbb{A}^1, S)$. In terms of categories, we have a diagram

$$
\begin{array}{ccc}
\mathcal{H}_0(\mathbb{A}^1) & \to & \mathcal{H}_0(\mathbb{G}_m) \\
\downarrow & & \downarrow \\
\mathcal{H}_0(\mathbb{A}^1) & \to & \mathcal{H}(S)
\end{array}
$$

Hence we have a diagram of spaces

$$
\begin{array}{ccc}
\mathcal{B} \mathcal{Q} \mathcal{P}(\{0\}) & \to & \mathcal{B} \mathcal{Q} \mathcal{H}_0(\mathbb{A}^1) \\
\downarrow & & \downarrow \\
F(i^*) & \to & \mathcal{B} \mathcal{Q} \mathcal{H}_0(\mathbb{A}^1) \to \mathcal{B} \mathcal{Q} \mathcal{H}(S)
\end{array}
$$

since the homotopy fiber in the localisation sequence is known to be homotopy equivalent to $\mathcal{B} \mathcal{Q} \mathcal{P}(\{0\})$. The induced map $\mathcal{B} \mathcal{Q} \mathcal{P}(\{0\}) \to \mathcal{B} \mathcal{Q} \mathcal{H}_0(\mathbb{A}^1)$ is the one which was used to define the cycle class of $[0]$ in $K_0(\mathbb{A}^1, S)$. So the lemma will follow if we can show that for $t \in K_1(k[t, t^{-1}]), \partial(t) = \pm [0] \in K_0(\{0\})$ in the localisation sequence. This is proved in Quillen [9].

We need one more lemma. Let $C \subset \mathbb{Y}$ be as before, and let $\Pi \in \mathcal{O}_{C \cap E, Y}$ be a local generator for the ideal sheaf of $C$ on $\mathbb{Y}$. Then we have a diagram of localisation sequences

$$
\begin{array}{ccc}
k_2(\mathcal{O}_{C \cap E, E}) & \to & k_2(F) \\
\uparrow & & \uparrow \\
k_2(\mathcal{O}_{C \cap E, Y}) & \to & k_2(\mathcal{O}_{C \cap E, Y}[\Pi^{-1}]) \to K_1(\mathcal{O}_{C \cap E, C}) \to 0
\end{array}
$$
(induced from the diagram)

\[
\begin{array}{c}
BQ \mathcal{M}(C \cap E, E) \rightarrow BQ \mathcal{M}(\mathcal{C}_{C \cap E, E}) \rightarrow BQ \mathcal{M}(F) \\
\uparrow \hspace{1cm} \uparrow \hspace{1cm} \uparrow \\
BQ \mathcal{P}(\mathcal{C}_{C \cap E, E}) \rightarrow BQ \mathcal{P}(\mathcal{C}_{C \cap E, Y}) \rightarrow BQ \mathcal{P}(\mathcal{C}_{C \cap E, Y}[II^{-1}])
\end{array}
\]

Here \( \mathcal{H}(\mathcal{C}_{C \cap E, Y}) \) is the category of coherent \( \mathcal{C}_{C \cap E, Y} \) modules \( M \) satisfying \( \text{Tor}(M, \mathcal{C}_{C \cap E, E}) = 0 \) for \( i > 0 \), and similarly for \( \mathcal{C}_{C \cap E, Y}[II^{-1}] \). Note that both rings are regular. There is another map

\[
\beta: K_1(\mathcal{C}_{C \cap E, C}) = \mathcal{C}^{*}_{C \cap E} \rightarrow K_1(\mathcal{C} \cap E) \rightarrow K_1(\mathcal{H})
\]

(where \( \mathcal{H} = \mathcal{H}(C \cap E, E) \)).

**Lemma (3.6):** \( \alpha = \beta \).

**Proof:** \( \mathcal{P}(\mathcal{C}_{C \cap E, E}) \rightarrow \mathcal{H}(C \cap E, E) \) factors through the full subcategory \( \mathcal{P}(C \cap E) \subset \mathcal{H}(C \cap E, E) \).

The point of Lemma (3.6) is to use symbols for calculations, to avoid dealing with \( \mathcal{H}(C \cap E, E) \). Lemmas (3.5) and (3.6) give the geometric description of the boundary map \( K_1(E) \rightarrow K_0(\mathcal{Y}, E) \), since we know that \( K_2(\mathcal{C}_{C \cap E, Y}[II^{-1}]) \rightarrow K_1(\mathcal{C}_{C \cap E, C}) \) is the tame symbol (see Quillen [9]).

Finally we are ready to prove the theorem. Let \( x_0, \ldots, x_n \) be homogeneous coordinates on \( \mathbb{P}^n_L \). Let \( a_0, \ldots, a_n \in \mathbb{C} \) be algebraically independent over \( L \), and let \( \Pi = a_0 + a_1 x_1/x_0 + \ldots + a_n x_n/x_0 \) be a rational function on \( \mathbb{P} \). The divisor of zeroes of \( \Pi \) consists of a union of fibres of the map \( p: \mathbb{P} \rightarrow X \); indeed, if \( H \subset \mathbb{P}^n_L \) is the hyperplane \( a_0 x_0 + \ldots + a_n x_n = 0 \), the divisor is just \( p^{-1}(H \cap X) \). Let \( D \) be a derivation of \( \mathbb{C} \) extending \( \partial/\partial a_0 \) of \( L[a_0, \ldots, a_n] \). Then \( (d\Pi, D) = 1 \), where we regard \( D \) as a derivation on functions on \( \mathbb{P} \) which is 0 on functions defined over \( L \).

Now the homogeneous coordinate \( x_0 \) may be regarded as a regular function on \( \mathbb{P} \) in the ideal of \( E_0 \), which generates that ideal at points of \( X - \{ x_0 = 0 \} \) (under the identification of \( X \) with \( E_0 \)). Since \( (\Pi = 0) \cap E_0 \) consists of points not defined over \( L \), \( x_0 \) generates \( I \) (and hence \( I/I^2 \)) at these points.

Let \( \phi \) be a rational function on \( \mathbb{P} \) which is regular at the points \( \{ t_1, \ldots, t_p \} = (x_0 = 0) \cap E_0 \). Then \( S = \phi(d\Pi/\Pi) \otimes x_0 \) (where \( \phi, \Pi \) are restricted to \( E_0 \)) represents an element of \( H^1(E_0, \Omega^1_{E_0} \otimes I/I^2) \subset SK_1(E) \), whose boundary is a relative 0-cycle. Hence Theorem 3 is proved if this...
class is nonzero. Clearly it is enough to show that \((S, D) \in H^1(E_0, I/I^2)\) is nonzero (where \((, D)\) denotes contraction with the derivation \(D\)). We will do this using Serre Duality, in its formulation in terms of residues (see [7]). Let \(\omega \in H^0(E_0, O_{E_0}(-1) \otimes \omega_{E_0})\), the dual vector space to \(H^1(E_0, I/I^2)\); assume \(\omega\) is defined over \(L\). Then \(\omega\) is nonzero at \(t_1, \ldots, t_p\), and \(x_0\omega/\Pi\) has a simple pole with nonzero residue at each \(t_i\). On the affine curve \(\Pi = 0\), we can find a regular function \(\phi\) with prescribed values at each \(t_i\). Thus, by properly choosing \(\phi\), we can arrange that \(\sum_{t=1}^p \text{res}_{t_i}[(\phi/\Pi) \otimes x_0\omega]\) is nonzero. This finishes the proof.

**Remarks:**

1) The proof in fact shows that \(K_0(Y)\) is uncountably generated, since there are uncountably many mutually algebraically independent choices of numbers \(a_0, \ldots, a_n\).

2) Since derivations of the form \(\delta/\partial a\) (with a running through a transcendence base for \(\mathbb{C}\)) span the dual of \(\Omega^1_C\), at least if we allow infinite linear combinations, one can show that \(\text{image}(SK_1(E) - K_0(\mathcal{Y}, E)) = F_0K_0(\mathcal{Y}, E)\). Hence \(K_0(\mathcal{Y}, E)\) is generated by algebraic cycles. This is no longer clear if the curve \(X\) is not defined over a number field, since the vector space \(\Omega^1_k\) may play some role, where \(k\) is a field of definition of \(X\).

However, theorem 3 is still valid; in the final step of the proof, choose \(a_0, \ldots, a_n\) to be algebraically independent over \(k\).

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