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ON ISOSPECTRAL DEFORMATIONS OF RIEMANNIAN METRICS. II

Ruishi Kuwabara

1. Introduction

Let M be an $n (\geq 2)$ dimensional compact oriented C^∞ manifold without boundary. Let g be a C^∞ Riemannian metric on M , and $\text{Spec}(M, g)$ denote the set of eigenvalues of the Laplace-Beltrami operator $\Delta_g = -g^{jk}\nabla_j\nabla_k$ acting on real C^∞ functions on M . A 1-parameter C^∞ deformation $g(t)$ ($-\varepsilon < t < \varepsilon$) of a Riemannian metric on M is called an *isospectral deformation* of $g(0)$ if $\text{Spec}(M, g(t)) = \text{Spec}(M, g(0))$ holds for every t . We call $g(t)$ to be *trivial* if there is a 1-parameter family $\eta(t)$ of diffeomorphisms of M such that $g(t) = \eta(t)^*g(0)$. We have considered in [1], [2] the following problem (given in [3, p. 233]).

PROBLEM A: *Is there a non-trivial isospectral deformation of a Riemannian metric?*

So far, we have few results concerning this problem except for special cases [1] ~ [6]. Among others the following is known.

THEOREM: *There are no non-trivial isospectral deformations of g , if*

(1) (M, g) is $(1/n)$ -pinched, that is, for each $x \in M$, there exists a positive number A (depending on x) such that $-1 - (1/n) < K/A < -1 + (1/n)$, K being the sectional curvature associated with any two dimensional subspace of T_xM , or

(2) (M, g) is of non-negative constant curvature.

The case (1) was proved by Guillemin and Kazhdan [4], [5], and (2) is due to Kuwabara [2] for flat case and to Tanno [6] for the case of positive constant curvature. Moreover, for the case (2), a stronger result

was shown as follows. Let \mathcal{R} be the manifold of C^∞ Riemannian metrics on M with C^∞ topology. If (M, g) is flat or a standard sphere, there is a neighborhood U of g in \mathcal{R} such that if $\text{Spec}(M, g) = \text{Spec}(M, g')$ and $g' \in U$ then (M, g') is isometric with (M, g) .

In the previous paper [1], [2] we studied the problem by considering the variations of Minakshisundaram’s coefficients under the deformation of the metric. We try in this paper a different approach to the problem based on Lax’s idea which plays a fundamental role in theory of nonlinear waves [7]. We consider the isospectral deformations confined to Lax’s sense which are called L -isospectral deformations, and set up the following problem.

PROBLEM B: *Is there a non-trivial L -isospectral deformation of a matrix?*

We see that there are no non-trivial L -isospectral deformations under suitable conditions.

In §2 we introduce the notion of L -isospectral deformations. In §3 we consider the non-existence of L -isospectral deformations and give a sufficient condition for it. It is shown in §4 that this condition is related to the non-existence of first integrals of the geodesic flow, and we give some results concerning the non-existence of L -isospectral deformations.

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2. L -isospectral deformations

Let $g(t)$ be a C^∞ isospectral deformation of $g = g(0)$, that is,

$$\Delta_{g(t)}\phi_j(t) \equiv \Delta_t\phi_j(t) = \lambda_j\phi_j(t), \tag{2.1}$$

and $\{\phi_j(t)\}_{j=0}^\infty$ is the system of real eigenfunctions orthonormal with respect to the inner product $(\cdot, \cdot)_t$ defined from the metric $g(t)$, namely, $(\phi, \psi)_t = \int \phi\psi \, dV(g(t))$, $dV(g(t)) = \sqrt{\det g(t)} \, dx^1 \dots dx^n$. Moreover by Browder’s theorem [8], we can choose $\phi_j(t)$ to be of C^∞ class with respect to t .

First, we give the following lemma.

LEMMA 2.1: *Let $g(t)$ be a C^∞ isospectral deformation of g , and $\mu = dV(g)$. Then, there is a C^∞ isospectral deformation $\tilde{g}(t)$ of g such that $\tilde{g}(t) = \eta(t)^*g(t)$ for a 1-parameter family $\eta(t)$ of diffeomorphisms of M , and $dV(\tilde{g}(t)) = \mu$.*

PROOF: It is well known that $\text{vol}(M, g(t))$ is left invariant under the isospectral deformation $g(t)$ (cf. [3, p.216]). Hence, the lemma is immediately obtained by the following lemma due to Moser [9].

LEMMA (Moser): *Let $\mu(t)$ be a C^∞ deformation of n -form on M which is non-degenerate and $\int_M \mu(t) = \int_M \mu(0)$ for each t . Then, there is a C^∞ family $\eta(t)$ of diffeomorphisms of M such that $\eta(t)^*\mu(t) = \mu(0)$.*

By Lemma 2.1, we consider hereafter only volume-element preserving deformations, for which the *infinitesimal deformation* (*i-deformation*, for short) $h(t) = dg(t)/dt$ satisfies (cf. [10])

$$\text{Tr}_{g(t)} h(t) = h_{jk}(t)g^{jk}(t) = 0.$$

We denote the set of all square integrable real functions on M by $L^2(M)$, the inner product being $(\cdot, \cdot) = (\cdot, \cdot)_t = (\cdot, \cdot)_0$, and the space of distributions on M by $\mathcal{E}'(M)$. For an isospectral deformation $g(t)$, we introduce a linear operator $B_t: L^2(M) \rightarrow \mathcal{E}'(M)$ for each t as follows. Suppose an element ϕ of $L^2(M)$ is expressed as $\sum_{j=0}^{\infty} a_j(t)\phi_j(t)$, $a_j(t) \in \mathbf{R}$. Then for $\psi \in C^\infty(M)$, we define

$$\langle B_t \phi, \psi \rangle = \sum_{j=0}^{\infty} a_j(t) (\phi'_j(t), \psi),$$

where $\phi'_j(t) \equiv d\phi_j(t)/dt$ and the domain $D(B_t)$ of the operator B_t is the set of all $\phi \in L^2(M)$ for which the right hand side of the above has a real finite value. Note that $B_t \phi_j(t) = \phi'_j(t) \in C^\infty(M)$ holds good.

Now, differentiate (2.1) with respect to t , and we have

$$\Delta'_t \phi_j(t) + \Delta_t B_t \phi_j(t) - \lambda_j B_t \phi_j(t) = 0,$$

hence,

$$(\Delta'_t + \Delta_t B_t - B_t \Delta_t) \phi_j(t) = 0.$$

Therefore, we get the following equation of operators on $D(B_t) \cap C^\infty(M)$;

$$\Delta'_t + [\Delta_t, B_t] = 0. \quad (2.2)$$

Thus we have

PROPOSITION 2.2: *If $g(t)$ is an isospectral deformation, there is a linear operator B_t satisfying (2.2), where*

$$\Delta'_t = h^{jk} \nabla_j \nabla_k + (\nabla_k h^{jk}) \nabla_j = \nabla_j (h^{jk} \nabla_k), \quad (2.3)$$

∇ being the covariant differentiation defined by $g(t)$.

PROOF: (2.3) is immediately derived from variational formulas of Riemannian structure [10]. Q.E.D.

REMARK: The operator B_t depends on the choice of the orthonormal basis of eigenfunctions $\{\phi_j(t)\}$.

The equation (2.2) may be called Lax's equation, which is originally studied concerning Korteweg-de Vries (KdV) equation (see Lax [7]):

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + u \frac{\partial u}{\partial x} = 0.$$

For the Schrödinger operator $L_t = (d^2/dx^2) + (1/6)u(x, t)$, consider a third order differential operator

$$B_t = -4 \frac{d^3}{dx^3} - u \frac{d}{dx} - \frac{1}{2} \frac{du}{dx}.$$

Then the equation $L'_t + [L_t, B_t] = 0$ is equivalent to the KdV equation and $\text{Spec}(L_t)$ is left invariant when u changes with t subject to the KdV equation. Moreover, for higher odd order differential operators B_t we get a series of higher order KdV equations, and $\text{Spec}(L_t)$ is invariant if u changes according to them.

On the basis of the above discussion, we introduce the following definition.

DEFINITION: Let $g(t)$ be an isospectral deformation. If B_t is a differential operator for each t , we call $g(t)$ an *isospectral deformation in Lax's sense*, or *L-isospectral deformation*. If B_t is a k -th order differential operator for each t , we call $g(t)$ an *L_k -isospectral deformation*. Note that $D(B_t) = L^2(M)$ for the L -isospectral deformation.

LEMMA 2.3: Let $g(t)$ be an L_k -isospectral deformation. Then, the k -th differential operator B_t is skew-symmetric, that is,

$$B_t + B_t^* = 0, \tag{2.4}$$

where B_t^* is the formal adjoint of B_t with respect to $(,)$.

PROOF: By differentiating $(\phi_j(t), \phi_k(t)) = \delta_{jk}$ with respect to t , we have

$$(B_t \phi_j(t), \phi_k(t)) + (\phi_j(t), B_t \phi_k(t)) = 0,$$

and (2.4) because the above holds for all ϕ_j 's. Q.E.D.

As a converse of Proposition 2.2, we have the following.

PROPOSITION 2.4: *Suppose there are a volume-element preserving C^∞ deformation $g(t)$ of a metric and a skew-symmetric k -th order differential operator B_t smoothly depending on t , which satisfy eq. (2.2). Assume that there exists a 1-parameter family of linear operators $T_t: C^\infty(M) \rightarrow C^\infty(M)$, $-\varepsilon < t < \varepsilon$, whose infinitesimal generator is B_t , that is, $T_t = \exp(\int_0^t B_s ds)$ and $T_0 = \text{Identity}$. Then the deformation $g(t)$ ($-\varepsilon < t < \varepsilon$) is an isospectral deformation of $g(0)$.*

PROOF: Let $\{\psi_j\}$ be a set of orthonormal eigenfunctions associated with $\text{Spec}(M, g(0)) = \{\lambda_j\}$, and set $\phi_j(t) = T_t \psi_j$. Then $\{\phi_j(t)\}_{j=0}^\infty$ forms an orthonormal basis of $L^2(M)$ for each t . In fact,

$$\frac{d}{dt}(\phi_j(t), \phi_k(t)) = (B_t \phi_j(t), \phi_k(t)) + (\phi_j(t), B_t \phi_k(t)) = 0,$$

hence $(\phi_j(t), \phi_k(t)) = (\psi_j, \psi_k) = \delta_{jk}$ holds. Set

$$\begin{aligned} \Delta_t \phi_j(t) &= \sum_{k=0}^\infty a_j^k(t) \phi_k(t), \\ a_j^k(t) &= (\Delta_t \phi_j(t), \phi_k(t)), \quad a_j^k(0) = \lambda_j \delta_j^k. \end{aligned}$$

The coefficients $a_j^k(t)$ are C^∞ functions and

$$\begin{aligned} \frac{d}{dt} a_j^k(t) &= (\Delta'_t \phi_j(t) + \Delta_t B_t \phi_j(t), \phi_k(t)) + (\Delta_t \phi_j(t), B_t \phi_k(t)) = \\ &= ((\Delta'_t + [\Delta_t, B_t]) \phi_j(t), \phi_k(t)) = 0. \end{aligned}$$

Therefore $a_j^k(t) = \lambda_j \delta_j^k$ and accordingly $\text{Spec}(M, g(t)) = \{\lambda_j\}$.

Q.E.D.

A fundamental example of L -isospectral deformation is a trivial deformation, that is,

LEMMA 2.5: *A trivial deformation is an L_1 -isospectral deformation.*

PROOF: Let $g(t) = \eta(t)^* g(0)$ for a 1-parameter family $\eta(t)$ of volume preserving diffeomorphisms of M . Then, we have for each eigenfunction,

$$\phi_j(x, s) = \phi_j(\eta(s - t)x, t) = \eta(s - t)^* \phi_j(x, t).$$

Therefore, we get $\phi'_j(t) = X_t \phi_j(t)$, where $X_t = d\eta(t)/dt$ is a vector field satisfying $\nabla_j X_t^j = 0$ (cf. [11]). Thus $B_t = X_t$ is a first order differential operator and satisfies (2.2) and (2.4).

Q.E.D.

3. Non-existence of L -isospectral deformations

Let $g(t)$ be a C^∞ deformation with $g(0) = g$. We consider the equation (2.2) at $t = 0$ (the suffix 0 being omitted). A k -th order differential operator B on (M, g) is expressed as

$$B = a_{(k)}^{i_1 \dots i_k} \nabla_{i_1} \dots \nabla_{i_k} + a_{(k-1)}^{j_1 \dots j_{k-1}} \nabla_{j_1} \dots \nabla_{j_{k-1}} + \dots + a_{(0)}, \tag{3.1}$$

where $a_{(m)}^{i_1 \dots i_m}$ are components of a contravariant symmetric m -tensor. For this operator B , we have

$$B^* = (-1)^k a_{(k)}^{i_1 \dots i_k} \nabla_{i_1} \dots \nabla_{i_k} + (\text{lower order terms}).$$

Therefore, k is odd because B is skew-symmetric (Lemma 2.3). Thus we have only to consider odd order differential operators B .

First, we deal with L_1 -isospectral deformations, and have the following which is the converse of Lemma 2.5.

PROPOSITION 3.1: *There are no non-trivial L_1 -isospectral deformations.*

PROOF: Let B is a first order skew-symmetric differential operator, namely, $B = a^i \nabla_i + (1/2)(\nabla_i a^i)$. Then, we have from (2.2),

$$(h^{jk} - 2\nabla^j a^k) \nabla_j \nabla_k + \{ \nabla^k h_k^i - \nabla_k \nabla^k a^j - \nabla^j \nabla_i a^i - a^k R_k^j \} \nabla_j + \frac{1}{2} \Delta(\nabla_i a^i) = 0,$$

where R_{jk} is the Ricci curvature tensor of (M, g) . Therefore, we get $h^{jk} = \nabla^j a^k + \nabla^k a^j$, that is, $h = (dg/dt)(0)$ is a trivial i -deformation (see [1]). Thus, if $g(t)$ is an L_1 -isospectral deformation, then $h(t)$ is trivial with respect to $g(t)$ for each t . Hence the proposition is obtained by the following lemma.

LEMMA (Koiso [12, Lemma 2.9]): *If $h(t) = dg(t)/dt$ is trivial for each t , then $g(t)$ is a trivial deformation.*

Next, we consider L_k -isospectral deformations for $k(\text{odd}) \geq 3$. Substituting the differential operator B given by (3.1) into eq. (2.2), we get a necessary and sufficient condition that the coefficients $a_{(m)}$ and h should be satisfied. The computation, however, is so complicated that we cannot write it explicitly.

As a necessary condition, we have the following.

PROPOSITION 3.2: *If $g(t)$ is an L_k -isospectral deformation for $k(\text{odd}) \geq 3$, then the highest order coefficients of B satisfy*

$$\nabla^\rho a_{(k)}^{j_1 \dots j_k} + \nabla^{j_1} a_{(k)}^{\rho j_2 \dots j_k} + \dots + \nabla^{j_k} a_{(k)}^{\rho j_1 \dots j_{k-1}} = 0. \tag{3.2}$$

PROOF: By straightforward calculations, eq. (2.2) leads to

$$(\nabla^\rho a_{(k)}^{j_1 \dots j_k}) \nabla_\rho \nabla_{j_1} \dots \nabla_{j_k} + (\text{lower order terms}) = 0.$$

Thus we get (3.2).

Q.E.D.

Let S_k be the space of all C^∞ contravariant symmetric k -tensor fields on M endowed with C^∞ topology. For a C^∞ Riemannian metric g , we define $\hat{\nabla}_g^k: S_k \rightarrow S_{k+1}$ by

$$(\hat{\nabla}_g^k a)^{i_1 \dots i_{k+1}} = \nabla^{i_1} a^{i_2 \dots i_{k+1}} + \nabla^{i_2} a^{i_1 i_3 \dots i_{k+1}} + \dots + \nabla^{i_{k+1}} a^{i_1 \dots i_k},$$

where ∇ is the covariant differentiation defined by g . Let \mathcal{R} be the manifold of all C^∞ Riemannian metrics with C^∞ topology, and

$$\mathcal{N}_k = \{g \in \mathcal{R}; (\hat{\nabla}_g^k)^{-1}(0) = \{0\}\}.$$

LEMMA 3.3:

- (1) \mathcal{N}_k is an open subset of \mathcal{R} .
- (2) $\mathcal{R} \supset \mathcal{N}_1 \supset \mathcal{N}_3 \supset \dots \supset \mathcal{N}_{2m-1} \supset \mathcal{N}_{2m+1} \supset \dots$

PROOF: (1) Define $\hat{\nabla}^k: \mathcal{R} \times (S_k \setminus \{0\}) \rightarrow S_{k+1}$ by $\hat{\nabla}^k(g, a) = \hat{\nabla}_g^k a$. Then we have $\mathcal{N}_k = \mathcal{R} \setminus \pi(\ker(\hat{\nabla}^k))$, where $\pi: \mathcal{R} \times (S_k \setminus \{0\}) \rightarrow \mathcal{R}$ is the projection. It is easy to see that $\hat{\nabla}^k$ is continuous and π is an open mapping. Hence \mathcal{N}_k is open in \mathcal{R} .

(2) We show $(\mathcal{R} \setminus \mathcal{N}_{2m-1}) \subset (\mathcal{R} \setminus \mathcal{N}_{2m+1})$. Let $g \in (\mathcal{R} \setminus \mathcal{N}_{2m-1})$ and $\hat{\nabla}_g^{2m-1} a = 0$. Then, obviously, $\hat{\nabla}_g^{2m+1}(a \hat{\otimes} g^{-1}) = 0$ holds, where $a \hat{\otimes} g^{-1}$ denotes the symmetrization of $a \otimes g^{-1}$. Q.E.D.

We have the following proposition by virtue of Proposition 3.2.

PROPOSITION 3.4: *If the metric g belongs to \mathcal{N}_k , $k(\text{odd}) \geq 3$, then there are no non-trivial L_k -isospectral deformations of g .*

PROOF: Assume B is the k -th order differential operator satisfying (2.2). If $g \in \mathcal{N}_k$, then it follows from Proposition 3.2 and Lemma 3.3, (2) that the operator B reduces to be of first order. Since the set \mathcal{N}_k is open, the isospectral deformation must be trivial by virtue of Proposition 3.1.

Q.E.D.

REMARK: We conjecture that for each positive odd integer k , the set \mathcal{N}_k is dense in \mathcal{R} . It is known that the statement is valid for the case of $k = 1$ (cf. Ebin [13, Proposition 8.3]).

Set $\mathcal{N}_\infty = \bigcap_{k:\text{odd}} \mathcal{N}_k$. Noting that \mathcal{N}_∞ is not necessarily open, we get the following.

PROPOSITION 3.5: *If the metric g belongs to \mathcal{N}_∞ , there are no non-trivial L -isospectral i -deformations of g .*

4. Relation with first integrals of geodesic flows

Consider the cotangent bundle T^*M with the natural symplectic structure. Let (x^i, p_i) be the local coordinate system of T^*M naturally induced from the coordinates (x^i) of M . For a Riemannian metric g on M , define a function H_g on T^*M by

$$H_g = \frac{1}{2}g^{jk}p_jp_k.$$

The Hamiltonian flow on T^*M defined by H_g is called the geodesic flow, and the image of its integral curves projected on M are geodesics of (M, g) .

Let P_k (k : positive integer) be the set of all polynomial functions on T^*M which are homogeneous of degree k in (p_i) . We define a one-one correspondence $\Phi: S_k \rightarrow P_k$ by

$$\Phi(a) = \frac{1}{k} a^{i_1 \dots i_k} p_{i_1} \dots p_{i_k}.$$

Then, we have the following (cf. [5, Proposition 3.1]).

LEMMA 4.1: *For each positive integer k , the equation $\widehat{V}_g^k a = 0$ is equivalent to*

$$\{\Phi(a), H_g\} = 0.$$

Here $\{, \}$ is the Poisson bracket defined from the symplectic structure of T^*M .

PROOF: For $\Phi(a) = (1/k)a^{i_1 \dots i_k} p_{i_1} \dots p_{i_k}$, we have

$$\begin{aligned} \{\Phi(a), H_g\} &= \frac{1}{k} \frac{\partial a^{i_1 \dots i_k}}{\partial x^j} p_{i_1} \dots p_{i_k} g^{jm} p_m - \\ &\quad - \frac{1}{2} a^{j i_1 \dots i_k} p_{i_1} \dots p_{i_k} \frac{\partial g^{km}}{\partial x^j} p_k p_m = \\ &= \frac{1}{k} (\nabla^m a^{i_1 \dots i_k}) p_m p_{i_1} \dots p_{i_k}. \end{aligned}$$

Thus the lemma is proved.

Q.E.D.

DEFINITION: A C^∞ function f on T^*M is called the *first integral* of the geodesic flow if $\{f, H_g\} = 0$, and f is not constant on any open set of any level surface of H_g . Moreover, if f belongs to P_k , we call f the *first integral of degree k* .

From the above lemma, we have for odd k ,

$$\mathcal{N}_k = \{g \in \mathcal{R}; \text{ the geodesic flow has no first integral of degree } k\}.$$

We have the following theorem from Propositions 3.4 and 3.5.

THEOREM 4.2: *There are no non-trivial L -isospectral i -deformations (resp. L_k -isospectral deformations for odd integer $k \geq 3$) of g , if the geodesic flow defined by g has no first integrals (resp. first integrals of degree k).*

By Anosov [14] the geodesic flow defined by the metric of negative curvature is ergodic and has no first integrals. Thus we have

COROLLARY 4.3: *If (M, g) is of negative sectional curvature, there are no non-trivial L -isospectral deformations of g .*

REMARK: In [4] Guillemin and Kazhdan showed that if (M, g) is of negative sectional curvature and $g(t)$ is an isospectral deformation of g , then there is a C^1 function f on T^*M such that

$$H'_g + \{H_g, f\} = 0, \tag{4.1}$$

where $H'_g = (1/2)h^{jk}p_j p_k$. Moreover if (M, g) is $(1/n)$ -pinched, it is shown that the function f satisfying (4.1) belongs to P_1 and accordingly $h = (dg/dt)(0)$ is trivial. We note that the equation (2.2) may be regarded as a quantum version of eq. (4.1).

Finally, we consider the case where the metric does not belong to \mathcal{N}_k , and have the following theorem.

THEOREM 4.4: *Let k be a positive odd integer, and assume that every first integral of odd degree $\leq k$ of the geodesic flow defined by the metric g is expressed as a linear combination of the products of the first integrals of degree one and H_g . Then there are no non-trivial L_k -isospectral i -deformations of g .*

PROOF: We prove the theorem by induction on k . For the case $k = 1$, the statement reduces to Proposition 3.1. For general odd k , suppose h

is an L_k -isospectral i -deformation of g , and

$$\Delta' + [\Delta, B] = 0,$$

where

$$\Delta' = \nabla_j(h^{jk}\nabla_k),$$

$$B = a^{i_1 \dots i_k} \nabla_{i_1} \dots \nabla_{i_k} + (\text{lower order terms}).$$

By Proposition 3.2, Lemma 4.1, and the assumption of the theorem, we have

$$a = \sum_{k=2r+s} g^{-1} \overbrace{\widehat{\otimes} \dots \widehat{\otimes}}^r g^{-1} \widehat{\otimes} \xi_1 \widehat{\otimes} \dots \widehat{\otimes} \xi_s,$$

where ξ_1, \dots, ξ_s are the Killing vectors on (M, g) . Set $\Omega_k = \xi_k^i \nabla_j$, $k = 1, \dots, s$, and

$$B_1 = \sum_{k=2r+s} (\Delta' \Omega_1 \dots \Omega_s)$$

corresponding to a , where $()$ denotes the symmetrization. We see easily that B_1 is a skew-symmetric k -th differential operator, and $[\Delta, B_1] = 0$. Moreover, we have $B = B_1 + B_2$, where B_2 is a skew-symmetric $(k - 2)$ -th differential operator, and

$$\Delta' + [\Delta, B_2] = 0$$

holds good. Thus h is an L_{k-2} -isospectral i -deformation of g . Therefore h is trivial by the assumption of induction. Q.E.D.

We conjecture that the assumption of the theorem is satisfied for every Riemannian symmetric spaces.

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