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WINFRIED BRUNS

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GENERIC MAPS AND MODULES

Winfried Bruns

0. Introduction

In this article we investigate generic maps of a given rank between free modules and the modules associated with these maps, in particular their cokernels. Let Z_{ij}, $1 \le i \le u$, $1 \le j \le v$, be indeterminates over the ring $\mathbb{Z}$ of integers, and r an integer, $0 \le r \le \min(u, v)$. Let further S be the residue class ring of the polynomial ring $\mathbb{Z}[Z_{ij}]$ with respect to the ideal generated by the $(r+1)$-minors of the matrix (Z_{ij}). Then the map ζ: $S^u \to S^v$ given by the matrix (z_{ij}) of residue classes, its cokernel C and a S-free resolution $\mathscr{X}$ of C have the following universal properties: If $\varphi : R^u \to R^v$ is a map of rank r between free modules over a commutative (noetherian) ring R, then $\varphi = \zeta \otimes R$, R made a S-algebra via the substitution $Z_{ij} \to x_{ij}$ and (x_{ij}) representing φ relative to bases of R^u and R^v. If M is a R-module given by v generators and u relations and of rang $v - r$, then $M = C \otimes R$, since M is represented by a map $\varphi : R^u \to R^v$ with rank $\varphi \le r$. Finally, to each R-free resolution $\mathscr{F}$ of M there exists a map $\mathscr{X} \otimes R \to \mathscr{F}$ of complexes: $\mathscr{X} \otimes R$ represents the generic part of the syzygies of M.

The rings S can be considered well-understood since Hochster and Eagon proved their perfection (relative to the polynomial ring $\mathbb{Z}[Z_{ij}]$) in the splendid article [15]. Perfection is often hunted for and usually found in generic situations. Not surprisingly, however, the perfection of the modules C depends on u and v: C is perfect if and only if $u \ge v$, whereas the images of the maps ζ are always perfect. In case $u < v$ the cokernels therefore deviate from perfection by the smallest non-zero value only. In regard to the homological properties of the S-modules C, the case $u = v$ contrasts sharply with the case $u \ne v$. In the latter case the most important homological invariants of C are grade-sensitive with

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respect to the ideal of the non-free locus of C (which coincides with the singular locus of the ring S).

In section 1 we formulate the results on the perfection of C and, more generally, of suitable specializations to which properties like perfection can be transferred by the theory of generic perfection ([12], [15]) and by exactness criteria for finite free resolutions ([7]); cf. section 2. In sections 3 and 4 we exploit the inductive methods of Hochster and Eagon to show the perfection of the modules $\operatorname{Im}\zeta$. A computation of the kernels of the maps ζ, which turn out to be generated by the determinantal relations, and the computation of the canonical modules of the rings S enable us to show that the only possibly remaining obstruction to the perfection of C is non-existent in case $u \geq v$ (sections 5 and 6). In the last section we investigate the homological properties of the modules C and conclude the article with a few remarks on the corresponding problems for symmetric and alternating matrices.

In certain cases the perfection (or almost perfection) of the modules C was known before ([11], [8], [3]). We give more detailed information in section 1.

Notations and terminology: All rings are assumed to be commutative. We refer the reader to [21] for the general theory of commutative algebra. The *grade* of a finitely generated module M over a noetherian ring R is the smallest integer i such that $\operatorname{Ext}^i_R(M, R) \neq 0$ ([22]). It equals the maximal length of a R-sequence contained in the annihilator ideal of M. By the usual abuse of language the grade of an ideal means the grade of the corresponding residue class ring. A module M is said to be *perfect* if its projective dimension, abbreviated by $\operatorname{pd} M$, coincides with its grade, which is always a lower bound of the projective dimension. An ideal is called perfect if the corresponding residue class ring is perfect.

We use the notion of rank in the somewhat restricted, but very useful sense of [23]: M has *rank* s if and only if $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module of constant rank s for all associated prime ideals $\mathfrak{p}$ of R. If M is represented as the cokernel of a map $\varphi: R^u \to R^v$ with matrix (x_{ij}) then M has rank $v - r$ if and only if the Ideal $\mathrm{I}_{r+1}(x_{ij})$ generated by the $(r+1)$-minors of (x_{ij}) is zero and $\mathrm{I}_r(x_{ij})$ does not consist entirely of zero divisors. (We adopt the usual conventions $\mathrm{I}_0(x_{ij}) = R$, $\mathrm{I}_r(x_{ij}) = 0$ for $r > \min(u, v)$.) In this case a localization $M_{\mathfrak{q}}$ is a free $R_{\mathfrak{q}}$-module if and only if $\mathfrak{q} \not\supset \mathrm{I}_r(x_{ij})$.

Finally we want to point out that a (u, v)-matrix (x_{ij}) represents a map $R^u \to R^v$, i.e. the *rows* of (x_{ij}) generate $\operatorname{Im}\varphi$.

1. The generic perfection of generic modules

Throughout the article we will use the following notations. Let u, v, and r be integers such that $u, v \geq 1$ and $0 \leq r \leq \min(u, v-1)$. Let Z_{ij}, $1 \leq i \leq u$, $1 \leq j \leq v$, denote algebraically independent elements over the ring $\mathbb{Z}$ of integers, let P be the polynomial ring $\mathbb{Z}[Z_{ij}\colon 1 \leq i \leq u, 1 \leq j \leq v]$ and S the factor ring $P/\mathrm{I}_{r+1}(Z_{ij})$. The map $\zeta\colon S^u \to S^v$ is given by the matrix (z_{ij}) of the residue classes z_{ij} of the indeterminates Z_{ij}. Finally, the complex

$$\mathscr{X}\colon \ldots \to F_k \xrightarrow{\zeta_k} F_{k-1} \to \ldots \to F_1 \xrightarrow{\zeta_1} F_0$$

with $F_0 = S^v$, $F_1 = S^u$, and $\zeta_1 = \zeta$ is a S-free resolution of $C := \operatorname{Coker} \zeta$. It is always understood that these data depend on u, v, and r.

THEOREM 1: *Let A be a noetherian ring, and u, v, r be integers such that $u, v \geq 1$ and $0 \leq r \leq \min(u, v-1)$. Let y_{ij}, $1 \leq i \leq u$, $1 \leq j \leq v$, be elements of A, and let R denote the factor ring $A/\mathrm{I}_{r+1}(y_{ij})$. Let $\varphi\colon R^u \to R^v$ be given by the matrix (x_{ij}) of the residue classes x_{ij} of the elements y_{ij}. Consider A as a P-algebra via the substitution $Z_{ij} \to y_{ij}$ and R as a S-algebra via the induced map $z_{ij} \to x_{ij}$. Suppose that* $\operatorname{grade} \mathrm{I}_{r+1}(y_{ij}) = (u-r)(v-r)$.

Assume further that $\mathrm{I}_r(x_{ij})$ contains an element which is not a zero divisor of R. Then:

(a) *$\mathscr{X} \otimes R$ is a R-free resolution of* $\operatorname{Coker} \varphi$.

(b) $\operatorname{Im} \varphi$ *and, hence, all higher syzygies of* $\operatorname{Coker} \varphi$ *are perfect A-modules.*

As long as $\mathrm{I}_{r+1}(y_{ij}) \neq A$, $\operatorname{grade} \mathrm{I}_{r+1}(y_{ij})$ is bounded above by $(u-r)(v-r)$. Hochster and Eagon ([16]) showed that $\mathrm{I}_{r+1}(y_{ij})$ is a perfect ideal, if $\operatorname{grade} \mathrm{I}_{r+1}(y_{ij})$ attains its maximal value $(u-r)(v-r)$. In particular, this is true if A is a polynomial ring $B[y_{ij}]$. To avoid tedious repetitions we will refer to the notations and hypotheses of the first paragraph of Theorem 1 as the *standard hypotheses on φ*. In regard to the perfection of $\operatorname{Coker} \varphi$ the cases $u \geq v$ and $u < v$ behave differently:

THEOREM 2: *Let φ satisfy the standard hypotheses.*

(a) *If $u \geq v$, then* $\operatorname{Coker} \varphi$, $\operatorname{Im} \varphi$ *and, hence, all higher syzygies of* $\operatorname{Coker} \varphi$ *in a R-free resolution are perfect A-modules.*

(b) *If $u < v$, $\mathrm{I}_r(x_{ij}) \neq R$, and $\mathrm{I}_r(x_{ij})$ contains an element which is not a zero divisor of R, then* $\operatorname{Coker} \varphi$ *is not a perfect A-module.*

As noted already, the standard hypotheses on φ are satisfied if A

$= B[y_{ij}]$ is a polynomial ring in the indeterminates y_{ij} (or, more generally, the y_{ij} form a A-regular sequence). It follows immediately from [16] that the additional assumptions on $\mathrm{I}_r(x_{ij})$ are likewise fulfilled in this case, where $\mathrm{I}_r(x_{ij}) \neq R$ requires $r \geq 1$ of course.

From the theory of generic perfection ([12], [15]) we obtain as an immediate consequence:

COROLLARY: (a) $\operatorname{Im} \zeta$ *is a (strongly) generically perfect module.* (b) *In case* $r \geq 1$ $\operatorname{Coker} \zeta$ *is (strongly) generically perfect if and only if* $u \geq v$.

It is easy to find examples which show that the additional assumptions on $\mathrm{I}_r(x_{ij})$ can not be omitted for part (a) of Theorem 1 and part (b) of Theorem 2, and they are obviously indispensable for part (b) of Theorem 1 in case $r = u$. In this case one simply has $S = P$ and $R = A$, (z_{ij}) is the matrix of indeterminates Z_{ij}, and $(y_{ij}) = (x_{ij})$ a matrix of linearly independent rows.

We have to exclude the case $r = v < u$ from Theorem 1 because part (b) does not longer hold then. In this case and also in case $r = v = u$ one studies $\operatorname{Coker}(y_{ij})$, which is of course annihilated by $\mathrm{I}_v(y_{ij}) \neq 0$, as a torsion module over A. These modules were called "generic torsion modules" by Buchsbaum and Eisenbud ([8]). Their perfection was proved by Buchsbaum and Rim who constructed an explicit generic free resolution ([11], Corollary 2.7, cf. also [10], Theorem 3.1, and [8]). Since $\operatorname{Coker}(y_{ij})$ is annihilated by $\mathrm{I}_v(y_{ij})$, the perfection of these modules is contained in the case $r = v - 1$ of part (a) of Theorem 2. A second case, in which A-free resolutions are known, is the case $v > u = r + 1$ (cf. [10], Theorem 5.2 and [3], Proposition 7).

We do not give P-free resolutions of the modules C. Their complexity is certainly comparable to the complexity of the resolutions of determinantal ideals ([19]. It would be more interesting to have an explicit description of the S-free resolutions $\mathscr{X}$ because they represent the generic part of the resolution of a module with v generators, u relations and rank $v - r$, as discussed in the introduction. As an auxiliary result for the proof of Theorem 2 we will compute the map ζ_2, i.e. compute a system of generators of $\operatorname{Ker} \zeta$. As one should expect, $\operatorname{Ker} \zeta$ is generated by the determinantal relations of the rows of (z_{ij}) (Theorem 3). The complexes $\mathrm{K}(\zeta, r)$ constructed by Buchsbaum in [6], pp. 281, 282 can perhaps provide resolutions of the generic modules $C = \operatorname{Coker} \zeta$, non-minimal ones however in general.

A second auxiliary result in the proof of Theorem 2 is the explicit representation of the canonical modules of the "determinantal rings" given in [5].

2. Reduction to the generic case

In this section we derive Theorems 1 and 2 from the following propositions.

PROPOSITION 1: *For all* $u, v \geq 1$ *and* r, $0 \leq r \leq \min(u, v-1)$, $\operatorname{Im}\zeta$ *is a perfect P-module.*

PROPOSITION 2: *For all* $u, v \geq 1$ *and* r, $1 \leq r \leq \min(u, v-1)$, $\operatorname{Coker}\zeta$ *is a perfect P-module if and only if* $u \geq v$.

We need a strengthening of Proposition 2 (which however will finally turn out to be just a special case of Proposition 2).

PROPOSITION 3: *Let* $u, v \geq 1$, $u < v$, $1 \leq r \leq \min(u, v-1)$. *Let* $\mathfrak{m}$ *denote the prime ideal* $\mathrm{I}_r(z_{ij})$ *of S and* $\mathfrak{M}$ *the prime ideal* $\mathrm{I}_r(Z_{ij})$ *of P. Then* $(\operatorname{Coker}\zeta)_{\mathfrak{m}}$ *is not a perfect* $R_{\mathfrak{M}}$*-module.*

PROOF OF THEOREM 1: Consider the S-free resolution of $C = \operatorname{Coker}\zeta$:

$$\mathscr{X}\colon \ldots \to F_k \xrightarrow{\zeta_k} F_{k-1} \to \ldots \to F_1 \xrightarrow{\zeta_1} F_0$$

Since $\operatorname{Im}\zeta$ is perfect by virtue of Proposition 1, and the free S-modules F_k are perfect P-modules too, the perfection of $\operatorname{Coker}\zeta_k$ follows immediately for all $k \geq 2$ by induction. Applying the theory of generic perfection ([12]) or the exactness criterion of Buchsbaum and Eisenbud ([7]) to $\mathscr{L} \otimes A$, where $\mathscr{L}$ is a P-free resolution of $\operatorname{Coker}\zeta_k$, one obtains that $(\operatorname{Coker}\zeta_k) \otimes R = \operatorname{Coker}(\zeta_k \otimes R)$ is a perfect A-module of grade $(u-r)(v-r)$ for all $k \geq 2$. Such a R-module is necessarily torsionfree.

Let $\mathfrak{r} \in \operatorname{Spec} R$ be an associated prime ideal of R. By hypothesis on $\mathrm{I}_r(x_{ij})$, its preimage $\mathfrak{s}$ in S does not contain $\mathrm{I}_r(z_{ij})$. Therefore $\mathscr{X} \otimes S_{\mathfrak{s}}$ and, consequently, $\mathscr{X} \otimes R_{\mathfrak{r}}$ are split-acyclic. The acyclicity of $\mathscr{X} \otimes R$ now follows from a trivial lemma:

LEMMA 1: *Let T be a commutative noetherian ring and*

$$\mathscr{C}\colon M \xrightarrow{\alpha} M' \to M''$$

a zero-sequence of finitely generated T-modules, such that $\operatorname{Coker}\alpha$ *is a torsionfree T-module. If* $\mathscr{C} \otimes T_{\mathfrak{t}}$ *is exact for all* $\mathfrak{t} \in \operatorname{Ass} T$, *then* $\mathscr{C}$ *is exact.*

PROOF OF THEOREM 2: $\operatorname{Coker}\zeta$ has positive rank over S (we will state this explicitly in Proposition 4) and thus has grade $(u-r)(v-r)$ over P.

Again by the theory of generic perfection, Coker $\varphi = (\operatorname{Coker} \zeta) \otimes R$ is a perfect A-module as a consequence of Proposition 2, if $u \geq v$.

The non-perfection of Coker ζ under the hypotheses of part (b) of Theorem 2 follows from a stronger statement:

SUPPLEMENT TO THEOREM 2: *Under the hypotheses of part* (b) *of Theorem* 2 *let* $\mathfrak{r}$ *be a prime ideal of* R, *which contains* $\mathrm{I}_r(x_{ij})$. *Then* $(\operatorname{Coker} \zeta)_\mathfrak{r}$ *is not a perfect module over the corresponding localization of* A.

In proving this supplement we may directly assume that R and A are local with maximal ideals $\mathfrak{r}$ and $\mathfrak{a}$ resp. The preimage $\mathfrak{s}$ of $\mathfrak{r}$ in S contains $\mathrm{I}_r(z_{ij})$. According to Proposition 3, $C_\mathfrak{s}$ is not perfect, but its deviation from perfection is small: From Proposition 1 we obtain $\operatorname{pd} C_\mathfrak{s} = \operatorname{grade} C_\mathfrak{s} + 1 = (u-r)(v-r) + 1$ over $P_\mathfrak{q}$, where $\mathfrak{q}$ is the preimage of $\mathfrak{s}$ in P. To simplify the notation let us write $S = S_\mathfrak{s}$, $P = P_\mathfrak{q}$ and $C = C_\mathfrak{s}$. Let

$$\mathscr{L}: 0 \to L_w \xrightarrow{\beta_w} L_{w-1} \to \ldots \to L_1 \xrightarrow{\beta_1} L_0$$

be a minimal P-free resolution of C. For every prime ideal $\mathfrak{q}$ of P such that $\mathfrak{q} \not\supset \mathrm{I}_{r+1}(z_{ij})$ the complex $\mathscr{L} \otimes P_\mathfrak{q}$ is split-acyclic. Hence $\mathscr{L} \otimes A$ is split-acyclic at all prime ideals $\mathfrak{n}$ of A with $\operatorname{grade} \mathfrak{n} < w - 1$ since $\operatorname{grade} \mathrm{I}_{r+1}(y_{ij}) = w - 1$ by hypothesis. The map β_w finally splits at primes $\mathfrak{q} \not\supset \mathrm{I}_r(Z_{ij})$ since $C_\mathfrak{q}$ is a free module over the corresponding localization of S and thus a perfect module over $P_\mathfrak{q}$. The hypothesis on $\mathrm{I}_r(x_{ij})$ implies $\operatorname{grade} \mathrm{I}_r(y_{ij}) \geq (u-r)(v-r) + 1 = w$. Hence $\beta_w \otimes A$ splits at all prime ideals $\mathfrak{n}$ of A with $\operatorname{grade} \mathfrak{n} < w$. The exactness criterion of Buchsbaum–Eisenbud applies and yields the acyclicity of $\mathscr{L} \otimes A$. Since the extension $P \to A$ is local, $\mathscr{L} \otimes A$ is a minimal resolution of Coker ζ. Since rank Coker $\zeta = v - r > 0$ over R, we obtain grade Coker $\zeta = (u - r)(v - r) = w - 1$.

3. The inductive system of Hochster and Eagon

In order to give an inductive proof of the perfection of determinantal ideals, Hochster and Eagon introduced a very large class of ideals. Let us first recall their notation, which has to be "transposed" for our purpose.

Throughout this section let B denote an integral domain, let X_{ij}, $1 \leq i \leq u$, $1 \leq j \leq v$, be indeterminates over B, and A the polynomial ring $B[X_{ij}]$. For a sequence $H = (u_0, \ldots, u_r)$ of integers u_i, $0 \leq u_0 < \ldots < u_r = u$, and an integer n, $0 \leq n \leq u$, let $\mathrm{I}(H, n)$ denote the ideal generated by the $(i+1)$-minors of the rows $1, \ldots, u_i$ of the matrix (X_{ij}), $i = 0, \ldots, r$, and the elements $X_{11}, \ldots, X_{n1}$. In [16] Hochster and Eagon proved the following propositions:

(HE-1) *If* $n = u_t$ *or* $n = u_t + 1$ *for a* $t \in \{0, \ldots, r\}$, *then* $\mathrm{I}(H, n)$ *is perfect.*

(HE-2) *If* $n = u_t$, $\mathrm{I}(H, n)$ *is a prime ideal.*

(HE-3) *If* $u_t < n < u_{t+1}$, *then* $\mathrm{I}(H, n)$ *is a radical ideal*:

$$\mathrm{I}(H, n) = \mathrm{I}(H', n) \cap \mathrm{I}(H, n')$$

where $H' = (u_0, \ldots, u_{t-1}, n, u_{t+1}, \ldots, u_r)$ *and* $n' = u_{t+1}$.

(HE-4) $$\operatorname{grade} \mathrm{I}(H, n) = uv - (u+v)r + h + \frac{r(r+1)}{2} + u_0 + \ldots + u_{r+1}$$

where h *is chosen such that* $u_{h-1} < n \leq u_h$.

(HE-5) *With the notations of* (HE-3), *in case* $u_t + 1 = n < u_{t+1}$ *one has* $\operatorname{grade} \mathrm{I}(H', n) = \operatorname{grade} \mathrm{I}(H, n') = \operatorname{grade} \mathrm{I}(H, n)$ *and* $\operatorname{grade} \mathrm{I}(H', n') = \operatorname{grade} \mathrm{I}(H, n) + 1$.

(HE-6) *If* B *is normal and* $n = u_t$, *then* $A/\mathrm{I}(H, n)$ *is normal.*

We are studying the homomorphism Φ: $A^u \to A^v$, which is given by the matrix (X_{ij}), the module $M := \operatorname{Coker} \Phi$, and in particular the modules

$$\mathrm{M}(H, n) := M \otimes_A (A/\mathrm{I}(H, n)).$$

Further let $\mathrm{R}(H, n) := A/\mathrm{I}(H, n)$. With (HE-3) it is almost trivial to compute the rank of $\mathrm{M}(H, n)$ over $\mathrm{R}(H, n)$.

PROPOSITION 4: *Let* $r \leq v$. (a) *As a* $\mathrm{R}(H, n)$*-module,* $\mathrm{M}(H, n)$ *has rank* $v - r$.

(b) *For all* $\mathfrak{r} \in \operatorname{Spec} \mathrm{R}(H, n)$ *the* $\mathrm{R}(H, n)_{\mathfrak{r}}$*-module* $\mathrm{M}(H, n)_{\mathfrak{r}}$ *is free if and only if* $\mathfrak{r} \not\supseteq \mathrm{I}_r(x_{ij})$, *where* x_{ij} *denotes the residue class of* X_{ij} *in* $\mathrm{R}(H, n)$.

PROOF: The first possibly non-vanishing Fitting ideal of $\mathrm{M}(H, n)$ is $\mathrm{I}_r(x_{ij})$. By (HE-2) and (HE-3) $\mathrm{I}((u_0, \ldots, u_{r-2}), n)$ is not contained in an associated prime ideal of $\mathrm{R}(H, n)$, and therefore $\mathrm{I}_r(x_{ij}) = \mathrm{I}((u_0, \ldots, u_{r-2}, u), n)/\mathrm{I}(H, n)$ contains an element which is not a zero divisor of $\mathrm{R}(H, n)$.

Let $\mathrm{U}(H,n)$ denote the kernel of the natural epimorphism $M \to \mathrm{M}(H,n)$ and $\mathrm{T}(H,n)$ the $\mathrm{R}(H,n)$-torsion submodule of $\mathrm{M}(H,n)$. Then as a consequence of Proposition 4, $\mathrm{T}(H,n)$ is annihilated by $\mathrm{I}_r(X_{ij})$, and thus the kernel $\mathrm{V}(H,n)$ of the natural epimorphism $M \to \mathrm{M}(H,n)/\mathrm{T}(H,n)$ is given by

$$\mathrm{V}(H,n) = \{x \in M : \mathrm{I}_r(X_{ij})x \subset \mathrm{U}(H,n)\} \tag{*}$$

The main result of this section is

PROPOSITION 5: *Let $u, v \geq 1$ be integers,*

$$\mathscr{H}_r := \{(H,n) \colon H = (u_0, \ldots, u_r),\ 0 \leq n \leq u\}$$

for $r = 0, \ldots, v-2$, and

$$\mathscr{H}_{v-1} := \{(H,n) \colon H = (u_0, \ldots, u_{v-1}),\ 0 \leq n < u\}.$$

Then for all $(H,n) \in \mathscr{H} := \bigcup_{r=0}^{v-1} \mathscr{H}_r$ the $\mathrm{R}(H,n)$-module $\mathrm{M}(H,n)$ is torsionfree.

In order to prove Proposition 5 we will show that the $\mathrm{U}(H,n)$ and $\mathrm{V}(H,n)$ form a module-theoretic version of Hochster and Eagon's *principal radical system*:

LEMMA 2: *Let T be a noetherian ring and M a finitely generated R-module. Let Λ be a partially ordered set with ascending chain condition and (U_λ) and (V_λ), $\lambda \in \Lambda$, two families of submodules of M with the following property: for all $\lambda, \mu \in \Lambda$ with $\lambda \leq \mu$ one has*

$$\begin{array}{ccc} U_\lambda & \subset & V_\lambda \\ \cap & & \cap \\ U_\mu & \subset & V_\mu \end{array}$$

Suppose further that for each $\lambda \in \Lambda$ at least one of the following conditions is satisfied:

(1) $U_\lambda = V_\lambda$.

(2) *There are a $\mu \in \Lambda$, $\mu > \lambda$, and an element $x \in T$, which is not a zero divisor of M/V_λ, such that $U_\lambda + xM = U_\mu$ and $(U_\lambda : V_\lambda) + Tx = T$ implies $U_\lambda = V_\lambda$.*

(3) *There are $\mu, \nu \in \Lambda$, $\mu, \nu > \lambda$, and an element $x \in T$ such that $U_\lambda + xM = U_\mu$, $U_\nu \supset U_\lambda \colon xT$, and $V_\lambda \colon xT \supset V_\nu$.*

Then $U_\lambda = V_\lambda$ for all $\lambda \in \Lambda$.

The proof of Lemma 2 is a straightforward generalization of the proof of Proposition 24 in [16]. Therefore we omit it. (We avoid the indexing by $\vartheta \in \Theta$; U_λ corresponds to the ideal I_λ and V_λ to its radical J_λ.)

Proof of Proposition 5: $\mathscr{H}_r$ is partially ordered in a natural way: $(H,n) \leq (\tilde{H},\tilde{n})$ if and only if $u_0 \leq \tilde{u}_0,\dots,u_r \leq \tilde{u}_r$ and $n \leq \tilde{n}$. Equation (*) guarantees that $\mathrm{V}(H,n) \subset \mathrm{V}(\tilde{H},\tilde{n})$ whenever $(H,n) \leq (\tilde{H},\tilde{n})$, whereas the inclusions $\mathrm{U}(H,n) \subset \mathrm{V}(H,n)$, $\mathrm{U}(\tilde{H},\tilde{n}) \subset \mathrm{V}(\tilde{H},\tilde{n})$ and $\mathrm{U}(H,n) \subset \mathrm{U}(\tilde{H},\tilde{n})$ are trivial. It further implies that $\mathrm{V}(H,n)$ is a homogeneous submodule of the graded A-module M (as is $\mathrm{U}(H,n)$ by definition).

We prove by induction on v that the families $(\mathrm{U}(H,n))$ and $(\mathrm{V}(H,n))$, $(H,n) \in \mathscr{H}_r$, satisfy the conditions of Lemma 5. The case $v = 1$ is trivial. Assume $v > 1$.

Case (1): $n = u$. Then the matrix (X_{ij}) looks like

$$\begin{bmatrix} 0 & x_{12} & \cdots & x_{1v} \\ \vdots & & & \vdots \\ 0 & x_{u2} & \cdots & x_{uv} \end{bmatrix}$$

modulo $\mathrm{I}(H,n)$. $\mathrm{M}(H,n)$ splits into the direct sum $\mathrm{R}(H,n) \oplus \mathrm{M}'(H,0)$ where M' is defined with respect to the matrix formed by the columns $2,\dots,v$ of (X_{ij}). Since, by the definition of $\mathscr{H}$, $n = u$ is only possible if $r < v - 1$, the induction hypothesis on v can be applied to give the torsionfreeness of $\mathrm{M}'(H,0)$ as a $\mathrm{R}(H,n) = \mathrm{R}'(H,0)$-module.

Case (2): $n = u_t < u$, $r < v - 1$ or $n = u_t < v - 1$, $r = v - 1$. The element $X_{n+1,1}$ is not a zero divisor modulo $\mathrm{I}(H,n)$ and thus not a zero divisor modulo $\mathrm{V}(H,n)$. Since $X_{n+1,1}$ is a form of positive degree and $\mathrm{U}(H,n)$ and $\mathrm{V}(H,n)$ are homogeneous, all the properties required in condition (2) of Lemma 2 are fulfilled with $(H, n+1)$ corresponding to μ.

Case (3): $u_t < n < u_{t+1}$, $r - v - 1$ or $u_t < n < u_{t+1}$, $n < u - 1$, $r = v - 1$: It is easy to check that condition (3) of Lemma 2 is satisfied with $x = X_{n+1,1}$, $(H, n+1)$ and (H', n) corresponding to μ and v resp., where $H' = (u_0,\dots,u_{t-1}, n, u_{t+1},\dots,u_r)$.

Case (4): $n = u_t = n - 1$, $r = v - 1$. Modulo $\mathrm{I}(H,n)$ the matrix (X_{ij}) becomes

$$(x_{ij}) = \begin{bmatrix} 0 & x_{12} & \cdots & x_{1v} \\ \vdots & & & \vdots \\ 0 & x_{u-1,2} & \cdots & x_{u-1,v} \\ x_{u1} & x_{u2} & & x_{uv} \end{bmatrix}$$

The elements $x_{u1},\dots,x_{uv}$ are algebraically independent over the subring

$$R' := B[X_{ij}\colon 1 \le i \le u-1,\ 2 \le j \le v]/\mathrm{I}(H',0),$$

where $H' = (u_0,\dots,u_{r-1})$ and $\mathrm{I}(H',0)$ is taken relative to the matrix $(X_{ij}\colon 1 \le i \le u-1,\ 2 \le j \le v)$. Let $M' := \mathrm{M}'(H',0)$ be the corresponding R'-module. Then

$$\mathrm{M}(H,n) = (R \oplus (M' \otimes_{R'} R))/R(x_{u1},\bar{x})$$

where $R := \mathrm{R}(H,n)$ and $\bar{x}$ is the residue class of $(x_{u2},\dots,x_{uv})$ in $M' \otimes_{R'} R$.

M' is a torsionfree R'-module by induction hypothesis on v, and thus $\tilde{M} = R \oplus (M' \otimes R)$ is a torsionfree R-module. The element $(x_{u1},\bar{x})$ is linearly independent, hence excludes primes $\mathfrak{p} \in \operatorname{Spec} R$ with $\operatorname{depth} R_{\mathfrak{p}} \ge 2$ from being associated to $M = \tilde{M}/R(x_{u1},\bar{x})$. It even generates a free direct summand of $M_{\mathfrak{p}}$ if $\mathfrak{p} \not\supset Rx_{u1}$. Therefore $\mathfrak{q} := Rx_{u1}$ could be the only non-zero prime ideal associated to M. But $M_{\mathfrak{q}}$ is a free $R_{\mathfrak{q}}$-module, since $\mathrm{I}_r(x_{ij}) \not\subset \mathfrak{q}$, and this excludes $\mathfrak{q}$ as an associated prime.

Case (5): $n = v-1$, $u_{r-1} < n$, $r = v-1$. Modulo $\mathrm{I}(H,n)$ the matrix (X_{ij}) has the same form as in case (4), and $\mathrm{M}(H,n) = R^v/U$, where U is the submodule generated by the rows $x_1,\dots,x_u$ of (x_{ij}) and $R = \mathrm{R}(H,n)$. Suppose $y = (y_1,\dots,y_v)$ is a torsion element modulo U:

$$ay = a_1x_1 + \dots + a_ux_u,$$

a not a zero divisor of R. By (HE-3) the element x_{u1} generates a minimal prime in R. Thus $a \notin Rx_{u1}$ and $y_1 \in Rx_{u1}$. Subtracting a suitable multiple of x_u from y we may assume that $y_1 = 0$ and $a_ux_{u1} = 0$. Again by (HE-3) $a_u \in \mathfrak{p} := \mathrm{I}_r(x_{ij}\colon 1 \le i \le u-1,\ 2 \le j \le v)$. The prime ideal $\mathfrak{p}$ annihilates R^{v-1} modulo the submodule $\tilde{U}$ generated by $\tilde{x}_1 := (x_{12},\dots,x_{1v})$, $\dots$, $\tilde{x}_{u-1} := (x_{u-1,2},\dots,x_{u-1,v})$. Consequently we may assume $a_u = 0$. The residue classes of $x_{u1},\dots,x_{uv}$ in $R/\mathfrak{p}$ are algebraically independent over R', the ring R' being defined as in case (4), and as in case (4) the module $M' \otimes R/\mathfrak{p}$ is torsionfree over $R/\mathfrak{p}$. Since $a \notin \mathfrak{p}$, we conclude that $(y_2,\dots,y_v) \in \mathfrak{p}R^{v-1} + \tilde{U} \subset \tilde{U}$ and $y \in U$.

4. Perfection of the image of a generic map

In order to prove Proposition 1 we will show that $\operatorname{Coker} \zeta$ is almost perfect:

DEFINITION: Let T be a noetherian ring and M a finitely generated T-module. We call M *almost perfect* if $\operatorname{pd} M \leq \operatorname{grade} M + 1$.

Again we will mimic Hochster and Eagon's inductive method. We need a straightforward (and elementary) generalization of Propositions 17 and 18 of [16].

LEMMA 3: *Let T be a noetherian ring, M a finitely generated T-module, and U and V submodules of M. Suppose that* $U \cap V = 0$, $\operatorname{grade} M/U = \operatorname{grade} M/U = \operatorname{grade} M$ *and* $\operatorname{grade} M/(U+V) = \operatorname{grade} M + 1$. *Then*:

(a) *If M/U, M/V and $M/(U+V)$ are (almost) perfect, M is (almost) perfect.*

(b) *If M/U and M/V are almost perfect and $M/(U+V)$ is not perfect, then M is not perfect.*

PROOF: One considers the behaviour of projective dimension along the exact sequences

$$0 \to U \to M \to M/U \to 0 \text{ and}$$
$$0 \to U \to M/V \to M/(U+V) \to 0.$$

In passing from M to M/xM, x not a zero divisor of M, it can happen that M/xM is (almost) perfect whereas M is not. This complication does not occur if x does not avoid all the prime ideals $\mathfrak{p}$ at which $\operatorname{pd} M_{\mathfrak{p}}$ is maximal. In particular we have as a substitute of [16], Proposition 19 and Corollary:

LEMMA 4: *Let K be a field and $T = \bigoplus_{i=0}^{\infty} T_i$ a graded noetherian K-algebra with $K = T_0$. Let $M \neq 0$ be a graded T-module and $x \in T$ a form of positive degree which is not a zero divisor of M. Further suppose that* $\operatorname{grade} M/xM = \operatorname{grade} M + 1$. *Then M is (almost) perfect if and only if M/xM is (almost) perfect.*

We finally need an analogue of Proposition 20 of [16]:

LEMMA 5: *Let $X_1, \ldots, X_n$ be indeterminates over the ring $\mathbb{Z}$ of integers, and M a finitely generated $\mathbb{Z}[X_1, \ldots, X_n]$-module which is torsionfree over $\mathbb{Z}$. Suppose that for each prime $p \in \mathbb{Z}$ the $(\mathbb{Z}/p\mathbb{Z})[X_1, \ldots, X_n]$-module $M \otimes \mathbb{Z}/p\mathbb{Z}$ has the same grade as M and is a (almost) perfect $(\mathbb{Z}/p\mathbb{Z})[X_1, \ldots, X_n]$-module. Then M is a (almost) perfect $\mathbb{Z}[X_1, \ldots, X_n]$-module.*

The proofs of Lemma 4 and Lemma 5 are straightforward generalizations of the corresponding proofs in [16].

In order to prove Proposition 1 it suffices to show that Coker ζ is almost perfect. For we have an exact sequence

$$0 \to \operatorname{Im}\zeta \to S^v \to \operatorname{Coker}\zeta \to 0,$$

S is perfect by (HE-1), and grade Coker ζ = grade S = grade Im ζ, since rank Coker ζ is positive in case $r < v$ (where, of course, the data ζ, S etc. are understood to depend on u, v, and r). By virtue of Lemma 5 we may replace the base $\mathbb{Z}$ by a field.

PROPOSITION 6: *Let B be a field and $u, v \geq 1$ integers. Let*

$$\mathscr{H}_r := \{(H,n)\colon H = (u_0,\ldots,u_r),\ n = u_t \text{ or } n = u_t + 1 \text{ for a } t \in \{0,\ldots,r\}\} \text{ for } r = 0,\ldots,v-2,$$

$$\mathscr{H}_{v-1} := \{(H,n)\colon H = (u_0,\ldots,u_{v-1}),\ n < u \text{ and } n = u_t \text{ or } n = u_t + 1 \text{ for a } t \in \{0,\ldots,r\}\}.$$

Then $\mathrm{M}(H,n)$ *is almost perfect for all* $(H,n) \in \mathscr{H} := \bigcup_{r=0}^{v-1} \mathscr{H}_r$.

PROOF: Again we use induction on v and descending induction on the partially ordered sets $\mathscr{H}_r$. The assertion is trivial for $v = 1$. As in the proof of Proposition 5 we distinguish five different cases:

Case (1): $n = u$. Then $\mathrm{M}(H,n) = \mathrm{R}(H,n) \oplus \mathrm{M}'(H,0)$ as in case (1) of the proof of Proposition 5, and $\mathrm{M}(H,n)$ is almost perfect by induction on v.

Case (2): $n = u_t < u$, $r < v - 1$ or $n = u_t < u - 1$, $r = v - 1$. Since $\mathrm{M}(H,n)$ is a torsionfree $\mathrm{R}(H,n)$-module, $X_{n+1,1}$ is not a zero divisor of $\mathrm{M}(H,n)$. We have $\mathrm{M}(H,n)/X_{n+1,1}\mathrm{M}(H,n) \cong \mathrm{M}(H,n+1)$. $\mathrm{M}(H,n+1)$ is almost perfect by induction on $\mathscr{H}_r$, and Lemma 4 shows that $\mathrm{M}(H,n)$ is almost perfect, too.

Case (3): $n = u_t + 1 < u_{t+1}$, $r < v - 1$ or $n = u_t + 1 < u_{t+1} < u$, $r = v - 1$. By virtue of (HE-3)

$$\mathrm{I}(H,n) = \mathrm{I}(H',n) \cap \mathrm{I}(H,n')$$

where $H' = (u_0,\ldots,u_{t-1},\ n,\ u_{t+1},\ldots,u_r)$ and $n' = u_{t+1}$. The minimal primes of the reduced ring $\mathrm{R}(H,n)$ are $\mathfrak{p} := \mathrm{I}(H',n)/\mathrm{I}(H,n)$ and $\mathfrak{q} := \mathrm{I}(H,n')/\mathrm{I}(H,n)$. As a torsionfree $\mathrm{R}(H,n)$-module, $\mathrm{M}(H,n)$ can be embedded in a free $\mathrm{R}(H,n)$-module. Therefore $\mathfrak{p}\mathrm{M}(H,n) \cap \mathfrak{q}\mathrm{M}(H,n) = 0$. By induction on $\mathscr{H}_r$ the modules $\mathrm{M}(H,n)/\mathfrak{p}\mathrm{M}(H,n) \cong \mathrm{M}(H',n)$,

$\mathrm{M}(H,n)/\mathfrak{q}\mathrm{M}(H,n) \cong \mathrm{M}(H,n')$ and $\mathrm{M}(H,n)/(\mathfrak{p}+\mathfrak{q})\mathrm{M}(H,n) \cong \mathrm{M}(H',n')$ are almost perfect and their grades satisfy the requirements of Lemma 3, which shows that $\mathrm{M}(H,n)$ is almost perfect, too.

Case (4): $n = u_t = u - 1$, $r = v - 1$. As in case (4) of the proof of Proposition 5 and with the notations introduced there, we have

$$\mathrm{M}(H,n) = (R \oplus (M' \otimes_{R'} R))/R(x_{u1}, x).$$

M' and, thus, $M' \otimes_{R'} R$ are almost perfect by induction on v, R and, thus, $R(x_{u1}, \bar{x})$ are perfect by (HE-1). They all have the same grade. Now the assertion results from the behaviour of projective dimension along the exact sequence associated with the preceding representation of $\mathrm{M}(H,n)$.

Case (5): $n = u_t + 1 < u_{t+1} = u$, $r = v - 1$. For the reason given in case (3) we have two exact sequences as in the proof of Lemma 3:

$$0 \to W - \mathrm{M}(H,n) \to \mathrm{M}(H',n) \to 0 \text{ and}$$
$$0 \to W \to \mathrm{M}(H,n') \xrightarrow{\pi} \mathrm{M}(H',n') \to 0,$$

where $W = \operatorname{Ker}\pi$. Since $(H',n) \in \mathscr{H}_r$, $\mathrm{M}(H',n)$ is almost perfect by induction on $\mathscr{H}_r$. Observe that $n' = u$. As in case (1) both the modules $\mathrm{M}(H',n)$ and $\mathrm{M}(H',n')$ split:

$$\mathrm{M}(H,n') = \mathrm{R}(H,n') \oplus \mathrm{M}'(H,0),$$
$$\mathrm{M}(H',n') = \mathrm{R}(H',n') \oplus \mathrm{M}'(H',0),$$

and the map π splits accordingly. The ideal $\mathfrak{a} := \mathrm{I}_{v-1}(X_{ij}\colon 1 \le i \le u, 2 \le v \le j)$ annihilates $\mathrm{M}'(H,0)$. Since $\mathrm{I}(H',n') = \mathrm{I}(H,n') + \mathfrak{a}$, the component map $\mathrm{M}'(H,0) \to \mathrm{M}'(H',0)$ is an isomorphism, and we have an exact sequence

$$0 \to W \to \mathrm{R}(H,n') \to \mathrm{R}(H',n') \to 0.$$

Thus W is perfect with annihilator $\mathrm{I}(H,n')$, and now the first of the two sequences above yields the assertion on $\mathrm{M}(H,n)$, since the grades of W, $\mathrm{M}(H,n)$, and $\mathrm{M}(H',n)$ coincide.

The proof of Theorem 1 is complete now.

5. The kernel of a generic map

In this section we show that the kernel of a generic map is generated by the determinantal relations of the rows of the corresponding matrix.

Simultaneously we derive a technical result which is one of the main ingredients of the proof of Theorem 2.

Let T be a ring and consider a (u, v)-matrix (x_{ij}) with $x_{ij} \in T$. Suppose $\mathrm{I}_{r+1}(x_{ij}) = 0$. Let

$$\Delta^{j_1 \dots j_k}_{i_1 \dots i_k} := \det \begin{bmatrix} x_{i_1 j_1} & \dots & x_{i_k j_k} \\ \vdots & & \vdots \\ x_{i_k j_1} & \dots & x_{i_k j_k} \end{bmatrix} \quad \text{and}$$

$$x_i := (x_{i1}, \dots, x_{iv}).$$

Then, for any collection of indices $i_1, \dots, i_{r+1}$ and $j_1, \dots, j_r$ one obviously has

$$\sum_{t=1}^{r+1} (-1)^{t+1} \Delta^{j_1 \dots j_r}_{i_1 \dots \hat{i_t} \dots i_{r+1}} x_{i_t} = 0. \qquad ($$

We refer to these relations and the corresponding elements of the kernel of the map represented by (x_{ij}) as the *determinantal relations of the rows* $x_1, \dots, x_u$.

THEOREM 3: *Let φ satisfy the standard hypotheses. Assume that $r \geq 1$ and that $\mathrm{I}_r(x_{ij})$ contains an element which is not a zero divisor of R. Then the kernel of φ is generated by the determinantal relations of the rows of φ.*

The only purpose of the condition $r \geq 1$ is to exclude the degenerate case $r = 0$. Part (a) of Theorem 1, whose proof has already been completed, reduces Theorem 3 to the generic case $A = \mathbb{Z}[Z_{ij}]$, and thus to part (c) of the following proposition for $\mathfrak{q} = 0$.

PROPOSITION 7: *Let B be an integral domain, let (X_{ij}) be a (u, v)-matrix of indeterminates over B, r an integer with $1 \leq r \leq \min(u, v-1)$, R the residue class ring $B[X_{ij}]/\mathrm{I}_{r+1}(X_{ij})$, and the homomorphism $R^u \to R^v$ given by the matrix (x_{ij}). Further let $\mathfrak{q}$ denote the zero ideal of R or the ideal generated by the r-minors of the first r columns of (x_{ij}), and $s \geq 1$ an integer. Then:*

(a) *The ideal generated by the r-minors of the first r rows of (x_{ij}) contains an element which is not a zero divisor of $R/\mathfrak{q}^s$.*

(b) *If $r \geq 2$ the element x_{11} is not a zero divisor of $R/(\mathfrak{q}^s + \mathrm{I}_r(x_{ij}\colon 1 \leq i \leq k,\ 1 \leq j \leq v))$ for $k = r, \dots, u-1$.*

(c) *Let $a_1, \dots, a_k$ be elements of R such that $a_1 x_1 + \dots + a_k x_k \in \mathfrak{q}^s R^v$,* *where x_i denotes the i-th row of (x_{ij}). Then there is a linear combination $(b_1,\dots,b_k)$ of the determinantal relations of the rows $x_1,\dots,x_k$ such that $a_i - b_i \in \mathfrak{q}^s$ for $i = 1,\dots,k$.*

Proof: Parts (a) and (b) are trivial in case $\mathfrak{q} = 0$. Their proof in the remaining case follows immediately from [5], Proposition 2. They serve as auxiliary arguments in the inductive proof of part (c).

Let $r = 1$. By part (a) the first row is linearly independent modulo $\mathfrak{q}^s$. Hence (c) holds for $k = 1$. Assume $k > 1$. By the same argument we have $a_k \in \mathfrak{b} + \mathfrak{q}^s$ where $\mathfrak{b}$ is the ideal generated by all the elements in the rows $1,\dots,k-1$. All these elements occur as coefficients of the k-th row in the determinantal relations of the rows $1,\dots,k$. Subtracting a suitable linear combination of these relations from $(a_1,\dots,a_k)$ we may assume $a_k = 0$. Then we are through by induction on k.

The proof of part (c) for $r > 1$ rests on a standard localization argument which, roughly, decreases the size of all minors by 1. Over $P[X_{11}^{-1}]$ we can transform the matrix (X_{ij}) by elementary row and column operations into

$$\begin{bmatrix} X_{11} & 0 & \dots & 0 \\ 0 & Y_{22} & \dots & Y_{2v} \\ \vdots & & & \\ 0 & Y_{u2} & \dots & Y_{uv} \end{bmatrix}$$

where $Y_{ij} = X_{ij} - X_{i1}X_{ij}X_{11}^{-1}$. The elements Y_{ij} are algebraically independent over B, and the elements $X_{11},\dots,X_{1v}, X_{21},\dots,X_{u1}$ are algebraically independent over $C := B[Y_{ij}]$. $R[x_{11}^{-1}]$ can be considered a flat overring of $C/\mathrm{I}_r(Y_{ij})$: $R[x_{11}^{-1}] = (C/\mathrm{I}_r(Y_{ij}))$ $[X_{11},\dots,X_{1v}, X_{21},\dots,x_{u1}][X_{11}^{-1}]$. The extension of the ideal $\tilde{\mathfrak{q}}$ of $C/\mathrm{I}_r(Y_{ij})$ generated by the $(r-1)$-minors of $(Y_{ij}\colon 2 \le i \le u,\ 2 \le j \le r)$ to $R[x_{11}^{-1}]$ is just $\mathfrak{q}R[x_{11}^{-1}]$, and the determinantal relations of the rows $2,\dots,k$ of the matrix above "extend" to determinantal relations of the rows $x_1,\dots,x_u$ (up to multiplication by x_{11}).

Let $r > 1$. In case $k \le r$ part (a) shows again that $x_1,\dots,x_k$ are linearly independent. In case $k > r$ the localization argument just explained shows that there is a linear combination $(b_1,\dots,b_k)$ of the determinantal relations of $x_1,\dots,x_k$ such that

$$x_{11}^N a_k - b_k \in \mathfrak{q}^s$$

for an integer $N \ge 0$. The element b_k is contained in the ideal $\mathfrak{b}$ generated by the r-minors of the first $k-1$ rows. By part (b) x_{11} is not a zero divisor modulo $\mathfrak{b} + \mathfrak{q}^s$, hence $a_k \in \mathfrak{b} + \mathfrak{q}^s$. Now the subtraction argument given for $r = 1$ and induction on k complete the proof of part (c).

6. The perfection of the cokernel of a generic map

In their monograph [1] Auslander and Bridger discuss various homological properties of modules. We need some of their results. Let T be a noetherian ring, M a finitely generated T-module and $\varphi: T^u \to T^v$ a homomorphism such that $M = \operatorname{Coker} \varphi$. Then one has exact sequences

$$0 \to N^* \to T^u \to T^v \to M \to 0 \text{ and}$$
$$0 \to M^* \to (T^v)^* \to (T^u)^* \to N \to 0$$

where $M^* = \operatorname{Hom}_T(M, T)$. N is determined by M up to projective equivalence, and we will only use properties of N which depend on the projective equivalence class of N. Therefore we may write $N = \mathrm{D}(M)$ and, correspondingly, $M = \mathrm{D}(N)$.

Auslander and Bridger call a module M *t-torsionless* if $\operatorname{Ext}^i_T(\mathrm{D}(M), T) = 0$ for $i = 1, \ldots, t$. One observes that M is torsionless, i.e. the canonical homomorphism $M \to M^{**}$ is injective, if and only if M is 1-torsionless and furthermore that M is reflexive, i.e. $M \to M^{**}$ is an isomorphism, if and only if M is 2-torsionless. A module M is t-torsionless for $t \geq 3$ if and only if it is reflexive and $\operatorname{Ext}^i_T(M^*, T) = 0$ for $i = 1, \ldots, t-2$. A t-torsionless module is clearly a t-th syzygy: there is an exact sequence

$$0 \to M \to F_1 \to \ldots \to F_t$$

with free T-modules F_i. A t-th syzygy certainly satisfies the Serre type condition

$(\tilde{\mathrm{S}}_t)$ $\operatorname{depth} M_{\mathfrak{t}} \geq \min(t, \operatorname{depth} T_{\mathfrak{t}})$ *for all* $\mathfrak{t} \in \operatorname{Spec} T$.

Proofs of these assertions can be found in [1]. We need partial converses which in the literature ([1], [13]) are usually given under conditions on T. Using the ideas from [1] and [13] the reader will be able to supply the proof of Lemma 6:

LEMMA 6: *Let T be a noetherian ring, M a finitely generated T-module, and t an integer. Suppose that $M_{\mathfrak{t}}$ is a free $T_{\mathfrak{t}}$-module for all* $\mathfrak{t} \in \operatorname{Spec} T$, $\operatorname{depth} T_{\mathfrak{t}} \leq t-1$.

(a) *If M satisfies* $(\tilde{\mathrm{S}}_t)$, *then M is a t-th syzygy.*
(b) *If M is a* $(t+1)$*-th syzygy, then M is* $(t+1)$*-torsionless.*

We now complete the proof of Propositions 2 and 3, in which we considered the map $\zeta\colon\ S^u \to S^v$ with $S = \mathbb{Z}[Z_{ij}]/\mathrm{I}_{r+1}(Z_{ij})$, $0 \le r \le \min(u, v-1)$, and ζ given by the matrix (z_{ij}) of the residue classes of the indeterminates Z_{ij}. We first show that $\operatorname{Coker}\zeta$ is perfect when $u \ge v$.

By Proposition 6 we already know that $C := \operatorname{Coker}\zeta$ is almost perfect. Translated into an assertion on the depth of the localizations $C_\mathfrak{p}$, $\mathfrak{p} \in \operatorname{Spec} S$, this means

$$\operatorname{depth} C_\mathfrak{p} \ge \operatorname{depth} S_\mathfrak{p} - 1$$

for all $\mathfrak{p} \in \operatorname{Spec} S$. (Note that $S_\mathfrak{p}$ is a Cohen–Macaulay ring). Perfection of $C_\mathfrak{p}$ means

$$\operatorname{depth} C_\mathfrak{p} = \operatorname{depth} S_\mathfrak{p}$$

for all $\mathfrak{p} \in \operatorname{Spec} S$, and is hence equivalent to $\operatorname{Ext}^1_S(C, \omega_S) = 0$ by the local duality theorem. (By ω_S we denote the canonical module of S, cf. [5].)

The dual $C^* = \operatorname{Hom}_S(C, S)$ of C is the kernel of ζ^*. Identifying S^u and $(S^u)^*$, S^v and $(S^v)^*$ we may consider ζ^* as the generic map $S^v \to S^u$ whose cokernel $\mathrm{D}(C)$ is almost perfect by Theorem 1. By virtue of Proposition 4, (b) $C_\mathfrak{p}$ and $\mathrm{D}(C)_\mathfrak{p}$ are free $S_\mathfrak{p}$-modules whenever $\mathfrak{p} \not\supset \mathrm{I}_r(z_{ij})$. Since $\operatorname{grade} \mathrm{I}_r(z_{ij}) = u + v - 2r + 1 \ge 3$, an application of Lemma 7 shows that C and $\mathrm{D}(C)$ are second syzygies and reflexive. C^* is even a fourth syzygy and hence $\operatorname{Ext}^1_S(C^{**}, S) = \operatorname{Ext}^1_S(C, S) = 0$. In case $u = v$ this already shows that C is perfect, since S is a Gorenstein ring then. So we may assume $u > v$.

In [5] we gave a representation of ω_S as an ideal of S. From the exact sequence $0 \to \omega_S \to S \to S/\omega_S \to 0$ one derives an exact sequence

$$0 \to \operatorname{Hom}_S(C, \omega_S) \to C^* \to \operatorname{Hom}_S(C, S/\omega_S) \to \operatorname{Ext}^1_S(C, \omega_S) \to 0$$

Thus it is enough to show that the natural homomorphism $\chi\colon C^* \to \operatorname{Hom}_S(C, S/\omega_S)$ is surjective. C^* can be identified with the submodule of all $(a_1, \ldots, a_v) \in (S^v)^*$ such that

$$a_1 z^1 + \ldots + a_v z^v = 0$$

where $z^1, \ldots, z^v$ are the columns of (z_{ij}). Each homomorphism

$\beta: C \to S/\omega_S$ can be lifted to an element $(b_1, \ldots, b_v) \in (S^v)^*$ such that

$$b_1 z^1 + \ldots + b_v z^v \in \omega_S(S^u)^*$$

We know from [5] that ω_S is the ideal $\mathfrak{p}^{u-v}$, where $\mathfrak{p}$ is generated by the r-minors of the rows $1, \ldots, r$ of (z_{ij}). Now Proposition 7 can be applied to the transpose of (z_{ij}). It shows that each relation $(b_1, \ldots, b_v)$ of $z^1, \ldots, z^v$ modulo ω_S is the sum of a relation $(a_1, \ldots, a_v) \in (S^v)^*$ and an element of $\omega_S(S^v)^*$, i.e. $(a_1, \ldots, a_v)$ is mapped to β by the natural homomorphism χ.

In order to prove Proposition 3 we have to show that in case $u < v$ and $r \geq 1$ the localization $C_{\mathfrak{m}}$ with respect to the prime ideal $\mathfrak{m} = \mathrm{I}_r(z_{ij})$ is not a perfect $S_{\mathfrak{m}}$-module. Since $\mathfrak{m} \cap \mathbb{Z} = \{0\}$ we can replace the base ring $\mathbb{Z}$ by the field $\mathbb{Q}$ of rational numbers. The localization argument of the proof of Proposition 7 reduces the assertion to the case $r = 1$, in which we invoke the induction machinery of Section 4: Lemma 4 shifts the problem to the S/Sz_{11}module $C/z_{11}C$, and part (b) of Lemma 3 shifts it to the $S/(\mathfrak{p} + \mathfrak{q})$-module $C/(\mathfrak{p} + \mathfrak{q})C$, where

$$\mathfrak{p} = \sum_{j=1}^{v} Sz_{ij} \text{ and } \mathfrak{q} = \sum_{i=1}^{u} Sz_{i1}$$

This is essentially the case $(u-1,\ v-1)$, and if finally $u = 1$, $C_{\mathfrak{m}}$ is a torsionfree module of projective dimension 1 over the regular local ring $S_{\mathfrak{m}}$, hence not perfect. (C is a perfect S-module if and only if $C_{\mathfrak{m}}$ is a perfect $S_{\mathfrak{m}}$-module; so Proposition 3 finally turned out to be a special case of Proposition 2.)

The proof of Theorem 2 is complete now.

7. Homological properties of generic modules

In this section we investigate the homological properties of the R-modules Coker φ, where φ, A, R etc. satisfy the standard hypotheses on φ. We will see that there is a sharp trichotomy between the cases $u = v$, $u < v$, and $u > v$, which is not apparent from Theorem 1 and 2. Since we will have to consider φ and φ^* simultaneously, we display the dependence of the generic objects ζ, C, S and $\mathscr{X}$ on u and v by suitable indices.

We first settle the case $u = v$. The acyclic complex $\mathscr{X}_{uu}$ resolves $C_{uu} = \operatorname{Coker} \zeta_{uu}$ as a S_{uu}-module. By $\tilde{\mathscr{X}}_{uu}$ we denote the complex which arises from $\mathscr{X}_{uu}$ under the substitution $Z_{ij} \to Z_{ji}$. Then $\tilde{\mathscr{X}}_{uu}$ is clearly a resolution of Coker $\zeta^*_{uu} = \mathrm{D}(C_{uu})$.

THEOREM 4: *Let φ satisfy the standard hypotheses. Suppose that $u = v$ and that $\mathrm{I}_r(x_{ij})$ contains an element which is not a zero divisor of R. Then:*
(a) $M := \operatorname{Coker} \varphi$ *is an infinite syzygy module, i.e. there is an infinite exact sequence*

$$0 \to M \to F_1 \to \ldots \to F_n \to F_{n+1} \to \ldots$$

with free R-modules F_i.
(b) $\mathrm{H}^i(\mathscr{X}^*_{uu} \otimes R) = 0$ *for all* $i \geq 1$.
(c) $\operatorname{Ext}^i_R(M, R) = \operatorname{Ext}^i_R(M^*, R) = 0$ *for all* $i \geq 1$.

PROOF: $\mathrm{D}(C_{uu})$ is a maximal Cohen–Macaulay module over the Gorenstein ring S_{uu}. Therefore $\mathrm{H}^i(\tilde{\mathscr{X}}^*_{uu}) = \operatorname{Ext}^i_{S_{uu}}(C_{uu}, S_{uu}) = 0$ for all $i \geq 1$. The argument, which we used to derive Theorem 1 from Proposition 1, shows that $\mathrm{H}^i(\tilde{\mathscr{X}}^*_{uu} \otimes R) = 0$ for $i \geq 1$. Since $\tilde{\mathscr{X}}_{uu} \otimes R$ is a free resolution of $\mathrm{D}(M)$ by Theorem 1, we conclude $\operatorname{Ext}^i_R(\mathrm{D}(M), R) = 0$ for $i \geq 1$. Hence M is an infinite syzygy and $\operatorname{Ext}^i_R(M^*, R) = 0$ for all $i \geq 1$. If we apply these arguments to ζ^*, we obtain the remaining claims.

In the case $u \neq v$ the homological invariants of $\operatorname{Coker} \zeta$ turn out to be grade-sensitive with respect to the ideal $\mathrm{I}_r(x_{ij})$.

THEOREM 5: *Let φ satisfy the standard hypotheses. Suppose that $\mathrm{I}_r(x_{ij})$ contains an element which is not a zero divisor of R, and that $\mathrm{I}_r(x_{ij}) \neq R$. Let $w :=$ grade $\mathrm{I}_r(x_{ij})$ (as an ideal of R). Then in case*
(a) $u > v$:
(i) $M = \operatorname{Coker} \varphi$ *is a w-th syzygy, but not a $(w+1)$-th syzygy.*
(ii) $\mathrm{H}^i(\mathscr{X}^*_{vu} \otimes R) = 0$ *for* $i = 1, \ldots, w$, $\mathrm{H}^{w+1}(\mathscr{X}^*_{vu} \otimes R) \neq 0$.
(iii) $\operatorname{Ext}^i_R(M, R) = 0$ *for* $i = 1, \ldots, w-1$, $\operatorname{Ext}^w_R(M, R) \neq 0$.

(b) $u < v$:
(i) *M is a $(w-1)$-th syzygy, but not a w-th syzygy.*
(ii) $\mathrm{H}^i(\mathscr{X}^*_{vu} \otimes R) = 0$ *for* $i = 1, \ldots, w-1$, $\mathrm{H}^w(\mathscr{X}^*_{vu} \otimes R) \neq 0$.
(iii) $\operatorname{Ext}^i_R M, R) = 0$ *for* $i = 1, \ldots, w$, $\operatorname{Ext}^{w+1}_R(M, R) \neq 0$.

Before we prove Theorem 5 let us recall that $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module if and only if $\mathfrak{p} \not\supset \mathrm{I}_r(x_{ij})$. Theorem 2 of [4] gives an upper bound for $w : w \leq u + v - 2r + 1$.

PROOF: It suffices to consider the case $u > v$ only and to prove the assertions of case (b) for the cokernel N of φ^*. Then $M = \mathrm{D}(N)$ and $N = \mathrm{D}(M)$. Further $\mathscr{X}_{uv} \otimes R$ is a resolution of M, whereas $\mathscr{X}_{vu} \otimes R$ is a resolution of N. Hence the following equivalences hold by virtue of Lemma 6: (a), (i) $\Leftrightarrow$ (a), (ii) $\Leftrightarrow$ (b), (iii) and (b), (i) $\Leftrightarrow$ (b), (ii) $\Leftrightarrow$ (a), (iii).

Part (i) of (b) is an easy consequence of Theorem 1 and the supplement of Theorem 2. N is almost perfect (as an A-module) hence $\operatorname{depth} N_\mathfrak{p} \geq \operatorname{depth} R_\mathfrak{p} - 1$ for all $\mathfrak{p} \in \operatorname{Spec} R$, and $N_\mathfrak{p}$ is free for all $\mathfrak{p} \in \operatorname{Spec} R$ such that $\operatorname{depth} R_\mathfrak{p} \leq w - 1$. By Lemma 6 N is a $(w-1)$-th syzygy. On the other hand, for a prime $\mathfrak{q} \supset \mathrm{I}_r(x_{ij})$ such that $\operatorname{depth} R_\mathfrak{q} = w$ the module $N_\mathfrak{q}$ is not perfect over the corresponding localization of A, whence $\operatorname{depth} N_\mathfrak{q} = w - 1$.

By Theorem 2 M is likewise a w-th syzygy. We consider $\mathscr{L}^*_{vu} \otimes R$ which starts out as

$$\begin{tikzcd}[column sep=small] R^u \arrow[r] \arrow[dr] & R^v \arrow[r, "\psi"] \arrow[d] & R^{r_2} \arrow[r] & \dots \arrow[r] & R^{r_k} \arrow[r] & \dots, \\ & M \arrow[ur] & & & & \end{tikzcd}$$

the map ψ factoring through M. According to Lemma 6, M is a $(w+1)$-th syzygy if and only if $\mathrm{H}^{w+1}(\mathscr{L}^*_{vu} \otimes R) = 0$, and this would force $\operatorname{Coker} \psi$ a w-th syzygy, especially $\operatorname{depth}(\operatorname{Coker} \psi)_\mathfrak{m} = w$ for all prime ideals $\mathfrak{m} \supset \mathrm{I}_r(x_{ij})$ with $\operatorname{depth} R_\mathfrak{m} = w$. Using again the argument, by which we derived the supplement of Theorem 2 from Proposition 3, we only need to disprove this in the generic case $R = S_{uv} = S$, $\varphi = \zeta_{uv} = \zeta$, $(x_{ij}) = (z_{ij})$, $\mathfrak{m} = \mathrm{I}_r(z_{ij})$.

The columns of a matrix representing ψ generate the relations on the columns of (z_{ij}). By Theorem 3 we can always choose the determinantal relations as generators. Then the formation of ψ commutes with the inversion of x_{11} (up to free direct summands), which inductively reduces the problem to the case $r = 1$. We may further replace the base ring $\mathbb{Z}$ by $\mathbb{Q}$, since $\mathfrak{m} \cap \mathbb{Z} = \{0\}$. Then $\mathfrak{m}$ is the irrelevant maximal ideal, and it suffices to show that $\operatorname{Coker} \psi$ itself is not perfect. The argument we will use parallels the one at the end of section 6.

One obtains a matrix of ψ by juxtaposing the "slices"

$$\begin{array}{cl} 0 \dots 0 & \\ \vdots \quad \vdots & \\ 0 \dots 0 & \\ -z_{1j} \dots -z_{uj} & \quad i\text{-th row} \\ 0 \dots 0 & \\ \vdots \quad \vdots & \\ 0 \dots 0 & \end{array}$$

$$\begin{array}{ll} z_{1i} \dots z_{ui} & j\text{-th row} \\ 0 \;\dots\; 0 & \\ \vdots \qquad \vdots & \\ 0 \;\dots\; 0 & \end{array}$$

for all i,j, $1 \le i < j \le v$. Let $G := \operatorname{Coker} \psi$. $G/x_{11}G$ is certainly torsionfree over $S/x_{11}S$, and therefore $(\mathfrak{p}G/z_{11}G) \cap (\mathfrak{q}G/z_{11}G) = 0$, where $\mathfrak{p}$ is the ideal generated by $z_{11}, \dots, z_{1v}$ and $\mathfrak{q}$ the ideal generated by $z_{11}, \dots, z_{u1}$ (cf. the proof of Proposition 6, case (3)). It easy to check that $G/\mathfrak{p}G$ is a direct sum of $\operatorname{Coker} \psi_{u-1,v}$ and a free direct summand (generated by the elements of the canonical basis of the target of ψ which correspond to the first column of each slice) whereas $G/\mathfrak{q}G$ is a direct sum of $\operatorname{Coker} \psi_{u,v-1}$ ($:= 0$ in case $v = 2$) and a module of projective dimension 1 and rank > 0 (arising from the slices which involve $z_{11}, \dots, z_{u1}$). Since $u - 1 \ge v$, $\operatorname{Im} \psi_{u-1,v} = C_{u-1,v}$ is perfect and, hence, $\operatorname{Coker} \psi_{u-1,v}$ is at least almost perfect. $\operatorname{Coker} \psi_{u,v-1}$ is likewise almost perfect, and a module of projective dimension 1 over $S/\mathfrak{q}$ is almost perfect, too: $G/\mathfrak{p}G$ and $G/\mathfrak{q}G$ are almost perfect. $G/(\mathfrak{p} + \mathfrak{q})G$ finally consists of three direct summands: $\operatorname{Coker} \psi_{u-1,v-1}$ ($:= 0$ in case $v = 2$), a free direct summand, and a direct summand of projective dimension 1 and rank > 0 over $S/(\mathfrak{p} + \mathfrak{q})$. The last summand already renders $G/(\mathfrak{p} + \mathfrak{q})G$ not perfect. Then by Lemma 3, (b) $G/x_{11}G$ and, by Lemma 4, G itself are not perfect.

Buchsbaum and Eisenbud showed in [9], Corollary 5.4 that the generic modules C_{uv} have infinite projective dimension in case $r < u$. They use the following argument: If C_{uv} had a finite free resolution then the ideal $\mathfrak{q}$ generated by the r-minors of the first r columns of (z_{ij}) would admit a greatest common divisor as a consequence of their structure theorem. This, however, is impossible for computational reasons as given in [9] and also because the class of $\mathfrak{q}$ generates the divisor class group of S_{uv} which is not zero. One can of course also use Theorems 1 and 2 to conclude that C_{uv} has infinite projective dimension. Even more: In case $u \ge v$ a localization $(C_{uv})_{\mathfrak{s}}$ is necessarily free when it has finite projective dimension, and in case $u < v$ its projective dimension is at most one and this again forces $(C_{uv})_{\mathfrak{s}}$ to be free, since $\operatorname{Ext}^1_{S_{uv}}(C_{uv}, S_{uv}) = 0$. Thus $(C_{uv})_{\mathfrak{s}}$ has finite projective dimension if and only $\mathfrak{s} \not\supset \mathrm{I}_r(z_{ij})$, i.e. if and only if it is free.

The methods of Hochster and Eagon were successfully applied to ideals of minors of symmetric matrices ([18]) and to ideals of pfaffians of alternating matrices ([20]). Therefore one should be able to analyze the modules associated to symmetric and alternating matrices in roughly the same way as we analyzed the modules associated to generic matrices. It should be helpful that the divisorial structure of the corresponding rings is very simple ([14], [2], [17]). We expect the following results:

(a) Let Z_{ij}, $1 \le i \le j \le u$, be indeterminates over $\mathbb{Z}$. Let r be an integer, $1 \le r \le u$, and I the ideals of $(r+1)$-minors of the symmetric matrix having the Z_{ij} as elements in the diagonal and above, and $S := \mathbb{Z}[Z_{ij}]/I$. Let $\sigma: S^u \to S^u$ be given by the matrix $(z_{ij}) = (z_{ji})$. Then $\operatorname{Im} \sigma$ is perfect, and $\operatorname{Coker} \sigma$ is perfect if and only if $r \not\equiv u \bmod 2$.

(b) Let Z_{ij}, $1 \le i < j \le u$, be indeterminates over $\mathbb{Z}$. Let s be an even integer, $2 \le s \le u$, and I the ideal of s-subpfaffians of the alternating matrix having the Z_{ij} as elements above the diagonal, and $S = \mathbb{Z}[Z_{ij}]/I$. Let $\alpha: S^u \to S^u$ be given by the matrix $(z_{ij}) = -(z_{ji})$. Then $\operatorname{Coker} \alpha$ is perfect.

REFERENCES

[1] M. Auslander and M. Bridger: Stable module theory. *Mem. Amer. Math. Soc. 94* (1969).

[2] L.L. Avramov: *A class of factorial domains*. Institut Mittag-Leffler, Report No. 2 (1979).

[3] L.L. Avramov: *Complete intersections and symmetric algebras*. Department of Mathematics, University of Stockholm, Report No. 7 (1980).

[4] W. Bruns: The Eisenbud–Evans generalized principal ideal theorem and determinantal ideals. *Proc. Amer. Math. Soc. 83* (1981) 19–24.

[5] W. Bruns: The canonical module of a determinantal ring. To appear in the *Proceedings of the Symposium on Commutative Algebra* at Durham, July 1981.

[6] D.A. Buchsbaum: Complexes associated with the minors of a matrix. *Symposia Math. IV* (1970) 255–283.

[7] D.A. Buchsbaum and D. Eisenbud: What makes a complex intact? *J. Algebra 25* (1973) 259–268.

[8] D.A. Buchsbaum and D. Eisenbud: Remarks on ideals and resolutions. *Symposia Math. XI* (1973) 193–204.

[9] D.A. Buchsbaum and D. Eisenbud: Some structure theorems for finite free resolutions. *Adv. Math. 12* (1974) 84–139.

[10] D.A. Buchsbaum and D. Eisenbud: Generic free resolutions and a family of generically perfect ideals. *Adv. Math. 18* (1975) 245–301.

[11] D.A. Buchsbaum and D.S. Rim: A generalized Koszul complex II. Depth and multiplicity. *Trans. Amer. Math. Soc. 111* (1964) 197–224.

[12] J.A. Eagon and D.G. Northcott: Generically acyclic complexes and generically perfect ideals. *Proc. Royal Soc. A 299* (1967) 147–172.

[13] H.-B. Foxby: *n*-Gorenstein rings. *Proc. Amer. Math. Soc. 42* (1974) 67–72.

[14] S. Goto: On the Gorensteinness of determinantal loci. *J. Math. Kyoto Univ. 19* (1979) 371–374.

[15] M. Hochster: Generically perfect modules are strongly generically perfect. *Proc. London Math. Soc. (3) 23* (1971) 477–488.

[16] M. Hochster and J.A. Eagon: Cohen–Macaulay rings, invariant theory, and the generic perfection of determinantal loci. *Amer. J. Math. 53* (1971) 1020–1058.

[17] H. Kleppe and D. Laksov: The algebraic structure and deformation of Pfaffian schemes. *J. Algebra 64* (1980) 167–189.

[19] A. Lascoux: Syzygies des variétés déterminantales. *Adv. Math. 30* (1978) 202–237.

[20] V. MARINOV: Perfection of ideals generated by the pfaffians of an alternating matrix. *C. R. Acad. Bulg. Sci. 31* (1979).
[21] H. MATSUMURA: *Commutative Algebra.* W.-A. Benjamin, New York 1970.
[22] D. REES: The grade of an ideal or module. *Proc. Cambridge Phil. Soc. 53* (1957) 28–42.
[23] G. SCHEJA and U. STORCH: Differentielle Eigenschaften der Lokalisierungen analytischer Algebren. *Math. Ann. 197* (1972) 137–170.

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Universität Osnabrück
– Abteilung Vechta –
Driverstraße 22
D-2848 Vechta
Federal Republic of Germany