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AN L_2 -ISOLATION THEOREM FOR YANG-MILLS FIELDS OVER COMPLETE MANIFOLDS

J. Dodziuk and Min-Oo

1. Introduction

In this note we extend the results of the preceding paper [4] to the case of non compact complete manifolds. Beyond the method of [4] we only make use of appropriate cut-off functions as in [2]. This cut-off trick is due to Andreotti and Vesentini (see [5, Th. 26]). It is the point of view of [2] that every vanishing theorem based on a Weitzenböck identity generalizes from the compact to the complete case for L_2 -forms. On the other hand the results in [4] are proved by applying a Sobolev inequality to a Weitzenböck formula for certain bundle valued harmonic forms. Thus it is not surprising that the L_2 -isolation theorem of the preceding paper extends to complete manifolds.

We shall use freely the notation and formulae of [4]. However the isoperimetric constant c_1 will have to be replaced by another isoperimetric constant $c_0 = c_0(M)$ defined as follows:

$$c_0 = \inf_D \frac{(\text{vol}(\partial D))^4}{(\text{vol}(D))^3},$$

where D ranges over all open, relatively compact subsets of M with smooth boundary. M is assumed from now on to be a noncompact, complete, oriented, 4-dimensional, Riemannian manifold.

We begin by stating the results. First, our method yields a simple proof of the following result of C.-L. Shen[6].

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THEOREM 1: Assume that

$$(i) \quad k_- - \frac{1}{\sqrt{3}} |\Omega_-| \geq 0$$

with strict inequality holding at some point of M .

Suppose further that β is a harmonic section of the bundle $\Lambda^2 \otimes E$ satisfying the decay condition:

$$(ii) \quad \lim_{R \rightarrow \infty} \frac{1}{R^2} \int_{B_{2R}(x_0) \setminus B_R(x_0)} |\beta|^2 = 0$$

for some point $x_0 \in M$. Then $\beta \equiv 0$.

In particular, if ω is a sourceless Yang-Mills field such that Ω_- satisfies (i) and (ii), then $\Omega_- \equiv 0$.

As a generalization of Theorem 2 of [4] we obtain the following

THEOREM 2: Assume that the curvature of M satisfies $k_- \geq 0$. If $\|\Omega_-\|_2^2 < c_0/108$, then every square integrable harmonic section of $\Lambda^2 \otimes E$ vanishes identically. In particular, if ω is a sourceless Yang-Mills field with $\|\Omega_-\|_2^2 < c_0/108$ then $\Omega_- \equiv 0$.

Of course, this theorem is of interest only if $c_0(M) > 0$. This is the case for $\mathbb{R}^4$ with the flat metric and we obtain the following:

COROLLARY: Let ω be a sourceless Yang-Mills field over $\mathbb{R}^4$ equipped with a complete conformally flat metric. If $\int_{\mathbb{R}^4} |\Omega_-|^2 < \pi^2 2^5/3^3$, then $\Omega_- \equiv 0$.

This corollary yields an improvement of the constant in Theorem 3 of [4].

THEOREM 3: Let ω be a sourceless Yang-Mills field over S^4 with a conformally flat metric. If

$$\frac{1}{2} \int_{S^4} |\Omega|^2 < 2\pi^2 \left(|p_1(E)| + \frac{16}{27} \right)$$

then ω is either self-dual or anti-self-dual.

Finally the following result gives a lower bound of the spectrum of the Laplacian Δ_-^ω .

THEOREM 4: Suppose $2k_- \geq \mu > 0$. If

$$\|\Omega_-\|_2^2 \leq \frac{c_0}{108}, \text{ then } \text{Spec}(\Delta^\omega) \subset [\mu, \infty).$$

We now prove the theorems stated above. For a given $x_0 \in M$ we can construct (cf. [5]) a family $\{\lambda_R\}_{R>0}$ of Lipschitz continuous function $\lambda_R: M \rightarrow \mathbb{R}$ with the following properties

$$(2) \quad \begin{aligned} & \text{(i) } \text{supp } \lambda_R \subset B_{2R}(x_0) \\ & \text{(ii) } 0 \leq \lambda_R \leq 1 \\ & \text{(iii) } \lambda_R|_{B_R(x_0)} \equiv 1 \\ & \text{(iv) } \lim_{R \rightarrow \infty} \lambda_R = 1 \\ & \text{(v) } |d\lambda_R| < \frac{C}{R} \text{ a.e.,} \end{aligned}$$

where $d\lambda_R$ exists almost everywhere since λ_R is Lipschitz and the constant C is independent of R . In what follows we shall write λ for λ_R . Set $\beta_- = \beta$ in the Weitzenböck identity (3.3) of [4] and take the inner product with $\lambda^2 \beta$. Integration by parts, which is permitted since $\text{supp } \lambda^2 \beta \subset B_{2R}(x_0)$ is compact, now yields

$$(3) \quad \begin{aligned} (\Delta^\omega \beta, \lambda^2 \beta) &= (\nabla \beta, \nabla(\lambda^2 \beta)) + \left(\frac{\kappa}{6} \beta, \lambda^2 \beta\right) \\ &\quad - (\beta \circ W_-, \lambda^2 \beta) - (\Omega_-, [\beta, \lambda^2 \beta]). \end{aligned}$$

Leibnitz rule shows that

$$(4) \quad (\nabla \beta, \nabla(\lambda^2 \beta)) = \|\nabla(\lambda \beta)\|_2^2 - \|d\lambda \otimes \beta\|_2^2.$$

Hence if $\Delta^\omega \beta = 0$, estimating the last three terms on the right hand side of (3) as in [4], we obtain

$$(5) \quad \begin{aligned} \|d\lambda \otimes \beta\|_2^2 &\geq \int_M \left(\frac{\kappa}{6} - \mu_-\right) |\lambda \beta|^2 - \frac{2}{\sqrt{3}} \int_M |\Omega_-| |\lambda \beta|^2 \\ &\geq 2 \int_M \left(k_- - \frac{1}{\sqrt{3}} |\Omega_-|\right) |\lambda \beta|^2. \end{aligned}$$

Observe that for $\lambda = \lambda_R$, (2) implies

$$(6) \quad \|d\lambda \otimes \beta\|_2^2 \leq \frac{c^2}{R^2} \int_{B_{2R}(x_0) \setminus B_R(x_0)} |\beta|^2.$$

Passing to the limit as $R \rightarrow \infty$ in (5) we see that under the assumptions of Theorem 1

$$\int_M \left(k_- - \frac{1}{\sqrt{3}} |\Omega_-| \right) |\beta|^2 = 0.$$

Hence $\beta = 0$ on an open set. By the unique continuation theorem of Aronszajn, Krzywicki and Szarski (cf. [5]) $\beta \equiv 0$ and Theorem 1 is proved.

To prove Theorem 2 we use the Sobolev inequality of P. Li [3, Lemma 6]

$$\|\nabla f\|_2^2 \geq \frac{1}{9} \sqrt{c_0} \|f\|_4^2$$

for compactly supported functions, which implies in our case that

$$(7) \quad \|\nabla(\lambda\beta)\|_2^2 \leq \frac{1}{9} \sqrt{c_0} \|\lambda\beta\|_4^2.$$

Now assuming $\Delta^\omega \beta = 0$, substituting (4) into (3), using (7) together with the definition of k_- and the pointwise estimate (3.8) of [4], we obtain

$$\begin{aligned} \|d\lambda \otimes \beta\|_2^2 &\geq \left(\frac{1}{9} \sqrt{c_0} - \frac{2}{\sqrt{3}} \|\Omega_-\|_2 \right) \|\lambda\beta\|_4^2 \\ &\quad + 2 \int k_- |\lambda\beta|^2. \end{aligned}$$

Theorem 2 now follows by passing to the limit as $R \rightarrow \infty$, since by (6) $\lim_{R \rightarrow \infty} \|d\lambda \otimes \beta\|_2^2 = 0$ if β is square integrable.

The corollary follows from Theorem 2 by substituting the value $c_0(\mathbb{R}^4) = 2^7 \pi^2$. Theorem 3 follows from the corollary since $\mathbb{R}^4$ with the flat metric is conformally equivalent to $S^4 \setminus \{\text{pt.}\}$ with the standard metric and because the Yang-Mills functional is conformally invariant.

We now turn to the proof of Theorem 4. The Laplacian Δ^ω is essentially self-adjoint on $C_0^\infty(\Lambda^2 \otimes E)$. This is a consequence of completeness (cf. [1]). Thus it suffices to estimate $(\Delta^\omega \beta, \beta)$ for compactly supported β . From the Weitzenböck identity (3.3) of [4], the Sobolev inequality (7), the definition of k_- and the estimate (3.8) of [4] we obtain through integration by parts the following estimate:

$$\begin{aligned} (\Delta^\omega \beta, \beta) &\geq \left(\frac{1}{9} \sqrt{c_0} - \frac{2}{\sqrt{3}} \|\Omega_-\|_2 \right) \|\beta\|_4^2 + 2 \int k_- |\beta|^2 \\ &\geq \mu \|\beta\|_2^2 \end{aligned}$$

provided the assumptions of Theorem 4 are satisfied. This proves the theorem.

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Added in proof

In “Best Constant in Sobolev Inequality”, *Ann. Mat. Pure Appl.* 110 (1976) 353–372, G. Talenti shows that the best constant in the Sobolev inequality $\|\nabla f\|^2 \geq c\|f\|_2^2$ for functions on R^4 is $c = (8\pi/\sqrt{6})$. Using this we can improve the Corollary of Theorem 2 and Theorem 3. In the corollary the constant $\pi^2 2^5/3^3$ can be replaced by $8\pi^2$ and in Theorem 2 $16/27$ may be replaced by 2. The statements obtained this way are *optimal*. In fact, Bourguignon and Lawson, in “Stability and Isolation Phenomena for Yang-Mills Fields”, *Commun. Math. Phys.* 79 (1981) 189–230, exhibit Yang-Mills fields on S^4 with its canonical metric for which the pointwise norm $|\Omega_-| \equiv \sqrt{3}$, and hence $\|\Omega_-\|_2^2 = 8\pi^2$. Analyzing the case of equality carefully, we can show that if $\|\Omega_-\|_2^2 = 8\pi^2$, then $|\Omega_-| \equiv \sqrt{3}$. Such fields have been classified by Bourguignon and Lawson.