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TADAYUKI MATSUZAWA

SHUKICHI TANNO

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## ESTIMATES OF THE FIRST EIGENVALUE OF A BIG CUP DOMAIN OF A 2-SPHERE

Tadayuki Matsuzawa and Shukichi Tanno

Let  $S^2 = (S^2, g_0)$  be the unit 2-sphere in the Euclidean 3-space  $E^3$  and let  $D(r)$  denote a cup domain of  $S^2$  which is the geodesic disk of radius  $r$  centered at the south pole  $Q$ . Let  $\lambda_1(r)$  denote the first eigenvalue of the Dirichlet problem for  $D(r)$ . For  $\pi/2 \leq r < \pi$ , we have  $2 \geq \lambda_1(r) > 0$ . However nice estimates of  $\lambda_1(r)$  seem to be unknown. For example, the value  $r$  for which  $\lambda_1(r) = 1$  is related to the study of stability of minimal surfaces in  $E^4$  (see [1], p. 27), and some estimate of  $r$  for which  $\lambda_1(r) > 1$  was given in [1].

In this note we give estimates of  $\lambda_1(r)$  for  $r$  which is near  $\pi$  (Theorem 1.1). If we let  $r$  tend to  $\pi$ , then

$$\lambda_1(r) \sim [-2 \log(\pi - r)]^{-1}.$$

This agrees with a recent result by S. Ozawa [5] (cf. Remark ii in the section 2).

In the section 3, as an example, we calculate our estimates for  $r = \pi - 10^{-50}$ ;

$$|\lambda_1(\pi - 10^{-50}) - 0.00438| < 0.00007.$$

Contrary to 2-dimensional case, the first eigenvalue of a cup domain of a unit 3-sphere  $S^3$  is explicitly calculatable (see [4], p. 154, [2], p. 201).

We express our thanks to Mr. Muto for some comments.

### 1. The first eigenvalue of a cup domain

Let  $(x, y, z)$  be the canonical coordinates of  $E^3$ . Then  $S^2 = \{(x, y, z); x^2 + y^2 + z^2 = 1\}$ . Put  $P = (0, 0, 1)$  and  $Q = (0, 0, -1)$ . We denote the

Laplacian on  $S^2$  by  $\Delta$ . Then it is known that

$$f_C(x, y, z) = f(z) = 2 - z \log \frac{1+z}{1-z} + Cz$$

satisfies  $\Delta f + 2f = 0$  on  $S^2 - \{P, Q\}$  for any constant  $C$ .  $f(z) = 0$  has just two solutions for  $-1 < z < 1$ , and solutions depend on  $C$ . Let  $r = r(x, y, z)$  denote the distance function from  $Q$  on  $S^2$ . Then

$$z = -\cos r.$$

We fix  $r_0 (0 < r_0 < \pi)$  and consider the corresponding cup domain  $D(r_0) = \{(x, y, z) \in S^2; r(x, y, z) \leq r_0\}$ . Then we have some constant  $C = C(r_0)$  such that the solutions  $\gamma = \gamma(r_0)$ ,  $\beta = \beta(r_0)$  of  $f_C(z) = 0$  satisfy  $\gamma < \beta = -\cos r_0$ . Namely  $C$  is determined by  $\beta$ ;

$$(1.1) \quad C = \log \frac{1+\beta}{1-\beta} - \frac{2}{\beta}.$$

Let  $z = \alpha$  be the point where  $f_C(z)$  takes its maximum. Then

$$(1.2) \quad C = \log \frac{1+\alpha}{1-\alpha} + \frac{2\alpha}{1-\alpha^2}$$

and the maximum value is equal to  $2/(1-\alpha^2)$ . We define a function  $F$  on  $D(r_0)$  by

$$\begin{aligned} F(x, y, z) &= ((1-\alpha^2)/2) f_C(x, y, z) & \alpha \leq z \leq \beta \\ &= 1 & -1 \leq z \leq \alpha. \end{aligned}$$

$F$  is a piecewise smooth  $C^1$ -function on  $D(r_0)$  vanishing on the boundary of  $D(r_0)$ . Therefore the first eigenvalue  $\lambda_1(r_0)$  of  $D = D(r_0)$  satisfies

$$\lambda_1(r_0) < \int_D (dF, dF) / \int_D F^2.$$

Since the volume element of  $D(r_0)$  is  $\sin r \, dr \wedge d\theta$  (where  $\theta$  denotes the rotation parameter around the  $z$ -axis), we get

$$\begin{aligned} \int_D F^2 &= 2\pi \int_{-1}^{\beta} F^2 \, dz \\ &= 2\pi(1+\alpha) + 2\pi \int_{\alpha}^{\beta} F^2 \, dz. \end{aligned}$$

Let  $D = D' \cup D''$ , where  $D'$  denotes the part of  $D$  for  $z \leq \alpha$  and  $D''$  corresponds to  $\alpha \leq z \leq \beta$ . Then

$$\begin{aligned} \int_D (dF, dF) &= \int_{D'} (dF, dF) = \int_{D'} (-\Delta F, F) \\ &= 2 \int_{D'} F^2 = 4\pi \int_{\alpha}^{\beta} F^2 dz, \end{aligned}$$

where the second equality follows from the fact that  $F$  vanishes on  $z = \beta$  and the normal derivative of  $F$  vanishes on  $z = \alpha$ . Therefore we get

$$(1.3) \quad \lambda_1(r_0) < 2A/(1 + \alpha + A).$$

**THEOREM 1.1:** *Let  $D(r_0)$  be a cup domain in  $S^2$ . Then the first eigenvalue  $\lambda_1(r_0)$  is estimated by*

$$(1.4) \quad 1/\log[2/(1 + \cos r_0)] < \lambda_1(r_0) < 2A/(1 + \alpha + A),$$

where  $\beta$ ,  $C$ ,  $\alpha$  and  $A$  are determined by the following:

$$(1.5) \quad \beta = -\cos r_0,$$

$$(1.6) \quad C = \log \frac{1 + \beta}{1 - \beta} - \frac{2}{\beta} = \log \frac{1 + \alpha}{1 - \alpha} + \frac{2\alpha}{1 - \alpha^2},$$

$$(1.7) \quad A = \frac{(1 - \alpha^2)^2}{4} \int_{\alpha}^{\beta} \left\{ 2 - z \log \frac{1 + z}{1 - z} + Cz \right\}^2 dz.$$

**PROOF:** The first inequality of (1.4) follows from a calculation applying  $\Theta(t) = \sin t$  to Theorem 3 in [3]. The second inequality of (1.4) is (1.3). q.e.d.

**REMARK i:** Our inequalities (1.4) are most effective for  $r_0 \doteq \pi$ . One merit of (1.4) is that it enables us to calculate approximated values of  $\lambda_1(r_0)$ ,  $r_0 \doteq \pi$ .

## 2. Calculation of $A = A(r_0)$

Here we calculate

$$I = \frac{2A}{(1 - \alpha^2)^2} = \frac{1}{2} \int_{\alpha}^{\beta} \left\{ 2 - z \log \frac{1 + z}{1 - z} + Cz \right\}^2 dz.$$

We put  $u = (1+z)/(1-z)$ ,  $a = (1+\alpha)/(1-\alpha)$  and  $b = (1+\beta)/(1-\beta)$ . Then,  $dz = 2 du/(u+1)^2$  and

$$\begin{aligned} I &= \left[ -\frac{C^2+4C+4}{u+1} + \frac{2C^2+4C}{(u+1)^2} - \frac{4C^2}{3(u+1)^3} \right]_a^b \\ &\quad + \left[ \left( -\frac{1}{u+1} + \frac{2}{(u+1)^2} - \frac{4}{3(u+1)^3} \right) (\log u)^2 \right]_a^b \\ &\quad + \int_a^b \left\{ -\frac{2C+4}{(u+1)^2} + \frac{8C+8}{(u+1)^3} - \frac{8C}{(u+1)^4} \right\} \log u \, du \\ &\quad + \int_a^b \left\{ \frac{1}{u+1} - \frac{2}{(u+1)^2} + \frac{4}{3(u+1)^3} \right\} \frac{2}{u} \log u \, du. \end{aligned}$$

Continuing calculations and applying

$$\begin{aligned} \int_a^b \frac{\log u}{u+1} \, du &= \frac{1}{2} [(\log(1-z))^2]_a^\beta - [\log(1+z) \log(1-z)]_a^\beta \\ &\quad + \int_\alpha^\beta \frac{\log(1-z)}{1+z} \, dz, \end{aligned}$$

we obtain the following

**PROPOSITION 2.1:** For  $a = (1+\alpha)/(1-\alpha)$  and  $b = (1+\beta)/(1-\beta)$ ,

$$\begin{aligned} \frac{2A}{(1-\alpha^2)^2} &= \left[ -\frac{C^2+8C/3+4/3}{u+1} + \frac{2C^2+8C/3}{(u+1)^2} - \frac{4C^2}{3(u+1)^3} \right. \\ &\quad + \left( -\frac{1}{u+1} + \frac{2}{(u+1)^2} - \frac{4}{3(u+1)^3} \right) (\log u)^2 \\ &\quad + \left( \frac{2C+8/3}{u+1} - \frac{4C+8/3}{(u+1)^2} + \frac{8C}{3(u+1)^3} \right) \log u + \frac{2C}{3} \log \frac{u+1}{u} \Big]_a^b \\ &\quad + \frac{1}{3} [(\log(1+z))^2]_a^\beta - \frac{2}{3} \int \frac{\log(1-z)}{1+z} \, dz. \end{aligned} \tag{2.1}$$

The last term of (2.1) is estimated by the following

**PROPOSITION 2.2:** For  $\alpha > 0$  we have

$$\begin{aligned} -\int_\alpha^\beta \frac{\log(1-z)}{1+z} \, dz &< \frac{1}{1+\alpha} [(1-z) \log(1-z)]_a^\beta + \frac{\beta-\alpha}{1+\alpha}, \\ -\int_\alpha^\beta \frac{\log(1-z)}{1+z} \, dz &> \frac{1}{1+\beta} [(1-z) \log(1-z)]_a^\beta + \frac{\beta-\alpha}{1+\beta}. \end{aligned}$$

Next we study the behavior of  $A(r_0)$  as  $r_0 \rightarrow \pi$ . We put  $\delta = 1 - \alpha$  and  $\rho = 1 - \beta$ . Then we get

$$C = -\log \rho - 2 + \log 2 + \dots,$$

$$\delta \log \rho = -1 + \delta \log \delta - 3\delta/2 + \dots$$

etc. By Propositions 2.1 and 2.2 we obtain

$$\lim_{r_0 \rightarrow \pi} \log(1 - \beta) A(r_0) = -1,$$

and hence for  $r_0 \rightarrow \pi$ ,

$$2A/(1 + \alpha + A) \sim [-\log(1 + \cos r_0)]^{-1}.$$

Therefore Theorem 1.3 implies

$$(2.2) \quad \lambda_1(r_0) \sim [-\log(1 + \cos r_0)]^{-1}.$$

**COROLLARY 2.3:** *Let  $D_\epsilon(P)$  be an  $\epsilon$ -neighborhood of a point  $P$  in  $S^2$ . Then the first eigenvalue  $\lambda_{1,\epsilon}$  of  $S^2 - D_\epsilon(P)$  satisfies*

$$(2.3) \quad \lambda_{1,\epsilon} \sim [-2 \log \epsilon]^{-1} \quad (\epsilon \downarrow 0).$$

**PROOF:** We put  $\epsilon = \pi - r_0$ . Then  $1 + \cos r_0 = \epsilon^2/2 + \dots$ , and  $\lambda_{1,\epsilon} = \lambda_1(r_0)$ . Thus, (2.3) follows from (2.2). q.e.d.

**REMARK ii.** A result of S. Ozawa [5] is stated as follows; Let  $(N, g)$  be a compact 2-dimensional Riemannian manifold and let  $D_\epsilon(p)$  be an  $\epsilon$ -neighborhood of a point  $p$  of  $N$ . Then the first eigenvalue  $\lambda_{1,\epsilon}$  of  $(N, g) - D_\epsilon(p)$  satisfies

$$\lambda_{1,\epsilon} = 2\pi[-\text{Vol}(N, g) \log \epsilon]^{-1} + o((\log \epsilon)^{-2}),$$

where  $\text{Vol}(N, g)$  denotes the volume of  $(N, g)$ . His proof depends on abstract analysis, and it is difficult to get the geometric shape of an eigenfunction corresponding to  $\lambda_{1,\epsilon}$ . To get an approximated shape of an eigenfunction the following observation is useful. Any 2-dimensional Riemannian manifold  $(N, g)$  is locally conformal to  $(S^2, g_0)$ . Let  $U$  be a neighborhood of  $p$  in  $N$  and let  $\phi$  be a conformal map from  $U$  into  $U^* = \phi U$  in  $S^2$ . We put  $P = \phi(p)$ . If  $\epsilon_0$  is sufficiently small, for any  $\epsilon < \epsilon_0$  we have some  $\mu = \mu(\epsilon)$  such that the boundary of an

$\epsilon$ -neighborhood  $D_\epsilon(p)$  of  $p$  in  $N$  is mapped by  $\phi$  into  $D_{\epsilon+\mu}(P) - D_{\epsilon-\mu}(P)$  in  $S^2$ . The converse is also true. In the 2-dimensional case we have another special property that, for conformally related metrics  $g_0$  and  $g^* = hg_0$  ( $h$ : positive function) the Laplacians  $\Delta$  and  $\Delta^*$  corresponding to  $g_0$  and  $g^*$  satisfy  $\Delta^*f = (1/h)\Delta f$  for each function  $f$ . Therefore, in an infinitesimal domain,  $h$  looks like a constant, and a function similar to  $F$  in the section 1 can be constructed in  $N$ .

### 3. An example; $r_0 = \pi - 10^{-50}$

As an example we calculate our estimate of  $\lambda_1(r_0)$  for  $r_0 = \pi - 10^{-50}$ .

$$229.64480 < C = \log b - 2/\beta < 229.64481$$

$$0.9955365 < \alpha < 0.9955366$$

$$0.00444348 < A < 0.00444382$$

$$0.0043169 < \lambda_1(r_0) < 0.0044439$$

Estimate in the introduction corresponds to

$$0.00431 < \lambda_1(r_0) < 0.00445.$$

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Department of Mathematics  
Tokyo Institute of Technology  
Ohokayama, Meguro, Tokyo, Japan