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THE VOLUMES OF SMALL GEODESIC BALLS FOR A METRIC CONNECTION

V. Miquel

§1. Introduction

Let M be a real-analytic Riemannian manifold of dimension n . Let $V_m^\nabla(r)$ denote the volume of the geodesic ball with center $m \in M$ and radius r , where ∇ denotes the Levi-Civita connection. Then $V_m^\nabla(r)$ can be expanded in a power series in r . In 1848 Bertrand–Diguët–Puisseux [3] computed the first two terms for surfaces in \mathbb{R}^3 . Vermeil [14] in 1917 and Hotelling [11] in 1939 generalized it to arbitrary Riemannian manifolds. Recently, the third and fourth term have been computed by A. Gray [5] and by A. Gray and L. Vanhecke [6], respectively.

To obtain that expansion, it is necessary to discuss general power expansions of tensor fields in normal coordinates as used for example for harmonic spaces (see [13]).

The volumes of tubes about submanifolds of \mathbb{R}^n , C^n , S^n , CP^n have been computed by H. Weyl [15], R.A. Wolf [17], F. J. Flaherty [4], P. A. Griffiths [9]. The expansions of volumes of tubes about submanifolds of arbitrary Riemannian manifolds are given in [11], [7], [8].

In this note we consider a metric connection D on M . Let $V_m^D(r)$ denote the volume of the D -geodesic ball $\bar{B}_r^D(m)$ of center m and radius r . Then $\bar{B}_r^D(m) \subseteq \bar{B}_r^\nabla(m)$ (see §2). We compute the first non trivial term C_1^D of the expansion of $V_m^D(r)$. This is our main theorem 5.4. If M is C^∞ , we can compute the Taylor expansion of $V_m^D(r)$, since it is the same as in the analytic case, although it may not be convergent.

We shall show that the difference $C_1^D - C_1^\nabla$ with the case $D = \nabla$ has constant sign and it vanishes only if ∇ and D have the same geodesics

(Corollary 5.5). On the total volume function, this result ($V_m^D(r) \leq V_m^\nabla(r)$) is a consequence of the inclusion of the D -balls in the ∇ -balls. This fact also implies that the volumes coincide only if ∇ and D have the same geodesics. The Corollary 5.5 shows that it is also true with the weaker hypothesis $C_1^D = C_1^\nabla$.

These results were announced in [12].

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§2. Geodesic balls for a metric connection

Let $\langle \cdot, \cdot \rangle$ be the metric tensor of M , $\chi(M)$ the algebra of vector fields over M and M_m the tangent space to M at the point $m \in M$.

A metric connection D over M is a linear connection which satisfies

$$(2.1) \quad X\langle Y, Z \rangle = \langle D_X Y, Z \rangle + \langle Y, D_X Z \rangle \text{ for every } X, Y, Z \in \chi(M).$$

By a normal coordinate system $(U; x^1, \dots, x^n)$ at m with respect to D we take a normal coordinate system in the sense of [10] such that the local vector fields $X_i = \partial/\partial x^i$ are orthonormal at m . Then, if $\exp_m: B_r(0) \rightarrow U$ is the exponential map associated to D , the normal coordinates are given by $x^i(\exp_m(\sum_{j=1}^n a^j e_j)) = a^i$, where $\{e_1, \dots, e_n\}$ is an orthonormal basis of M_m .

In this paper we always work in the domain U of a normal coordinate system.

The injectivity radius r_D of (M, D) at m is the supremum of the positive real numbers r such that \exp_m is a diffeomorphism of $B_r(0)$ onto its image.

Let \mathcal{U} be the open set $\mathcal{U} = \exp_m B_{r_D}(0)$. For any p in \mathcal{U} , there exists a unique D -geodesic arc joining m and p . Then, we define $\delta^D(m, p)$ as the length of this geodesic arc. Then, since the velocity vector of a geodesic for a metric connection has constant length,

$$(2.2) \quad \delta^D(m, p) = \|\exp_m^{-1}(p)\|.$$

Let r be a positive real number such that $r < r_D$. We call a

D -geodesic ball of center m and radius r , the set $\bar{B}_r^D(m) = \{p \in \mathcal{U} / \delta^D(m, p) \leq r\}$. By (2.2) we have $\bar{B}_r^D(m) = \exp_m(\bar{B}_r(0))$.

Now, we examine the inclusion relation between $\bar{B}_r^D(m)$ and $\bar{B}_r^\nabla(m)$.

It is well known that, if d is the standard distance function for the Riemannian manifold M , and $d(m, p) = r < r_\nabla$, there exists a unique arc of ∇ -geodesic σ from m to p of length r . Moreover, if α is another arc of curve from m to p , then the length of α is greater or equal than r .

Let r be a real number such that $r < \min(r_D, r_\nabla)$. If $p \in \bar{B}_r^D(m)$, there exists an arc of D -geodesic α joining m and p , and another arc of ∇ -geodesic σ from m to p . As we have indicated above, we have $d(m, p) = \text{length of } \sigma \leq \text{length of } \alpha = \delta^D(m, p) \leq r$. Then $p \in \bar{B}_r^\nabla(m)$ and, consequently, $\bar{B}_r^D(m) \subseteq \bar{B}_r^\nabla(m)$. It implies $V_m^D(r) \leq V_m^\nabla(r)$.

We are going to obtain an integral formula for $V_m^D(r)$, the volume of $\bar{B}_r^D(m)$. In [6], it was done for the Levi-Civita connection ∇ , by using the Gauss lemma. This approach fails for a general metric connection, and we require the use of polar coordinates as defined in [1] and [2] for a new proof of this formula.

2.1. PROPOSITION: *Let M be orientable, ω , the standard volume form on M and $\omega_{1\dots n} = \omega(X_1, \dots, X_n)$. For any $r_0 < r_D$ we have*

$$(2.3) \quad V_m^D(r_0) = \int_0^{r_0} r^{n-1} \left(\int_{S^{n-1}} \omega_{1\dots n}(\exp_m(ru)) \sigma \right) dr,$$

where σ is the standard volume form on S^{n-1} .

PROOF: The definition of $V_m^D(r_0)$ gives

$$V_m^D(r) = \int_{\bar{B}_{r_0}^D(m)} \omega = \int_{\bar{B}_{r_0}(0)} \exp_m^* \omega = \int_{B_{r_0}(0)} (\omega_{1\dots n} \circ \exp_m) \theta,$$

where θ is the standard volume form on M_m .

Let be $f: S^{n-1} \times]0, r_0[\rightarrow B_{r_0}(0) - \{0\}$ the map defining the polar coordinates (u, r) . It is well known [2] that $f^* \theta = r^{n-1} dr \wedge \sigma$, so

$$V_m^D(r) = \int_{S^{n-1} \times]0, r_0[} (\omega_{1\dots n} \circ \exp_m(ru)) r^{n-1} dr \wedge \sigma.$$

From this, (2.3) follows immediately. \square

§3. Power expansions in normal coordinates of a r -covariant tensor

Let S be the curvature operator of D given by

$$S_{XY} = D_{[X,Y]} - [D_X, D_Y], \quad S_{XYZW} = \langle S_{XY}Z, W \rangle.$$

We denote by T the torsion of D , and

$$D_{X_1 \dots X_p}^p Y = D_{X_1}(D_{X_2} \dots (D_{X_p} Y) \dots).$$

We say that $X \in \chi(M)$ is a coordinate vector field at m if there exists constants a^1, \dots, a^n such that, in \mathcal{U} , $X = \sum_{i=1}^n a^i X_i$. From now on X, Y, Z, \dots will denote coordinate vector fields and a, b, c, \dots their corresponding integral curves with initial conditions $a(0) = b(0) = c(0) = \dots = m$. Thus, a, b, c, \dots are geodesics starting at m , and $a'(t) = X_{a(t)}$ wherever $a(t)$ is defined. Moreover we have $S_{XY}Z = -D_{XY}^2 Z + D_{YX}^2 Z$, $T_X Y = D_X Y - D_Y X$, and $T_{XYZ} = \langle T_X Y, Z \rangle$.

Then, we have the following results, whose proofs follow closely the ones given in [5] for the corresponding ones.

3.1. LEMMA:

$$(3.1.1) \quad (D_{X \dots X}^p X)_{a(t)} = 0 \quad p = 1, 2, \dots$$

$$(3.1.2) \quad (D_X Y)_m = \frac{1}{2} (T_X Y)_m.$$

3.2. LEMMA:

$$(3.2.1) \quad (D_{X \dots X}^p Y)_m + \sum_{k=1}^p (D_{X \dots \overset{k}{Y} \dots X}^p X)_m = 0.$$

$$(3.2.2) \quad \sum_{k=1}^p (D_{X \dots \overset{k}{Y} \dots X}^p Y)_m + \sum_{k \neq 1}^p (D_{X \dots \overset{k}{XYX} \dots \overset{1}{XYX} \dots X}^p X)_m = 0.$$

.....

$$(3.2.p-1) \quad \sum_{k \neq 1}^p (D_{Y \dots \overset{k}{YXY} \dots \overset{1}{YXY} \dots Y}^p Y)_m + \sum_{k=1}^p (D_{Y \dots \overset{k}{X} \dots Y}^p X)_m = 0.$$

$$(3.2.p) \quad \sum_{k=1}^p (D_{Y \dots \overset{k}{X} \dots Y}^p Y)_m + (D_{Y \dots Y}^p X)_m = 0.$$

3.3. LEMMA: *At m , we have*

$$(3.3) \quad (p+1)D_{X \dots X}^p Y - pD_{X \dots X}^{p-1}(T_X Y) + (p-1)D_{X \dots X}^{p-2}(S_{XY} X) = 0.$$

From (3.1.1), (3.1.2) and (3.3) we have

$$(3.4) \quad (D_{XX}^2 Y)_m = \left\{ -\frac{1}{3} S_{XY} X + \frac{2}{3} D_X(T)_X Y + \frac{1}{3} T_X T_X Y \right\}_m.$$

The same method works for $p \geq 3$ to get $D_{X \dots X}^p Y$.

From now on, we assume that the manifold M and any mathematical object defined on M are real-analytic. (The expansions are the same for the C^∞ case).

Let W be a r -covariant tensor field on a neighbourhood of m . We denote $W(X_{a_1}, \dots, X_{a_r})$ by $W_{a_1 \dots a_r}$ and D_{X_i} by D_i . The power series expansion of $W_{a_1 \dots a_r}$ is then

$$(W_{a_1 \dots a_r})_x = \sum_{k=0}^{\infty} \sum_{i_1, \dots, i_k=1}^n \frac{1}{k!} (X_{i_1} \dots X_{i_k} W_{a_1 \dots a_r})_m x^{i_1} \dots x^{i_k},$$

where x^1, \dots, x^n are the coordinates of the point $x \in M$.

Notice that

$$(3.5) \quad (X^p W_{a_1 \dots a_r})_m = \sum_{\nu_1 + \dots + \nu_{r+1} = p} \frac{p!}{\nu_1! \dots \nu_{r+1}!} \\ \times D_{X \dots X}^{\nu_{r+1}}(W) (D_{X \dots X}^{\nu_1} X_{a_1}, \dots, D_{X \dots X}^{\nu_r} X_{a_r})_m.$$

Then, it is possible to determine (3.5) as a function of S, T and their covariant derivatives. We can also determine the coefficients of the power series expansion of $W_{a_1 \dots a_r}$ by linearizing the left hand side of (3.5).

3.4. THEOREM: *For any point x in U we have the following expansion:*

$$W_{a_1 \dots a_r}(x) = W_{a_1 \dots a_r}(m) + \sum_{i=1}^n \left\{ D_i(W)_{a_1 \dots a_r} \right. \\ \left. + \frac{1}{2} \sum_{s=1}^r \sum_{q=1}^n T_{i a_s q} W_{a_1 \dots a_{s-1} q a_{s+1} \dots a_r} \right\}(m) \\ \times x^i + \frac{1}{2} \sum_{i,j=1}^n \left\{ D_{ij}^2(W)_{a_1 \dots a_r} \right.$$

$$\begin{aligned}
& + \sum_{s=1}^r \sum_{q=1}^n T_{ia_sq} D_j(W)_{a_1 \dots a_{s-1} q a_{s+1} \dots a_r} \\
& + \frac{1}{3} \sum_{s=1}^r \sum_{q=1}^n \left(-S_{ia_jq} + 2D_i(T)_{ja_sq} + \sum_{\beta=1}^n T_{i\beta q} T_{ja_s\beta} \right) \\
& \times W_{a_1 \dots a_{s-1} q a_{s+1} \dots a_r} + \frac{1}{4} \sum_{s \neq t=1}^r \sum_{q,h=1}^n T_{ia_sq} T_{ja_th} \\
& \times W_{a_1 \dots a_{s-1} q a_{s+1} \dots a_{t-1} h a_{t+1} \dots a_r} \Big\} (m) x^i x^j.
\end{aligned}$$

PROOF: From (3.1.2), (3.4) and (3.5) we get

$$\begin{aligned}
X_i(W_{a_1 \dots a_r})(m) & = \left\{ D_i(W)_{a_1 \dots a_r} + \sum_{s=1}^r W(X_{a_1}, \dots, D_i X_{a_s}, \dots, X_{a_r}) \right\}_m \\
& = \left\{ D_i(W)_{a_1 \dots a_r} + \sum_{s=1}^r \sum_{q=1}^n \frac{1}{2} T_{ia_sq} W_{a_1 \dots a_{s-1} q a_{s+1} \dots a_r} \right\}_m,
\end{aligned}$$

which is the coefficient of x^i , and

$$\begin{aligned}
X_i^2(W_{a_1 \dots a_r})(m) & = \left\{ D_{ii}^2(W)_{a_1 \dots a_r} + \sum_{s=1}^r \sum_{q=1}^n T_{ia_sq} D_i(W)_{a_1 \dots a_{s-1} q a_{s+1} \dots a_r} \right. \\
& + \frac{1}{3} \sum_{s=1}^r \sum_{q=1}^n \left(-S_{ia_siq} + 2D_i(T)_{ia_sq} \right. \\
& \left. + \sum_{\beta=1}^n T_{i\beta q} T_{ia_s\beta} \right) W_{a_1 \dots a_{s-1} q a_{s+1} \dots a_r} \\
& \left. + \frac{1}{4} \sum_{s \neq t=1}^r \sum_{q,h=1}^n T_{ia_sq} T_{ia_th} W_{a_1 \dots a_{s-1} q a_{s+1} \dots a_{t-1} h a_{t+1} \dots a_r} \right\}_m.
\end{aligned}$$

Linearizing the last expression, we get the coefficient of $x^i x^j$. \square

We apply this expansion to the metric tensor. Let $g_{ij} = \langle X_i, X_j \rangle$, then $g_{ij}(m) = \delta_{ij}$ and, since $D(\ , \) = 0$, we get

3.5. PROPOSITION: For any x in U and $A, B = 1, \dots, n$, we have

$$\begin{aligned}
g_{AB}(x) & = \delta_{AB} + \frac{1}{2} \sum_{i=1}^n (T_{iAB} + T_{iBA})_m x^i \\
& + \frac{1}{6} \sum_{i,j=1}^n \left\{ -(S_{iAjB} + S_{iBjA}) + 2(D_i(T)_{jAB} + D_i(T)_{jBA}) \right. \\
& + \sum_{\beta=1}^n (T_{i\beta A} T_{jB\beta} + T_{i\beta B} T_{jA\beta}) \\
& \left. + \frac{3}{4} \sum_{\beta=1}^n (T_{iA\beta} T_{jB\beta} + T_{iB\beta} T_{jA\beta}) \right\}_m x^i x^j + \dots.
\end{aligned}$$

In the remainder of this paper we assume that M is orientable. This is not a real restriction, since we are always working locally.

We choose the normal coordinates in such a way that $\{X_1, \dots, X_n\}$ is a positively-oriented local frame. As X_1, \dots, X_n are orthonormal at m , we have $\omega_{1\dots n}(m) = 1$. Clearly then $D\omega = 0$.

Let ρ be the Ricci tensor of the connection D . Then, for any local orthonormal frame $\{E_1, \dots, E_n\}$, $\rho(X, Y) = \sum_{i=1}^n S_{XE_iYE_i}$.

3.6. PROPOSITION: *Applying 3.4 to ω , we get, for any x in U ,*

$$\begin{aligned} \omega_{1\dots n}(x) = & 1 + \frac{1}{2} \sum_{i=1}^n \left(\sum_{\beta=1}^n T_{i\beta\beta} \right)_m x^i \\ & + \frac{1}{6} \sum_{ij=1}^n \left(-\rho_{ij} + 2 \sum_{\beta=1}^n D_i(T)_{j\beta\beta} + \frac{1}{4} \sum_{\beta,\delta=1}^n T_{i\beta\beta} T_{j\beta\delta} \right. \\ & \left. + \frac{3}{4} \sum_{\beta,\delta=1}^n T_{i\beta\beta} T_{j\delta\delta} \right)_m x^i x^j + \dots \end{aligned}$$

§4. Relationship between T and B

Let B be the difference tensor of the connections D and ∇ , $B_X Y = D_X Y - \nabla_X Y$. We define $B_{XYZ} = \langle B_X Y, Z \rangle$. Then

$$(4.1) \quad T_{XYZ} = B_{XYZ} - B_{YXZ}$$

It is well known (see [10]) that the connections D and ∇ have the same geodesics if and only if $B_X Y = -B_Y X$. Moreover [18], a connection D is a metric connection if and only if $B_{XYZ} = -B_{XZY}$. Then the connections D and ∇ have the same geodesics if and only if $T_{XYZ} = -T_{XZY}$. In fact, D and ∇ have the same geodesics if and only if $B_X Y = -B_Y X$, i.e., $\langle T_X Y, Y \rangle = \langle B_X Y - B_Y X, Y \rangle = 2\langle B_X Y, Y \rangle = 0$. In this case the torsion T and the tensor B belong to the irreducible subspace $\Lambda^3 V^*$ of $\Lambda^2 V^* \otimes V^*$ and $V^* \otimes \Lambda^2 V^*$, respectively, where $V = M_m$.

It is also useful to have an expression of B in terms of T . Yano [18] proved

$$(4.2) \quad B_{XYZ} = \frac{1}{2} (T_{XYZ} + T_{ZXY} - T_{YZX}).$$

Here, we give another proof of this formula by using elementary representation theory. Later we shall use the same method to obtain a good formula for the expansion of $V_m^D(r)$.

The tensor T_m belongs to $\Lambda^2 V^* \otimes V^*$ and B_m to $V^* \otimes \Lambda^2 V^*$. Since the map $\alpha : V^* \otimes \Lambda^2 V^* \rightarrow \Lambda^2 V^* \otimes V^*$ given by $\alpha(B_m) = T_m$ is $\text{Gl}(n)$ -invariant, it is a multiple of the intertwining operator between the $\text{Gl}(n)$ -irreducible subspaces of $V^* \otimes \Lambda^2 V^*$ and those of $\Lambda^2 V^* \otimes V^*$. The space $W = \Lambda^2 V^* \otimes V^* \cong V^* \otimes \Lambda^2 V^*$ decomposes in the form

$$W = \Lambda^3 V^* \otimes Y_1^2,$$

where Y_1^2 is the irreducible representation of $\text{Gl}(n)$ with Young diagram $\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}$. The projection β on $\Lambda^3 V^*$ is given by

$$\beta(A)_{XYZ} = \frac{1}{3}(A_{XYZ} + A_{ZXY} + A_{YZX})$$

and $\ker \beta$ identifies itself with Y_1^2 .

To obtain the inverse of α , we observe that the restriction to $\Lambda^3 V^*$ is given by

$$\alpha(A)_{XYZ} = A_{XYZ} - A_{YXZ} = 2A_{XYZ},$$

and, the restriction to $Y_1^2 = \ker \beta$ is

$$\alpha(A)_{XYZ} = A_{XYZ} - A_{YXZ} = A_{XYZ} + A_{ZYX} + A_{XZY} = -A_{ZXY}.$$

Then

$$\begin{aligned} B_{XYZ} &= \alpha^{-1}(T)_{XYZ} = \frac{1}{2} \beta(T)_{XYZ} - (T - \beta(T))_{YZX} \\ &= \frac{1}{2} (T_{XYZ} + T_{ZXY} - T_{YZX}). \end{aligned}$$

§5. Power series expansion of the volume function of a D -geodesic ball

5.1. PROPOSITION: For any r such that $0 < r < r_D$, we have:

$$\begin{aligned} V_m^D(r) &= \frac{(\pi r^2)^{n/2}}{(n/2)!} \left\{ 1 + \frac{1}{n+2} \left(-\frac{1}{6} \tau_D - \frac{1}{3} \partial \bar{T} \right. \right. \\ &\quad \left. \left. + \frac{1}{24} \check{T} + \frac{1}{8} \|\bar{T}\|^2 \right) r^2 + O(r^4) \right\} \end{aligned}$$

where

$$\begin{aligned}\tau_D &= \sum_{i,j=1}^n S_{ijj}(m) \quad (\text{scalar curvature of } D \text{ at } m), \\ -\partial\bar{T} &= \sum_{i,j=1}^n D_i(T)_{ij}(m), \quad \|T\|^2 = \sum_{i,j=1}^n T_{ij}^2(m) \\ \check{T} &= \sum_{i,j,k=1}^n T_{ijk}T_{ikj}(m), \quad \|\bar{T}\|^2 = \sum_{i=1}^n \bar{T}_i^2(m),\end{aligned}$$

\bar{T} being the one-form defined by $\bar{T}_X = \sum_{j=1}^n T_{XE_jE_j}$ for any local orthonormal frame $\{E_1, \dots, E_n\}$.

The proof – which makes use of 2.1 and 3.6 – follows closely the one given in [6] for the Levi-Civita connection. \square

Some geometric formulas will prove useful to eliminate $\delta\bar{T}$ in 5.1. For this, we need the following lemmas:

5.2. LEMMA: *If R is the curvature tensor of ∇ , at m , we have:*

$$\begin{aligned}S_{XYXY} &= R_{XYXY} - \frac{1}{2}(\|B_XY\|^2 + \|B_YX\|^2) + \langle B_XX, B_YY \rangle + \|T_XY\|^2 \\ &\quad + \frac{1}{2}(\langle T_XT_XY, Y \rangle + \langle T_YT_YX, X \rangle) + \langle D_X(T)_XY, Y \rangle \\ &\quad + \langle D_Y(T)_YX, X \rangle.\end{aligned}$$

PROOF:

$$\begin{aligned}S_{XYXY} &= -\langle D_{XY}^2X, Y \rangle + \langle D_{YX}^2X, Y \rangle \\ &= R_{XYXY} - \langle B_X\nabla_YX, Y \rangle - \langle \nabla_X(B_YX), Y \rangle \\ &\quad - \langle B_XB_YX, Y \rangle + \langle B_Y\nabla_XX, Y \rangle + \langle \nabla_Y(B_XX), Y \rangle + \langle B_YB_XX, Y \rangle.\end{aligned}$$

But, at m , by (3.1.2)

$$\nabla_YX = D_YX - B_YX = \frac{1}{2}T_YX - B_YX = -\frac{1}{2}(B_XY + B_YX).$$

Then, using (4.1) and (4.2) we have

$$\begin{aligned}S_{XYXY} &= R_{XYXY} - \frac{1}{2}\langle B_XY, B_YX + B_XY \rangle - X\langle T_YX, Y \rangle \\ &\quad - \frac{1}{2}\langle B_YX, B_XY + B_YX \rangle \\ &\quad + \langle B_XY, B_YX \rangle + \langle B_YY, B_XX \rangle + Y\langle T_YX, X \rangle;\end{aligned}$$

and the lemma follows by a direct computation, using (3.1.2). \square

5.3. LEMMA: If τ_∇ is the scalar curvature of ∇ , at m , we have

$$\tau_D = \tau_\nabla + \frac{1}{4} \|T\|^2 + \frac{1}{2} \dot{T} + \|\bar{T}\|^2 - 2 \partial \bar{T}.$$

PROOF: From 5.2 we see that

$$(5.1) \quad \tau_D = \tau_\nabla - \|B\|^2 - \|\bar{B}\|^2 + \|T\|^2 + \dot{T} - 2 \partial \bar{T},$$

where $\|B\|^2 = \sum_{i,j,k=1}^n B_{ijk}^2(m)$ and $\|\bar{B}\|^2 = \sum_{i=1}^n (\sum_{j=1}^n B_{ij})^2(m)$. The result is then immediate, since from (4.2) we have

$$(5.2) \quad \|B\|^2 = \frac{3}{4} \|T\|^2 + \frac{1}{2} \dot{T},$$

$$(5.3) \quad \|\bar{B}\|^2 = \|\bar{T}\|^2. \quad \square$$

Now, we consider the decomposition of $\Lambda^2 V^* \otimes V^*$ into $O(n)$ -irreducible subspaces. $\Lambda^3 V^*$ is already $O(n)$ -irreducible, but Y_1^2 decomposes into two subspaces, namely $\bar{Y}_1^2 = \{A \in Y_1^2 / \bar{A}_X = 0 \text{ for any } X \in V\}$ and

$$\bar{Y}_1^2 = \left\{ A \in Y_1^2 / A_{XYZ} = \frac{1}{n-1} (-\langle X, Z \rangle \bar{A}_Y + \langle Y, Z \rangle \bar{A}_X) \right\}$$

(where $\bar{A}_X = \sum_{i=1}^n A_{Xe_i e_i}$, $\{e_i\}$ being an orthonormal basis of V). (cfr [16]).

If we split $T = T^1 + T^2 + T^3$, with T^1 belonging to $\Lambda^3 V^*$, T^2 to \bar{Y}_1^2 and T^3 to \bar{Y}_1^2 , obviously $\|T\|^2 = \|T^1\|^2 + \|T^2\|^2 + \|T^3\|^2$.

If $\tilde{\alpha}: \Lambda^2 V^* \otimes V^* \rightarrow \Lambda^2 V^* \otimes V^*$ is the map given by $\tilde{\alpha}(A)_{XYZ} = A_{ZYX} - A_{ZXY}$, then $\tilde{\alpha}|_{\Lambda^3 V^*} = -2I$ and $\tilde{\alpha}|_{Y_1^2} = I$ (here I is the identity map). Moreover, $\tilde{\alpha}$ is $Gl(n)$ -invariant, and $\dot{T} = (1/2)\langle T, \tilde{\alpha}(T) \rangle$. Then

$$(5.4) \quad \dot{T} = -\|T^1\|^2 + \frac{1}{2} \|T^2\|^2 + \frac{1}{2} \|T^3\|^2.$$

We also get $\bar{T} = \bar{T}^1 + \bar{T}^2 + \bar{T}^3 = \bar{T}^3$, hence

$$(5.5) \quad \|\bar{T}\|^2 = \|\bar{T}^3\|^2 = \frac{n-1}{2} \|T^3\|^2.$$

Now 5.1 can be reformulated as follows:

5.4. THEOREM: For any r such that $0 < r < r_D$ we have

$$V_m^D(r) = \frac{(\pi r^2)^{(n/2)}}{(n/2)!} \left\{ 1 - \frac{1}{6(n+2)} \left(\tau_\nabla + \frac{3}{8} \|T^2\|^2 + \frac{n+2}{8} \|T^3\|^2 \right) r^2 + O(r^4) \right\}.$$

If $V_0 = \frac{\pi^{(n/2)}}{(n/2)!}$ (the volume of the unit ball in \mathbb{R}^n), 5.4 can be rewritten

$$V_m^D(r) = V_0 r^n \{ 1 + C_1^D r^2 + C_2^D r^4 + \dots + C_n^D r^{2n} + \dots \}$$

and we can state the following corollaries:

5.5. COROLLARY: D and ∇ have the same geodesics if and only if $C_1^D = C_1^\nabla$ for any m in M .

PROOF: $C_1^D = C_1^\nabla$ implies $T^2 = T^3 = 0$, so $T = T^1$, i.e., T lies on $\Lambda^3 V^*$ and, as we have indicated in §4, D and ∇ have the same geodesics. \square

5.6. COROLLARY: If M has non-negative Ricci curvature ρ_∇ and $C_1^D = C_2^D = 0$ for any m in M , then M is locally flat ($R = 0$).

PROOF: $\rho_\nabla(X, X) \geq 0$ gives $\tau_\nabla \geq 0$. Since $C_1^D = 0$, from 5.4 we have $\tau_\nabla = 0$ and $T = T^1$. Then D and ∇ have the same geodesics, $V_m^D(r) = V_m^\nabla(r)$ and $C_1^D = C_2^D = C_1^\nabla = C_2^\nabla = 0$. But in [6] it is proved that if $\rho_\nabla(X, X) \geq 0$ and $C_1^\nabla = C_2^\nabla = 0$, then $R = 0$. \square

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