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UNIFORM MEASURES ON TOPOLOGICAL GROUPS

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Summary

The theory of uniform measures is applied to the problem of uniqueness of invariant means on topological groups. The result: If LIM, the set of left-invariant means defined on right uniformly continuous functions on a separable metrizable topological group, contains a G_δ point then the group is precompact (and therefore LIM contains a unique mean).

1. Overview

Let X be a topological group, and denote by $U_b(rX)$ the space of bounded right uniformly continuous real-valued functions on X. Consider LIM(X), the set of left-invariant means defined on $U_b(rX)$, endowed with the topology $w^* = w(U_b(rX)^*, U_b(rX))$; clearly LIM($X$) is compact. Granirer [10] investigated the size of LIM(X) for separable (not necessarily locally compact) X.

In section 4 of the present paper we prove (Theorem 3): If X is separable metrizable and LIM(X) contains a G_δ point then X is precompact (and hence LIM(X) has cardinality 1). For metrizable groups, this improves Granirer's results ([10], pp. 65–67) as well as ([4], Th. 7).

Here (as in [10]) the existence of a G_δ point in LIM is a symptom of smallness of LIM. For example, if LIM is finite dimensional (i.e. very small) or, more generally, metrizable (i.e. quite small) then every point in LIM is G_δ. Thus our result can be given this meaning: If X is separable metrizable and LIM(X) $\neq \emptyset$ then either there is a unique left

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invariant mean or the set LIM(X) is large. Other results of this kind, with other criteria of smallness, are known. Two of the more recent ones are [11], [3]; they estimate the dimension and the cardinality of LIM on discrete semigroups and groups, respectively.

Our proof of Theorem 3 is phrased in terms of uniform measures. Uniform measures were first studied by Fedorova [7], Berezanskiĭ [1] and Le Cam [12]. Later developments are surveyed in [6] and [8]. Section 2 of the present paper recalls the definition of a uniform measure and one result that will be needed in section 4.

Section 3 describes invariant uniform measures on topological groups: they can be identified with the Haar measures on compact groups (modulo the completion of the group). In the particular case where the group is locally compact or complete metric this result is obvious, since in that case every uniform measure is Radon (see [1], [7], [12]). However, in a general topological group uniform measures need not even be countably additive.

In section 4 we apply Theorems 1 and 2 to prove the result about LIM(X). Our method of proof is close to Granirer's but, with the machinery of uniform measures available, we are able to derive stronger conclusions.

My thanks are due to E. Granirer who showed me the problem and discussed it with me.

2. Uniform measures

All vector spaces are over $\mathbf{R}$, the field of real numbers. All topological spaces are Hausdorff. We describe uniform spaces by uniformly continuous pseudometrics ([9], Chap. 15).

When d is a pseudometric on a set X, put

$$\operatorname{Lip}(d) = \{f : X \to \mathbf{R} \mid |f(x)| \le 1 \quad \text{and} \quad |f(x) - f(y)| \le d(x, y) \text{ for } x, y \in X\}.$$

With the pointwise operations and topology (i.e. as a subset of $\mathbf{R}^X$), Lip(d) is a compact convex lattice.

When X is a uniform space, denote by $U_b(X)$ the space of bounded uniformly continuous real-valued functions on X. A linear form on $U_b(X)$ is called a *uniform measure on* X if it is continuous on Lip(d) for every uniformly continuous pseudometric d on X. In other words, a linear form $\mu : U_b(X) \to \mathbf{R}$ is a uniform measure if for every uniformly continuous pseudometric d on X and for every net of functions $f_\alpha \in \mathrm{Lip}(d)$ such that $\lim_\alpha f_\alpha(x) = f(x)$ for all $x \in X$, we have $\lim_\alpha \mu(f_\alpha) = \mu(f)$. The space of uniform measures on X is denoted $M_u(X)$. Put

$$M_u^+(X) = \{\mu \in M_u(X) \mid \mu(f) \geq 0 \text{ for every } f \geq 0 \text{ in } U_b(X)\}.$$

A linear form μ on $U_b(X)$ is called a *molecular measure* if there are finitely many points $x_1, x_2, \ldots, x_n \in X$ and real numbers $r_1, r_2, \ldots, r_n$ such that

$$\mu(f) = \sum_{i=1}^{n} r_i f(x_i) \text{ for } f \in U_b(X).$$

The space of molecular measures on X is denoted $\mathrm{Mol}(X)$. Obviously, $\mathrm{Mol}(X) \subseteq M_u(X)$.

THEOREM 1: *For every uniform space X, the space $M_u(X)$ is $w(M_u(X), U_b(X))$ sequentially complete. In other words, if $\mu_n \in M_u(X)$ for $n = 1, 2, \ldots$ and $\lim_n \mu_n(f) = \mu(f)$ for all $f \in U_b(X)$ then $\mu \in M_u(X)$.*

PROOF: See [5] or [13].

3. Invariant uniform measures

A pseudometric d on a group X is *right-invariant* if $d(x, y) = d(xz, yz)$ for all $x, y, z \in X$; it is *left-invariant* if $d(x, y) = d(zx, zy)$ for all $x, y, z \in X$.

If X is a topological group, its *right uniformity* rX is generated by all right-invariant continuous pseudometrics on X; the *left uniformity* lX on X is generated by all left-invariant continuous pseudometrics on X.

Say that a topological group is *precompact* if it satisfies one of the equivalent conditions in the following proposition.

PROPOSITION: *For a topological group X, these conditions are equivalent*:

(i) *lX is precompact*;
(ii) *rX is precompact*;
(iii) *X is a topological subgroup of a compact group.*

PROOF: See ([2], 3.2 and Ex. 3.8) and ([15], Th.X).

For a function f on a topological group X and $x \in X$, put $L_x f(y) = f(xy)$. If $Y = lX$ or $Y = rX$ then each L_x, $x \in X$, maps $U_b(Y)$ into $U_b(Y)$; say that a linear functional μ on $U_b(Y)$ is *left-invariant* if $\mu(f) = \mu(L_x f)$ for all $x \in X$ and $f \in U_b(Y)$.

THEOREM 2: *For a topological group X, the following are equivalent*:

(i) *X is precompact*;

(ii) *X admits a left-invariant* $\mu \in M_u^+(lX)$, $\mu \neq 0$;

(iii) *X admits a left-invariant* $\mu \in M_u^+(rX)$, $\mu \neq 0$.

PROOF: The implications (i)⇒(ii) and (i)⇒(iii) are easy: If X is precompact then $lX = rX$ and the completion $\widehat{lX}$ of lX is a compact group. Let m be the Haar measure on $\widehat{lX}$. Every f in $U_b(lX) = U_b(rX)$ extends to a unique continuous function $\hat{f}$ on $\widehat{lX}$ and $\mu(f) = m(\hat{f})$ is a left-invariant uniform measure on $lX = rX$.

The proofs of (ii)⇒(i) and (iii)⇒(i) follow the same pattern: Let $\mu \neq 0$ be a positive left-invariant uniform measure. Take a left- or right-invariant continuous pseudometric d on X. (We want to show that (X, d) is precompact.) Choose an arbitrary $\epsilon > 0$. For every finite set $F \subseteq X$ put

$$g_F(x) = \max\left\{\left(1 - \frac{1}{\epsilon}\, d(x, y)\right)^+ \,\middle|\, y \in F\right\},$$

and write $g_y = g_{\{y\}}$ for $y \in X$. Each g_F is in $\mathrm{Lip}((1/\epsilon)d)$ and $\lim_F g_F(x) = 1$ for all $x \in X$. Since μ is a uniform measure and $\mu(1) > 0$, it follows that $\mu(g_F) > 0$ for some finite $F \subseteq X$. Since $\Sigma_{v \in F}\, g_v \geq g_F$ and $\mu \geq 0$, we have $\mu(g_v) > 0$ for some $v \in F$.

Now in the case $\mu \in M_u^+(lX)$, i.e. for the implication (ii)⇒(i), we apply the previous construction with an arbitrary left-invariant continuous pseudometric d and proceed as follows: Let e be the identity element in X. Since μ and d are left-invariant, we have

$$\mu(g_e) = \mu(L_v g_v) = \mu(g_v) > 0.$$

Take a maximal set $H \subseteq X$ such that $\min(g_x, g_y) = 0$ for $x, y \in H$, $x \neq y$. The set H is finite because

$$\mathrm{card}\ H \cdot \mu(g_e) = \mu\left(\sum_{x \in H} g_x\right) \leq \mu(1).$$

We show that every point $y \in X$ is within the distance 2ϵ from H:

there are $z \in X$ and $x \in H$ such that

$$\min(g_x(z), g_y(z)) > 0$$

(because H is maximal). Hence $1-(1/\epsilon)d(z, y)>0$ and $1-(1/\epsilon)d(z, x)>0$, and therefore $d(x, y)<2\epsilon$. Thus (X, d) is precompact.

Finally, we prove that (iii)$\Rightarrow$(i). In this case $\mu \in M_u^+(rX)$, $\mu \neq 0$, and we prove that (X, d) is precompact for an arbitrary right-invariant continuous pseudometric d on X. Apply again the construction above to get $v \in X$ such that $\mu(g_v)>0$. Take a maximal set $H \subseteq X$ such that $\min(L_x g_v, L_y g_v)=0$ for $x, y \in H$, $x \neq y$. Again, H is finite because

$$\text{card } H \cdot \mu(g_v) = \mu\Big(\sum_{x \in H} L_x g_v\Big) \leq \mu(1).$$

For any $y \in X$ there are $z \in X$ and $x \in H$ such that

$$\min(L_x g_v(z), L_y g_v(z)) > 0$$

(because H is maximal). Hence $1-(1/\epsilon)d(xz, v)>0$ and $1-(1/\epsilon)d(yz, v)>0$, and so $d(x, y)=d(xz, yz)<2\epsilon$. As before, this shows that (X, d) is precompact.

4. Invariant means

Let again X be a topological group. A linear form μ on $U_b(rX)$ is a *left-invariant mean* if $\mu \geq 0$ (i.e. $\mu(f) \geq 0$ for $f \geq 0$), $\mu(1)=1$, and $\mu(L_x f)=\mu(f)$ for all $f \in U_b(rX)$ and $x \in X$. The set of left-invariant means on $U_b(rX)$, endowed with the topology $w^* = w(U_b(rX)^*, U_b(rX))$, will be denoted LIM$(X)$; it is a compact and convex set.

Theorems 1 and 2 of the present paper together with Corollary 2.1 in [10] yield: If X contains a countable dense subsemigroup which is left-amenable as a discrete semigroup and if LIM(X) has a G_δ point then X is precompact. Here we prove another version of this result. Namely, we show that if X is separable metrizable then the assumption about a discretely amenable subsemigroup can be omitted. The proof is similar to the proof of ([10], Th.2).

LEMMA 1: *Let K be a compact space and D its dense subset. If B_n, $n=1,2,\ldots$, are subsets of K such that $B_1 \supseteq B_2 \supseteq \ldots$ then every G_δ point of $\bigcap_{n=1}^{\infty} cl(B_n \cap D)$ is the limit of a sequence of points in D.*

PROOF: See ([10], Lemma M_1, p. 13).

LEMMA 2: *Let X be a topological group whose topology is defined by a right-invariant metric d, and let $S = \{s_1, s_2, \ldots\}$ be a countable subset of X. Put*

$$B_n = \{\nu \in U_b(rX)^* \mid \nu \geq 0,\ \nu(1) = 1 \text{ and } |\nu(L_{s_i}g) - \nu(g)| \leq 1/n \text{ for all } g \in \mathrm{Lip}(d),\ 1 \leq i \leq n\}.$$

Then $\bigcap_{n=1}^{\infty} w^*cl(B_n \cap \mathrm{Mol}(rX)) = \bigcap_{n=1}^{\infty} B_n$.

PROOF: Each B_n is w^* closed. Therefore

$$\bigcap_{n=1}^{\infty} w^*cl(B_n \cap \mathrm{Mol}(rX)) \subseteq \bigcap_{n=1}^{\infty} B_n.$$

Conversely, take any k and $\mu \in \bigcap_{n=1}^{\infty} B_n$. We have to prove that $\mu \in w^*cl(B_k \cap \mathrm{Mol}(rX))$. Since $\mu \in \bigcap_n B_n$, we have $\mu(L_s f) = \mu(f)$ for $s \in S$ and $f \in \mathrm{Lip}(d)$; moreover, the set $\bigcup_{n=1}^{\infty} n\,\mathrm{Lip}(d)$ is norm dense in $U_b(rX)$ (see e.g. [4], Lemma 3.3), and therefore $\mu(L_s f) = \mu(f)$ for all $s \in S$ and $f \in U_b(rX)$.

Let U be an arbitrary convex w^* neighborhood of μ in $U_b(rX)^*$. The set

$$\mathrm{Mol}^{+1}(rX) = \{\nu \in \mathrm{Mol}(rX) \mid \nu \geq 0 \text{ and } \nu(1) = 1\}$$

is w^* dense in $\{\nu \in U_b(rX)^* \mid \nu \geq 0 \text{ and } \nu(1) = 1\}$; hence there is a net $\{\mu_\alpha\}_\alpha$ of molecular measures $\mu_\alpha \in U \cap \mathrm{Mol}^{+1}(rX)$ that w^* converges to μ.

Write $L_y^*\nu(f) = \nu(L_y f)$ for $y \in X$, $\nu \in U_b(rX)^*$ and $f \in U_b(rX)$. We have seen that $\mu = L_s^*\mu$ for each $s \in S$. Put $E = [\mathrm{Mol}(rX)]^k$ and $F = [U_b(rX)]^k$ and define the duality between E and F by

$$\langle(\nu_1, \nu_2, \ldots, \nu_k), (f_1, f_2, \ldots, f_k)\rangle = \sum_{i=1}^{k} \nu_i(f_i)$$

for $\nu_i \in \mathrm{Mol}(rX)$, $f_i \in U_b(rX)$, $1 \leq i \leq k$. Put

$$W = \{(L_{s_1}^*\nu - \nu, L_{s_2}^*\nu - \nu, \ldots, L_{s_k}^*\nu - \nu) \mid \nu \in U \cap \mathrm{Mol}^{+1}(X)\};$$

the set $W \subseteq E$ is convex and $0 = (0, 0, \ldots, 0) \in E$ belongs to the

$w^* = w(E, F)$ closure of W, because

$$w^* \lim_\alpha (L^*_{s_1}\mu_\alpha - \mu_\alpha, L^*_{s_2}\mu_\alpha - \mu_\alpha, \ldots, L^*_{s_k}\mu_\alpha - \mu_\alpha) =$$

$$= (L^*_{s_1}\mu - \mu, L^*_{s_2}\mu - \mu, \ldots, L^*_{s_k}\mu - \mu) = 0.$$

It follows that 0 belongs to the $\tau(E, F)$ closure of W, where $\tau(E, F)$ is the Mackey topology ([14], IV-3.1). Since the set $[\mathrm{Lip}(d)]^k \subseteq F$ is convex and $w(F, E)$ compact, its polar is a $\tau(E, F)$ neighborhood of 0 ([14], IV-3.2). Hence there exists $\nu_0 \in U \cap \mathrm{Mol}^{+1}(rX)$ such that

$$|L_{s_i}\nu_0(g) - \nu_0(g)| \leq 1/k$$

for all $g \in \mathrm{Lip}(d)$, $1 \leq i \leq k$. This shows that $\nu_0 \in U \cap B_k \cap \mathrm{Mol}^{+1}(rX)$; hence $U \cap B_k \cap \mathrm{Mol}(rX) \neq \emptyset$, q.e.d.

THEOREM 3: *Let X be a separable metrizable topological group. If* LIM(X) *contains a G_δ point then X is precompact.*

PROOF: Let d be a right-invariant metric that defines the topology of X, and let $S = \{s_1, s_2, \ldots\}$ be a countable dense subset of X. Define B_n, $n = 1, 2, \ldots$, as in Lemma 2; then $\mathrm{LIM}(X) = \bigcap_n B_n$.

Let μ be a G_δ point of LIM(X). By Lemmas 1 and 2 (with $K = \{\nu \in U_b(rX)^* \mid \nu \geq 0$ and $\nu(1) = 1\}$ and $D = \{\nu \in \mathrm{Mol}(rX) \mid \nu \geq 0$ and $\nu(1) = 1\}$), there is a sequence of molecular measures w^* converging to μ. By Theorem 1, μ belongs to $M_u(rX)$. By Theorem 2, X is precompact.

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