COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 45, nº 2 (1982), p. 199-205
<http://www.numdam.org/item?id=CM_1982__45_2_199_0>

COMPOSITIO MATHEMATICA, Vol. 45, Fasc. 2, 1982, pag. 199–205

ALGEBRAIC CHARACTERIZATIONS OF THE ALGEBRA OF FUNCTIONS AND OF THE LIE ALGEBRA OF VECTOR FIELDS OF A MANIFOLD

M. De Wilde and P. Lecomte

1. Introduction

Let M be a smooth, connected, Hausdorff and second countable manifold. Denote by $\mathscr{X}(M)$ the Lie algebra of smooth vector fields of M and by $C_\infty(M)$ the space of smooth functions on M. Recall that the Lie derivative $\mathscr{L}_X(X \in \mathscr{X}(M))$ is defined on $\otimes^p C_\infty(M)$ by

$$\mathscr{L}_X(f_1 \otimes \cdots \otimes f_p) = \sum_{j=1}^{p} f_1 \otimes \cdots \otimes (\mathscr{L}_X f_j) \otimes \cdots \otimes f_p.$$

If $L : \otimes^p C_\infty(M) \to C_\infty(M)$ is a p-linear map, the adjoint action of $\mathscr{L}_X$ on L, $ad(\mathscr{L}_X)L$ is the commutator $\mathscr{L}_X \circ L - L \circ \mathscr{L}_X$. We intend to show that the only symmetric p-linear maps $L : \otimes^p C_\infty(M) \to C_\infty(M)$ for which $ad(\mathscr{L}_X)L = 0$ for each $X \in \mathscr{X}(M)$ are the multiples of the product of p factors: $f_1 \otimes \cdots \otimes f_p \to f_1 \cdots f_p$. It provides thus a characterization of the product by means of its derivations.

A similar question is investigated for the Lie bracket of vector fields, which is shown to be determined up to a constant by the fact that its derivations are the Lie derivatives. This question has been solved by Van Strien in [4] under stronger assumptions which will be discussed in §3.

2. Characterization of the algebraic structure of $C_\infty(M)$

LEMMA 2.1: *There exist a finite partition of the unity $\lambda_t (t \le r)$ of M, vector fields $X_t (t \le r) \in \mathscr{X}(M)$ and functions $\mu_t (t \le r) \in C_\infty(M)$ such that $\lambda_t = \lambda_t X_t \cdot \mu_t$ for each $t \le r$.*

0010-437X/82020199-07$00.20/0

PROOF: It is well known from dimension theory [2], p. 20 that M admits an open cover $U_t(t \leq r)$ such that each U_t is the disjoint union of domains of charts $U_{it}(i \in \mathbb{N})$, as well as a locally finite partition of the unity ρ_{it} $(t \leq r, i \in \mathbb{N})$ such that each ρ_{it} has compact support in U_{it}. We set $V_{it} = \{x \in U_{it} : \rho_{it}(x) > 0\}$ and choose a partition of the unity $\rho'_{it}(t \leq r; i \in \mathbb{N})$ such that supp ρ'_{it} is compact in V_{it} for all i, t.

For each $t \leq r$ and $i \in \mathbb{N}$, we may find $\alpha_{it}, \beta_{it} \in C_\infty(M)$ with compact supports in U_{it} such that $\alpha_{it} \mid V_{it} = 1$ and $\beta_{it} = 1$ in some neighborhood of supp α_{it}. If $(x^1_{it}, \ldots, x^n_{it})$ are local coordinates in U_{it},

$$\lambda_t = \sum_i \rho'_{it},\ X_t = \sum_i \alpha_{it} D_{x^1_{it}} \text{ and } \mu_t = \sum_i \beta_{it} x^1_{it}$$

have the required properties. Hence the lemma.

Denote by A_p the subspace of $\otimes^p C_\infty(M)$ spanned by the tensors which are antisymmetric in at least two arguments.

LEMMA 2.2: *For all $f_i \in C_\infty(M)$ $(i \leq p)$, there exist $N \in \mathbb{N}$, vector fields $X_i \in \mathscr{X}(M)$ $(i \leq N)$ and $f_{ik} \in C_\infty(M)$ $(i \leq N, k \leq p)$, such that*

$$f_1 \otimes \cdots \otimes f_p - \sum_{i=1}^N \mathscr{L}_{f_1 X_i}(f_{i1} \otimes \cdots \otimes f_{ip}) \in A_p.$$

Moreover the vector fields $X_i(i \leq N)$ and the functions $f_{ik}(i \leq N, k \leq p)$ can be choosen independently of f_1.

PROOF: We have

$$f_1 \otimes \cdots \otimes f_p = \sum_{t \leq r} \lambda_t f_1 \otimes f_2 \cdots \otimes f_p. \tag{*}$$

Since $\lambda_t = \lambda_t X_t \cdot \mu_t$,

$$\lambda_t f_1 \otimes f_2 \cdots \otimes f_p = \mathscr{L}_{f_1 \lambda_t X_t}(\mu_t \otimes f_2 \otimes \cdots \otimes f_p)$$
$$- \sum_{i>1} \mu_t \otimes f_2 \otimes \cdots \otimes f_1 \lambda_t X_t \cdot f_i \otimes \cdots \otimes f_p. \tag{**}$$

Let us consider for instance the term $i = 2$. Setting $X_t \cdot f_2 = g_2$, it reads

$$\begin{aligned}&\mu_t \otimes f_1 g_2 \lambda_t X_t \cdot \mu_t \otimes f_3 \otimes \cdots \otimes f_p \\ &\quad = \tfrac{1}{2}(\mu_t \otimes f_1 g_2 \lambda_t X_t \cdot \mu_t \otimes f_3 \otimes \cdots + f_1 g_2 \lambda_t X_t \cdot \mu_t \otimes \mu_t \otimes f_3 \otimes \cdots) \\ &\quad + \tfrac{1}{2}(\mu_t \otimes f_1 g_2 \lambda_t X_t \cdot \mu_t \otimes f_3 \otimes \cdots - f_1 g_2 \lambda_t X_t \cdot \mu_t \otimes \mu_t \otimes f_3 \cdots).\end{aligned}$$

The last term is antisymmetric in the two first arguments, hence may be neglected. The other one can be written, setting $\frac{1}{2}g_2X_t = X'_t$,

$$\mathscr{L}_{f_1\lambda_t X'_t}(\mu_t \otimes \mu_t \otimes f_3 \otimes \cdots)$$

$$-\sum_{i>2} \mu_t \otimes \mu_t \otimes f_3 \otimes \cdots \otimes f_2\lambda_t X'_t \cdot f_i \otimes \cdots$$

In the terms we had to consider in (**), one of the arguments was μ_t, another one had λ_t as a factor. In the terms which we are left to consider, we now have two arguments equal to μ_t and one more divisible by λ_t. This shows the outline of the proof, which will now be achieved by proving the following, by induction on k: if one of the f_i's is divisible by $\lambda_t f$ ($f \in C_\infty(M)$) and $p-k$ others are equal to μ_t, then

$$f_1 \otimes \cdots \otimes f_p - \sum_i \mathscr{L}_{fX_i}(f_{i1} \otimes \cdots \otimes f_{ip}) \in A_p$$

for suitable $X_i \in \mathscr{H}(M)$ and $f_{ij} \in C_\infty(M)$, independent of f.

For $k = 1$, assuming for instance that $f_1 = \lambda_t fg$ and $f_i = \mu_t (i > 1)$,

$$\begin{aligned} f_1 \otimes \cdots \otimes f_p &= fg\lambda_t X_t \cdot \mu_t \otimes \mu_t \otimes \cdots \otimes \mu_t \\ &= \frac{1}{p} \mathscr{L}_{fg\lambda_t X_t}(\mu_t \otimes \cdots \otimes \mu_t) \end{aligned}$$

$$+\frac{1}{p}\sum_{i>1} (fg\lambda_t X_t \cdot \mu_t \otimes \cdots \otimes \mu_t - \underbrace{\mu_t \otimes \cdots \otimes fg\lambda_t X_t \mu_t}_{i} \otimes \cdots)$$

has the required form.

In general, if the property holds true for k, assuming for simplicity that $f_{k+1} = \lambda_t f g_{k+1}$ and $f_i = \mu_t$ for $i > k+1$, and setting $f_1 \otimes \cdots \otimes f_k = T$,

$$f_1 \otimes \cdots \otimes f_{k+1} \otimes \cdots \otimes \mu_t = \frac{1}{p-k} \mathscr{L}_{fg_{k+1}\lambda_t X_t}(T \otimes \mu_t \otimes \cdots \otimes \mu_t)$$

$$+\frac{1}{p-k}\sum_{i>k+1} (T \otimes f_{k+1} \otimes \cdots \otimes \mu_t - T \otimes \mu_t \otimes \cdots \otimes \underset{(i)}{f_{k+1}} \otimes \cdots)$$

$$-\frac{1}{p-k}\sum_{i\leq k} f_1 \otimes \cdots \otimes fg_{k+1}\lambda_t X_t \cdot f_i \otimes \cdots \otimes f_k \otimes \mu_t \otimes \cdots \otimes \mu_t$$

hence the conclusion, by induction.

Applying this to (*) for $f = f_1$ and $k = p$ yields the lemma.

Proposition 2.3: *Let $P : C_\infty(M) \times \cdots \times C_\infty(M) \to C_\infty(M)$ be a p-linear symmetric map. If $ad(\mathscr{L}_X)\ P = 0$ for each $X \in \mathscr{H}(M)$, then*

$$P(f_1, \ldots, f_p) = k\, f_1 \cdots f_p$$

for some $k \in \mathbb{R}$.

Proof: Since P is symmetric, P vanishes on A_p. Using lemma 2.2.,

$$P(f_1, \ldots, f_p) = \sum_i P \circ \mathscr{L}_{f_1 X_i}(f_{i1} \otimes \cdots \otimes f_{ip}) = \sum_i \mathscr{L}_{f_1 X_i} \circ P(f_{ij}, \ldots, f_{ip})$$
$$= f_1 \cdot \left[\sum_i X \cdot P(f_{i1}, \ldots, f_{ip})\right],$$

where the X_i's and f_{ij}'s do not depend on f_1. Therefore

$$P(f_1, \ldots, f_p) = f_1 P(1, f_2, \ldots, f_p).$$

It is clear that $P(1, f_2, \ldots, f_p) = P'(f_2, \ldots, f_p)$ satisfies again $ad(\mathscr{L}_X)P' = 0$, thus, by induction,

$$P(f_1, \ldots, f_p) = f_1 \ldots f_p \cdot P(1, \ldots, 1).$$

The condition $ad(\mathscr{L}_X)P = 0$ clearly shows that $P(1, \ldots, 1)$ is constant, hence the result.

Result 2.4: *The symmetry assumption on P is necessary.*

Indeed, consider a compact, connected, oriented manifold M of dimension $p - 1$ and take the map P:

$$(f_1, \ldots, f_p) \to \int f_1\, \mathrm{d}f_2 \wedge \cdots \wedge \mathrm{d}f_p.$$

It is clear that p is such that $ad(\mathscr{L}_X)P = 0$ for all $X \in \mathscr{H}(M)$. However, $P \neq 0$, thus P is not even local.

3. Characterization of the algebraic structure of $\mathscr{H}(M)$

Proposition 3.1: *Let $B : \mathscr{H}(M) \times \mathscr{H}(M) \to \mathscr{H}(M)$ be a bilinear map such that*

$$\mathscr{L}_X B(Y, Z) = B(\mathscr{L}_X Y, Z) + B(Y, \mathscr{L}_X Z),\ \forall\ X, Y, Z \in \mathscr{H}(M). \qquad (*)$$

Then there exists $k \in \mathbb{R}$ *such that*

$$B(X, Y) = k[X, Y], \forall X, Y \in \mathcal{H}(M).$$

A similar result is proved by Van Strien in [4]. The assumption (*) is replaced by the "naturality" of B, which means the following: B is supposed to be defined on every manifold M and, for every smooth open imbedding φ, the diagram

$$\begin{array}{ccc} \mathcal{H}(M) \times \mathcal{H}(M) & \xrightarrow{B_M} & \mathcal{H}(M) \\ \Big\downarrow \varphi_* \times \varphi_* & & \Big\downarrow \varphi_* \\ \mathcal{H}(N) \times \mathcal{H}(N) & \xrightarrow{B_N} & \mathcal{H}(N) \end{array}$$

commutes (see [3]). It is clear that the naturality implies the locality of B and, using the pseudo-group of X, it also implies (*).

LEMMA 3.2: *If* $X \in \mathcal{H}(M)$ *vanishes in an open subset* U *of* M, *for each* $x \in U$, *there exists a neighborhood* ω *of* x *and vector fields* $X_i, X'_i (i \leq N)$ *such that* $X = \sum_{i \leq N} [X_i, X'_i]$ *and* X_i, X'_i *vanish in* ω.

PROOF: Fix ω relatively compact in U. Choose ρ_{it} as in lemma 2.1 and $\varphi_{it} \in C_\infty(M)$ with compact support in $U_{it} \setminus \omega$ and equal to 1 in a neighborhood of supp $\rho_{it} X$. Then, if $(x^1, \ldots, x^n)$ are local coordinates in U_{it},

$$\rho_{it} X = \sum_{k \leq n} [X_{tik}, X'_{tik}],$$

where

$$X_{tik} = \varphi_{it} D_{x^k} \text{ and } X'_{tik} = \varphi_{it} \int_0^{x^k} \rho_{it} X^k \varphi_{it}^{-2} \, dx^k \cdot D_{x^k},$$

X^k being the k's component of the corresponding local form of X. In the integral, the function is extended by 0 outside the image of supp $\rho_{it} X$.

The vector fields

$$X_{tk} = \sum_{i \in \mathbb{N}} X_{tik} \text{ and } X'_{tk} = \sum_{i \in \mathbb{N}} X'_{tik}$$

are vanishing in ω and moreover,

$$X = \sum_{t \le r} \sum_{i \in \mathbb{N}} \rho_{it} X = \sum_{t \le r} \sum_{k \le m} [X_{tk'} X'_{tk}],$$

hence the lemma.

PROOF OF PROPOSITION 3.1: We first prove that B is a local map. By Peetre's theorem (for the multilinear version, see [1]), it will then be a differential bilinear operator.

Suppose that $X \in \mathcal{H}(M)$ vanishes in an open subset U. For each $x_0 \in U$, there exist ω such that $x_0 \in \omega \subset U$ and X_i, X'_i vanishing in ω such that

$$X = \sum_i [X_i, X'_i].$$

Choose $Z \in \mathcal{H}(M)$ with support in ω and in a domain of chart V of M, such that $Z_{x_0} = 0$ and $D_{x_0} Z = \mathrm{I}$ for a coordinate system of V. Then

$$\begin{aligned} B(X, Y) &= \sum_i [\mathcal{L}_{X_i} B(X'_i, Y) - B(X'_i, \mathcal{L}_{X_i} Y)] \\ &= -\sum_i B(X'_i, \mathcal{L}_{X_i} Y) \end{aligned}$$

in ω and

$$\begin{aligned} B(X, Y)_{x_0} &= \mathcal{L}_Z B(X, Y)_{x_0} = \sum_i \mathcal{L}_Z B(X'_i, \mathcal{L}_{X_i} Y)_{x_0} \\ &= \sum_i [B(\mathcal{L}_Z X'_i, \mathcal{L}_{X_i} Y)_{x_0} - B(X'_i, \mathcal{L}_Z \mathcal{L}_{X_i} Y)_{x_0}] = 0 \end{aligned}$$

because $\mathcal{L}_Z Z' = 0$ whenever $Z' = 0$ in ω. Thus $X = 0$ on U implies that $B(X, Y) = 0$ on U and B is local.

The end of the proof consists in computations on the differential operator B. The arguments of Van Strien would easily adapt to the present situation. A slightly different approach is given here for the sake of completeness.

Let us decompose B in its symmetric and antisymmetric parts, which both verify the assumption of prop. 3.1.

Fix a coordinate system $(U, x^1, \ldots, x^n)$ of M and compute B in these coordinates. Taking for X the fields D_{x^i} and $\sum_i x^i D_{x^i}$ shows that, for some A independent of $(x^1, \ldots, x^n)$,

$$B(X, Y) = A(X, D_x Y) \pm A(Y, D_x X)$$

(everything is now written in local coordinates) according to B is symmetric or antisymmetric.

The assumption on B reads then

$$\begin{aligned} &D_xX \cdot A(Y, D_xZ) - A(D_xX \cdot Y, D_xZ) - A(Y, [D_xX, D_xZ]) \\ &\quad \pm \{D_xX \cdot A(Z, D_xY) - A(D_xX \cdot Z, D_xY) - A(Z, [D_xX, D_xY])\} \qquad (*) \\ &\quad = A(Y, Z \cdot D_xX) \pm A(Z, Y \cdot D_xX). \end{aligned}$$

If we choose $X, Y \in \mathscr{H}(M)$ such that $D_{x^i}D_{x^j}X = 0$ at x and $Y_x = 0$, it follows that

$$D_xX \cdot A(Z, D_xY) = A(D_xX \cdot Z_x, D_xY) + A(Z_x, [D_xX, D_xY])$$

and thus that the bilinear form $A : \mathbb{R}^n \times gl(n, \mathbb{R}) \to \mathbb{R}^n$ verifies

$$P\,A(u, Q) = A(Pu, Q) + A(u, [P, Q]).$$

In other words, $Q \to A(., Q) \in L(gl(n, \mathbb{R}))$ belongs to the centralizer of the adjoint action of $gl(n, \mathbb{R})$. It is then easily seen that

$$A(u, Q) = k\left(Q - \frac{1}{n}\operatorname{tr} Q \cdot I\right) + \frac{1}{n}\operatorname{tr} Q \cdot I,$$

for some $k, l \in \mathbb{R}$. Substituting this in (*), it follows that $k = l = 0$ if B is symmetric and that $k = l$ if B is antisymmetric. Thus

$$B(X, Y) = k[X, Y],$$

hence the result.

REFERENCES

[1] M. Cahen, M. De Wilde and S. Gutt: Local cohomology of the algebra of C^∞ functions on a connected manifold. *Lett. in Math. Physics 4* (1980), 157–167.

[2] W. Greub, S. Halperin and R. Vanstone: *Connections, curvature and cohomology, Vol. I*, Academic Press, New York and London (1973).

[3] R. Palais: Natural operations on differential forms. *Trans. A.M.S. 92* (1959), 125–141.

[4] S. Van Strien: Unicity of the Lie Product. *Comp. Math. 40* (1980), 79–85.

(Oblatum 5-I-1981 & 27-IV-1981)

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