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**ALGEBRAIC CHARACTERIZATIONS OF THE ALGEBRA
 OF FUNCTIONS AND OF THE LIE ALGEBRA OF VECTOR
 FIELDS OF A MANIFOLD**

M. De Wilde and P. Lecomte

1. Introduction

Let M be a smooth, connected, Hausdorff and second countable manifold. Denote by $\mathcal{H}(M)$ the Lie algebra of smooth vector fields of M and by $C_\infty(M)$ the space of smooth functions on M . Recall that the Lie derivative $\mathcal{L}_X(X \in \mathcal{H}(M))$ is defined on $\otimes^p C_\infty(M)$ by

$$\mathcal{L}_X(f_1 \otimes \cdots \otimes f_p) = \sum_{j=1}^p f_1 \otimes \cdots \otimes (\mathcal{L}_X f_j) \otimes \cdots \otimes f_p.$$

If $L : \otimes^p C_\infty(M) \rightarrow C_\infty(M)$ is a p -linear map, the adjoint action of \mathcal{L}_X on L , $ad(\mathcal{L}_X)L$ is the commutator $\mathcal{L}_X \circ L - L \circ \mathcal{L}_X$. We intend to show that the only symmetric p -linear maps $L : \otimes^p C_\infty(M) \rightarrow C_\infty(M)$ for which $ad(\mathcal{L}_X)L = 0$ for each $X \in \mathcal{H}(M)$ are the multiples of the product of p factors: $f_1 \otimes \cdots \otimes f_p \rightarrow f_1 \cdots f_p$. It provides thus a characterization of the product by means of its derivations.

A similar question is investigated for the Lie bracket of vector fields, which is shown to be determined up to a constant by the fact that its derivations are the Lie derivatives. This question has been solved by Van Strien in [4] under stronger assumptions which will be discussed in §3.

2. Characterization of the algebraic structure of $C_\infty(M)$

LEMMA 2.1: *There exist a finite partition of the unity $\lambda_t (t \leq r)$ of M , vector fields $X_t (t \leq r) \in \mathcal{H}(M)$ and functions $\mu_t (t \leq r) \in C_\infty(M)$ such that $\lambda_t = \lambda_t X_t \cdot \mu_t$ for each $t \leq r$.*

PROOF: It is well known from dimension theory [2], p. 20 that M admits an open cover $U_t (t \leq r)$ such that each U_t is the disjoint union of domains of charts $U_{it} (i \in \mathbb{N})$, as well as a locally finite partition of the unity $\rho_{it} (t \leq r, i \in \mathbb{N})$ such that each ρ_{it} has compact support in U_{it} . We set $V_{it} = \{x \in U_{it} : \rho_{it}(x) > 0\}$ and choose a partition of the unity $\rho'_{it} (t \leq r; i \in \mathbb{N})$ such that $\text{supp } \rho'_{it}$ is compact in V_{it} for all i, t .

For each $t \leq r$ and $i \in \mathbb{N}$, we may find $\alpha_{it}, \beta_{it} \in C_\infty(M)$ with compact supports in U_{it} such that $\alpha_{it} \mid V_{it} = 1$ and $\beta_{it} = 1$ in some neighborhood of $\text{supp } \alpha_{it}$. If $(x^1_{it}, \dots, x^n_{it})$ are local coordinates in U_{it} ,

$$\lambda_t = \sum_i \rho'_{it}, X_t = \sum_i \alpha_{it} D_{x^1_{it}} \text{ and } \mu_t = \sum_i \beta_{it} x^1_{it}$$

have the required properties. Hence the lemma.

Denote by A_p the subspace of $\otimes^p C_\infty(M)$ spanned by the tensors which are antisymmetric in at least two arguments.

LEMMA 2.2: For all $f_i \in C_\infty(M)$ ($i \leq p$), there exist $N \in \mathbb{N}$, vector fields $X_i \in \mathcal{H}(M)$ ($i \leq N$) and $f_{ik} \in C_\infty(M)$ ($i \leq N, k \leq p$), such that

$$f_1 \otimes \cdots \otimes f_p - \sum_{i=1}^N \mathcal{L}_{f_i X_i} (f_{i1} \otimes \cdots \otimes f_{ip}) \in A_p.$$

Moreover the vector fields $X_i (i \leq N)$ and the functions $f_{ik} (i \leq N, k \leq p)$ can be chosen independently of f_1 .

PROOF: We have

$$f_1 \otimes \cdots \otimes f_p = \sum_{t \leq r} \lambda_t f_1 \otimes f_2 \cdots \otimes f_p. \quad (*)$$

Since $\lambda_t = \lambda_t X_t \cdot \mu_t$,

$$\begin{aligned} \lambda_t f_1 \otimes f_2 \cdots \otimes f_p &= \mathcal{L}_{f_1 \lambda_t X_t} (\mu_t \otimes f_2 \otimes \cdots \otimes f_p) \\ &- \sum_{i>1} \mu_t \otimes f_2 \otimes \cdots \otimes f_1 \lambda_t X_t \cdot f_i \otimes \cdots \otimes f_p. \end{aligned} \quad (**)$$

Let us consider for instance the term $i = 2$. Setting $X_t \cdot f_2 = g_2$, it reads

$$\begin{aligned} &\mu_t \otimes f_1 g_2 \lambda_t X_t \cdot \mu_t \otimes f_3 \otimes \cdots \otimes f_p \\ &= \frac{1}{2} (\mu_t \otimes f_1 g_2 \lambda_t X_t \cdot \mu_t \otimes f_3 \otimes \cdots + f_1 g_2 \lambda_t X_t \cdot \mu_t \otimes \mu_t \otimes f_3 \otimes \cdots) \\ &+ \frac{1}{2} (\mu_t \otimes f_1 g_2 \lambda_t X_t \cdot \mu_t \otimes f_3 \otimes \cdots - f_1 g_2 \lambda_t X_t \cdot \mu_t \otimes \mu_t \otimes f_3 \otimes \cdots). \end{aligned}$$

PROPOSITION 2.3: *Let $P : C_\infty(M) \times \cdots \times C_\infty(M) \rightarrow C_\infty(M)$ be a p -linear symmetric map. If $ad(\mathcal{L}_X)P = 0$ for each $X \in \mathcal{H}(M)$, then*

$$P(f_1, \dots, f_p) = k f_1 \cdots f_p$$

for some $k \in \mathbb{R}$.

PROOF: Since P is symmetric, P vanishes on A_p . Using lemma 2.2.,

$$\begin{aligned} P(f_1, \dots, f_p) &= \sum_i P \circ \mathcal{L}_{f_i X_i}(f_{i1} \otimes \cdots \otimes f_{ip}) = \sum_i \mathcal{L}_{f_i X_i} \circ P(f_{ij}, \dots, f_{ip}) \\ &= f_1 \cdot \left[\sum_i X \cdot P(f_{i1}, \dots, f_{ip}) \right], \end{aligned}$$

where the X_i 's and f_{ij} 's do not depend on f_1 . Therefore

$$P(f_1, \dots, f_p) = f_1 P(1, f_2, \dots, f_p).$$

It is clear that $P(1, f_2, \dots, f_p) = P'(f_2, \dots, f_p)$ satisfies again $ad(\mathcal{L}_X)P' = 0$, thus, by induction,

$$P(f_1, \dots, f_p) = f_1 \cdots f_p \cdot P(1, \dots, 1).$$

The condition $ad(\mathcal{L}_X)P = 0$ clearly shows that $P(1, \dots, 1)$ is constant, hence the result.

RESULT 2.4: *The symmetry assumption on P is necessary.*

Indeed, consider a compact, connected, oriented manifold M of dimension $p - 1$ and take the map P :

$$(f_1, \dots, f_p) \rightarrow \int f_1 df_2 \wedge \cdots \wedge df_p.$$

It is clear that p is such that $ad(\mathcal{L}_X)P = 0$ for all $X \in \mathcal{H}(M)$. However, $P \neq 0$, thus P is not even local.

3. Characterization of the algebraic structure of $\mathcal{H}(M)$

PROPOSITION 3.1: *Let $B : \mathcal{H}(M) \times \mathcal{H}(M) \rightarrow \mathcal{H}(M)$ be a bilinear map such that*

$$\mathcal{L}_X B(Y, Z) = B(\mathcal{L}_X Y, Z) + B(Y, \mathcal{L}_X Z), \quad \forall X, Y, Z \in \mathcal{H}(M). \quad (*)$$

Then there exists $k \in \mathbb{R}$ such that

$$B(X, Y) = k[X, Y], \forall X, Y \in \mathcal{H}(M).$$

A similar result is proved by Van Strien in [4]. The assumption (*) is replaced by the “naturality” of B , which means the following: B is supposed to be defined on every manifold M and, for every smooth open imbedding φ , the diagram

$$\begin{array}{ccc} \mathcal{H}(M) \times \mathcal{H}(M) & \xrightarrow{B_M} & \mathcal{H}(M) \\ \downarrow \varphi_* \times \varphi_* & & \downarrow \varphi_* \\ \mathcal{H}(N) \times \mathcal{H}(N) & \xrightarrow{B_N} & \mathcal{H}(N) \end{array}$$

commutes (see [3]). It is clear that the naturality implies the locality of B and, using the pseudo-group of X , it also implies (*).

LEMMA 3.2: *If $X \in \mathcal{H}(M)$ vanishes in an open subset U of M , for each $x \in U$, there exists a neighborhood ω of x and vector fields $X_i, X'_i (i \leq N)$ such that $X = \sum_{i \leq N} [X_i, X'_i]$ and X_i, X'_i vanish in ω .*

PROOF: Fix ω relatively compact in U . Choose ρ_{it} as in lemma 2.1 and $\varphi_{it} \in C_\infty(M)$ with compact support in $U_{it} \setminus \omega$ and equal to 1 in a neighborhood of $\text{supp } \rho_{it} X$. Then, if (x^1, \dots, x^n) are local coordinates in U_{it} ,

$$\rho_{it} X = \sum_{k \leq n} [X_{tik}, X'_{tik}],$$

where

$$X_{tik} = \varphi_{it} D_x^k \text{ and } X'_{tik} = \varphi_{it} \int_0^{x^k} \rho_{it} X^k \varphi_{it}^{-2} dx^k \cdot D_x^k,$$

X^k being the k 's component of the corresponding local form of X . In the integral, the function is extended by 0 outside the image of $\text{supp } \rho_{it} X$.

The vector fields

$$X_{ik} = \sum_{i \in \mathbb{N}} X_{tik} \text{ and } X'_{ik} = \sum_{i \in \mathbb{N}} X'_{tik}$$

are vanishing in ω and moreover,

$$X = \sum_{i \leq r} \sum_{i \in N} \rho_{ii} X = \sum_{i \leq r} \sum_{k \leq m} [X_{ik} X'_{ik}],$$

hence the lemma.

PROOF OF PROPOSITION 3.1: We first prove that B is a local map. By Peetre's theorem (for the multilinear version, see [1]), it will then be a differential bilinear operator.

Suppose that $X \in \mathcal{H}(M)$ vanishes in an open subset U . For each $x_0 \in U$, there exist ω such that $x_0 \in \omega \subset U$ and X_i, X'_i vanishing in ω such that

$$X = \sum_i [X_i, X'_i].$$

Choose $Z \in \mathcal{H}(M)$ with support in ω and in a domain of chart V of M , such that $Z_{x_0} = 0$ and $D_{x_0} Z = I$ for a coordinate system of V . Then

$$\begin{aligned} B(X, Y) &= \sum_i [\mathcal{L}_{X'_i} B(X_i, Y) - B(X'_i, \mathcal{L}_{X_i} Y)] \\ &= - \sum_i B(X'_i, \mathcal{L}_{X_i} Y) \end{aligned}$$

in ω and

$$\begin{aligned} B(X, Y)_{x_0} &= \mathcal{L}_Z B(X, Y)_{x_0} = \sum_i \mathcal{L}_Z B(X'_i, \mathcal{L}_{X_i} Y)_{x_0} \\ &= \sum_i [B(\mathcal{L}_Z X'_i, \mathcal{L}_{X_i} Y)_{x_0} - B(X'_i, \mathcal{L}_Z \mathcal{L}_{X_i} Y)_{x_0}] = 0 \end{aligned}$$

because $\mathcal{L}_Z Z' = 0$ whenever $Z' = 0$ in ω . Thus $X = 0$ on U implies that $B(X, Y) = 0$ on U and B is local.

The end of the proof consists in computations on the differential operator B . The arguments of Van Strien would easily adapt to the present situation. A slightly different approach is given here for the sake of completeness.

Let us decompose B in its symmetric and antisymmetric parts, which both verify the assumption of prop. 3.1.

Fix a coordinate system (U, x^1, \dots, x^n) of M and compute B in these coordinates. Taking for X the fields D_{x^i} and $\sum_i x^i D_{x^i}$ shows that, for some A independent of (x^1, \dots, x^n) ,

$$B(X, Y) = A(X, D_x Y) \pm A(Y, D_x X)$$

(everything is now written in local coordinates) according to B is symmetric or antisymmetric.

The assumption on B reads then

$$\begin{aligned} D_x X \cdot A(Y, D_x Z) - A(D_x X \cdot Y, D_x Z) - A(Y, [D_x X, D_x Z]) \\ \pm \{D_x X \cdot A(Z, D_x Y) - A(D_x X \cdot Z, D_x Y) - A(Z, [D_x X, D_x Y])\} \quad (*) \\ = A(Y, Z \cdot D_x X) \pm A(Z, Y \cdot D_x X). \end{aligned}$$

If we choose $X, Y \in \mathcal{H}(M)$ such that $D_{x^i} D_{x^j} X = 0$ at x and $Y_x = 0$, it follows that

$$D_x X \cdot A(Z, D_x Y) = A(D_x X \cdot Z_x, D_x Y) + A(Z_x, [D_x X, D_x Y])$$

and thus that the bilinear form $A : \mathbb{R}^n \times \mathfrak{gl}(n, \mathbb{R}) \rightarrow \mathbb{R}^n$ verifies

$$P A(u, Q) = A(Pu, Q) + A(u, [P, Q]).$$

In other words, $Q \rightarrow A(\cdot, Q) \in L(\mathfrak{gl}(n, \mathbb{R}))$ belongs to the centralizer of the adjoint action of $\mathfrak{gl}(n, \mathbb{R})$. It is then easily seen that

$$A(u, Q) = k \left(Q - \frac{1}{n} \operatorname{tr} Q \cdot I \right) + \frac{1}{n} \operatorname{tr} Q \cdot I,$$

for some $k, l \in \mathbb{R}$. Substituting this in (*), it follows that $k = l = 0$ if B is symmetric and that $k = l$ if B is antisymmetric. Thus

$$B(X, Y) = k[X, Y],$$

hence the result.

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