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MINIMAL MODELS FOR ELLIPTIC CURVES WITH COMPLEX MULTIPLICATION

Benedict H. Gross

Let R be the ring of integers in an algebraic number field F . An abelian variety A of dimension g over F determines an element c_A in the ideal class group R in the following manner. Let N denote the Néron model of A over R [4]; the space $\omega_{N/R}$ of invariant differentials on N is a projective R -module of rank g . We may define c_A to be the class of $\overset{g}{\omega}_{N/R}$ in $\text{Pic}(R)$.

When $\dim A = 1$ Tate has given an alternate description of the class c_A in terms of minimal Weierstrass models [5]. We use this formulation, and some classical results of Deuring [1] and Hasse, to calculate c_A for some elliptic curves with complex multiplication.

§1. Minimal models of elliptic curves

Let A be an elliptic curve over F , a number field with ring of integers R . The space $\omega_{A/F} = H^0(A, \Omega^1/F)$ of invariant differentials is an F -vector space of dimension 1. Associated to any non-zero differential ω we have its discriminant $\Delta_\omega \in F^*$ [5]. If $\omega' = u^{-1}\omega$ then $\Delta_{\omega'} = u^{12}\Delta_\omega$; hence A determines a coset $\Delta_A \in F^*/F^{*12}$.

For any discrete valuation v of F , let ω_v and $\Delta_v = \Delta_{\omega_v}$ be the differential and discriminant of a minimal Weierstrass equation for A at v [5]. We define the discriminant ideal $\mathcal{D}_A$ by the formula:

$$(1.1) \quad \mathcal{D}_A = \prod_v \mathcal{P}_v^{v(\Delta_v)},$$

where $\mathcal{P}_v$ is a prime ideal at the place v . For any non-zero differential

ω on A over F we define the ideal δ_ω by the formula:

$$(1.2) \quad \delta_\omega = \prod_v \mathcal{P}_v^{v(\omega/\omega_v)}.$$

One then has the equality of ideals in R :

$$(1.3) \quad (\Delta_\omega) \delta_\omega^{12} = \mathcal{D}_A.$$

The class of the ideal δ_ω in $\text{Pic}(R)$ is independent of the choice of ω . We denote this class by δ_A ; then A has a global differential ω with $(\Delta_\omega) = \mathcal{D}_A$ if and only if $\delta_A \sim 1$ in $\text{Pic}(R)$. In this case one can find a global minimal model for A : i.e., an equation for A over R which is simultaneously minimal at all places v .

By (1.3) one has:

$$(1.4) \quad \delta_A^{12} \sim \mathcal{D}_A \quad \text{in } \text{Pic}(R).$$

Hence a necessary condition for the existence of a global minimal model is that the ideal $\mathcal{D}_A$ be principal. By (1.4) this is also sufficient when the group $\text{Pic}(R)$ has no 12-torsion.

It is not difficult to compare δ_A with the class c_A of Néron differentials defined in the introduction. Let X be the minimal regular model for A over R_v ; X is a regular projective scheme over R_v which can be obtained by resolving the possible singularity on a minimal Weierstrass equation for A over R_v [4, pp. 94–101]. The Néron minimal model N is a smooth group scheme over R_v ; it is obtained by removing all fibres of multiplicity greater than one on X and all singular points in the remaining fibres. The pull-back of a minimal Weierstrass differential ω_v on A/R_v is everywhere non-zero on N . Hence we find:

$$(1.5) \quad \varpi_{N/R_v} = \omega_v R_v \subset \varpi_{A/F_v},$$

so globally we have the identity:

$$(1.6) \quad \varpi_{N/R} = \omega \delta_\omega^{-1} \subset \varpi_{A/F}.$$

To sum up, we have the following

PROPOSITION 1.7:

- (1) $c_A \sim \delta_A^{-1}$ in $\text{Pic}(R)$.
- (2) The following statements are equivalent
 - (a) $c_A \sim \delta_A \sim 1$ in $\text{Pic}(R)$.
 - (b) A has a global minimal Weierstrass model over R .
 - (c) A has a non-zero differential ω with $(\Delta_\omega) = \mathcal{D}_A$.
 - (d) $\omega_{N/R}$ is a free R -module of rank 1.

§2. Elliptic curves with complex multiplication

We now assume that A is an elliptic curve with complex multiplication by the ring of integers $\mathcal{O}$ of an imaginary quadratic field K . We assume further that the field F of definition for A is H , the Hilbert class field of K . Then all endomorphisms of A are defined over H , and the curve A is determined up to isomorphism by its modular invariant j_A and the associated Hecke character χ_A on the idèles I_H of H [2; 9.1.3].

PROPOSITION 2.1: *Both the ideal $\mathcal{D}_A$ and the class δ_A depend only on the character χ_A , and not on the modular invariant j_A .*

PROOF: Let B be another elliptic curve over F with $\chi_B = \chi_A$; then $j_B = j_A^\sigma$ with $\sigma \in \text{Aut}(H)$. The group $\text{Hom}_H(B, A)$ is described in [2, 9.4.2]: for any integral ideal $\mathfrak{a}$ of K such that $\sigma = \sigma_{\mathfrak{a}}^{-1}$ in $\text{Aut}(H)$ we have an isogeny $\phi_{\mathfrak{a}}: B \rightarrow A$ with kernel isomorphic to $\mathcal{O}/\mathfrak{a}$. More precisely, we may choose an embedding of H into $\mathbb{C}$ so that the following diagram commutes:

$$(2.2) \quad \begin{array}{ccc} B(\mathbb{C}) & \xrightarrow{\phi_{\mathfrak{a}}} & A(\mathbb{C}) \\ \int \phi_{\mathfrak{a}}^* \omega \downarrow & & \downarrow \int \omega \\ \mathbb{C}/\Omega \mathfrak{a} & \xrightarrow{p} & \mathbb{C}/\Omega \mathcal{O} \end{array}$$

where ω is a non-zero differential on A , $\Omega \in \mathbb{C}^*$ is a fixed integral period of ω , and p is the natural projection.

Now let v be a fixed place of H and choose $\mathfrak{a}$ with $\sigma_{\mathfrak{a}}^{-1} = \sigma$ and $N\mathfrak{a}$ prime to v (this is always possible). Then the induced map $\phi_{\mathfrak{a}}^*: \omega_{B/R_v} \rightarrow \omega_{A/R_v}$ on the spaces of local Néron differentials is an isomorphism. Hence to show that $\mathcal{D}_A = \mathcal{D}_B$ it suffices to show that $v(\Delta_{\omega_v}) = v(\Delta_{\phi_{\mathfrak{a}}^* \omega_v})$. But by (2.2), if we compute over $\mathbb{C}$,

$$(2.3) \quad \Delta_{\omega} = \frac{\Delta(\mathcal{O})}{\Delta(\mathfrak{a})} \Delta_{\phi_{\mathfrak{a}}^* \omega}.$$

It is well-known that $\Delta(\mathcal{O})/\Delta(\mathfrak{a})$ is an algebraic integer in H which generates the ideal $\mathfrak{a}^{12}$ [1, p. 33], [3, p. 165]. Since this is prime to v , the minimal discriminants have the same valuation.

Now let ω be any non-zero differential on A over H and put $\nu = \phi_{\mathfrak{a}}^*(\omega)$. Then by (1.3) and the above paragraph:

$$(\Delta_{\omega})\delta_{\omega}^{12} = \mathcal{D}_A = \mathcal{D}_B = (\Delta_{\nu})\delta_{\nu}^{12}.$$

Since $\Delta_{\omega}/\Delta_{\nu} = \Delta(\mathcal{O})/\Delta(\mathfrak{a})$ by (2.3), we have

$$(\delta_{\nu}/\delta_{\omega})^{12} = (\Delta(\mathcal{O})/\Delta(\mathfrak{a})) = \mathfrak{a}^{12}.$$

Hence $\delta_{\nu} = \delta_{\omega} \cdot \mathfrak{a}$ as ideals of H . But the ideal $\mathfrak{a}$ of K capitulates in H ; hence $\delta_A \sim \delta_B$ in $\text{Pic}(R)$.

Note: If we assume that the Hecke character $\chi_A: I_H \rightarrow K^*$ is $\text{Gal}(H/K)$ -equivariant, then by Proposition 2.1 the ideal $\mathcal{D}_A$ is fixed by $\text{Gal}(H/K)$. Since H is unramified over K , any fixed ideal is represented by an ideal of K . But all ideals of K capitulate in H , so $\mathcal{D}_A \sim 1$ in $\text{Pic}(R)$. Is $\delta_A \sim 1$ in $\text{Pic}(R)$? We will show this is the case when K has prime discriminant.

§3. A global minimal model for $A(p)$

We now specialize to the case where the multiplication field $K = \mathbb{Q}(\sqrt{-p})$ has *prime* discriminant.

LEMMA 3.1: *For any fractional ideal $\mathfrak{a}$ of K , the ratio $\Delta(\mathcal{O})/\Delta(\mathfrak{a})$ is a 12^{th} power in H^* .*

PROOF: By Deuring [1, p. 14, 41] the ratio $\Delta(\mathcal{O})/\Delta(\mathfrak{b}^2)$ is a 24^{th} power in H^* when $(6, \mathfrak{b}) = 1$. When K has prime discriminant, its class group has *odd* order. Hence we may find an ideal $\mathfrak{b}$ prime to 6 such that $(\alpha)\mathfrak{a} = \mathfrak{b}^2$. Then

$$\Delta(\mathcal{O})/\Delta(\mathfrak{a}) = \alpha^{12} \cdot \Delta(\mathcal{O})/\Delta(\mathfrak{b}^2) \equiv 1 \pmod{H^{*12}}.$$

We can now answer affirmatively a question posed by D. Zagier. Assume that $p > 3$ and let $A(p)$ denote $\mathbb{Q}$ -curve over the field $F = \mathbb{Q}(j_{A(p)})$ studied in chapter 5 of [2]. Recall that $A(p)$ has good reduction outside p and has minimal discriminant ideal $\mathcal{D}_{A(p)} = (-p^3)$. The

fact that this ideal is principal raises the possibility of a global minimal model.

PROPOSITION 3.2: *The curve $A(p)$ has a global minimal model over the field $F = \mathbb{Q}(j_{A(p)})$ with discriminant $\Delta = -p^3$. The associated differential $\omega(p)$ is determined up to sign.*

PROOF: In §23 of [2] we constructed a pair (A, ω) over F with $j_A = j_{A(p)}$, $\Delta_\omega = -p^3$, and $\text{sign } c_6 = \left(\frac{2}{p}\right)$. Recall that A is given by the equation

$$(3.3) \quad y^2 = x^3 + \frac{mp}{2^4 \cdot 3} x - \frac{np^2}{2^5 \cdot 3^3}$$

where

$$(3.4) \quad m^3 = j_{A(p)} \\ n^2 = (j_{A(p)} - 1728)/-p, \quad \text{sign } n = \left(\frac{2}{p}\right),$$

The differential $\omega = dx/2y$ on A has $\Delta_\omega = -p^3$. To prove Proposition 3.2 we will show that A is *isomorphic* to $A(p)$ over F . We will then have a global minimal model by Proposition 1.7, as $(\Delta_\omega) = \mathcal{D}_{A(p)}$. The differential $\omega = \omega(p)$ with $\Delta_\omega = -p^3$ is determined up to sign, as $\mu(F^*) = \langle \pm 1 \rangle$.

In summary, we are reduced to proving:

PROPOSITION 3.5: *The elliptic curve A defined by equations (3.3–3.4) is a $\mathbb{Q}$ -curve which is isomorphic over F to the curve $A(p)$.*

PROOF: Consider the map

$$f_A : \text{Gal}(H/\mathbb{Q}) \rightarrow \text{Hom}(I_H, K^*) \\ \sigma \mapsto \chi_A^{\sigma-1}$$

where all Homs refer to continuous homomorphisms of topological groups. Then f_A is a 1-cocycle, which takes values in the group $\text{Hom}(I_H/H^*, K^*)$. Since K^* is totally disconnected, this group may be identified with the group $\text{Hom}(\text{Gal}(\bar{H}/H), K^*)$ via the Artin homomorphism of global class field theory. Since $\text{Gal}(\bar{H}/H)$ is compact and K^* is discrete, any continuous homomorphism takes values

in the finite group $\mu(K^*) = \langle \pm 1 \rangle$. Finally, we may identify

$$\text{Hom}(\text{Gal}(\bar{H}/H), \pm 1) \simeq H^*/H^{*2},$$

by Kummer theory, and view f_A as a map

$$(3.5) \quad f_A : \text{Gal}(H/\mathbb{Q}) \rightarrow H^*/H^{*2}.$$

To show A is a $\mathbb{Q}$ -curve is equivalent to showing that $f_A(\sigma) \equiv 1$ for all $\sigma \in \text{Gal}(H/\mathbb{Q})$. Since A is defined over F we have $f_A(\tau) \equiv 1$. Hence, it suffices to show $f_A(\sigma) = 1$ for all $\sigma \in \text{Gal}(H/K)$.

For this, we need a concrete description of $f_A(\sigma)$ in H^*/H^{*2} . Embed F in $\mathbb{C}$ via its real place, and let $\mathfrak{a}$ be an integral ideal of K with $\sigma = \sigma_{\mathfrak{a}}^{-1}$. There is an isogeny $\phi_{\mathfrak{a}}$ defined over $\bar{\mathbb{Q}}$ which makes the following diagram commutative:

$$\begin{array}{ccc} A^{\sigma} & \xrightarrow{\quad \phi_{\mathfrak{a}} \quad} & A \\ \int \phi_{\mathfrak{a}}^* \omega \downarrow & & \downarrow \int \omega \\ \mathbb{C}/\Omega \mathfrak{a} & \xrightarrow[p]{} & \mathbb{C}/\Omega \mathcal{O}. \end{array}$$

If we write $\phi_{\mathfrak{a}}^*(\omega) = h_{\mathfrak{a}} \cdot \omega^{\sigma}$ with $h_{\mathfrak{a}} \in \bar{\mathbb{Q}}^*$, then the isogeny $\phi_{\mathfrak{a}}$ is defined over the extension $H(h_{\mathfrak{a}})$. The identities:

$$\begin{aligned} c_4(\mathcal{O})/c_4(\mathfrak{a}) &= h_{\mathfrak{a}}^4 \cdot c_4^{1-\sigma} \\ c_6(\mathcal{O})/c_6(\mathfrak{a}) &= h_{\mathfrak{a}}^6 \cdot c_6^{1-\sigma} \end{aligned}$$

show that $h_{\mathfrak{a}}^2 \in H^*$ [3, p. 158]. In fact, we have the formula

$$(3.6) \quad f_A(\sigma) \equiv h_{\mathfrak{a}}^2 \pmod{H^{*2}}.$$

On the other hand, we have the identity:

$$\Delta(\mathcal{O})/\Delta(\mathfrak{a}) = h_{\mathfrak{a}}^{12} \cdot \Delta^{1-\sigma} = h_{\mathfrak{a}}^{12}$$

as $\Delta = -p^3$ is fixed by $\text{Gal}(H/\mathbb{Q})$. By Lemma 3.1, $h_{\mathfrak{a}}^{12}$ is a 12th power in H^* . Since $h_{\mathfrak{a}}^2 \in H^*$, we must have $h_{\mathfrak{a}} \in H^* \mu_4$ and $f_A(\sigma) \equiv \pm 1 \pmod{H^{*2}}$. But f_A is a cocycle and the order of $\text{Gal}(H/K)$ is odd. Hence $f_A(\sigma) \equiv 1$ and A is a $\mathbb{Q}$ -curve.

Since $v_{\mathcal{P}}(\Delta_{\omega}) = 3$ we see $A \simeq A(p)^d$ with $(p, d) = 1$ [2, 12.3.2]. But $\mathcal{D}_A = \mathfrak{b}^{12}(-p^3)$ and $\mathcal{D}_{A(p)^d} = \mathfrak{c}^{12}(-p^3 d^6)$ where $\mathfrak{b}$ and $\mathfrak{c}$ are ideals of H .

Hence $(d) = (\mathfrak{b}/\mathfrak{c})^2$ is the square of an ideal of H . Since H is unramified over K and d is a quadratic discriminant, there are only two possibilities: $d = 1$ and $d = -4$. But the curve $A(p)^{-4}$ has the wrong sign of c_6 , so $A \simeq A(p)$.

§4. Global minimal models for K-curves

Let $\omega(p)$ be one of the differentials on $A(p)$ given by Proposition 3.2. For any integral ideal $\mathfrak{a}$ of K we may define $h_{\mathfrak{a}}$ in $H^*/\pm 1$ by the formula:

$$(4.1) \quad \phi_{\mathfrak{a}}^*(\omega(p)) = h_{\mathfrak{a}} \cdot \omega(p)^{\sigma_{\mathfrak{a}}^{-1}}.$$

The ambiguity in sign is caused by the ambiguity in the choice of isogeny $\phi_{\mathfrak{a}}$; we will discuss a choice of the sign in §5. In $H^*/\pm 1$ we have the cocycle relations

$$(4.2) \quad \begin{aligned} h_{\mathfrak{a}\mathfrak{b}} &= h_{\mathfrak{a}}^{\sigma_{\mathfrak{b}}^{-1}} \cdot h_{\mathfrak{b}} \\ h_{\mathfrak{a}^{\tau}} &= h_{\mathfrak{a}}^{\tau} \end{aligned}$$

We have seen in §3 that when F is embedded into $\mathbb{C}$ via its real place we have the complex identity:

$$h_{\mathfrak{a}}^{12} = \Delta(\mathcal{O})/\Delta(\mathfrak{a}).$$

Hence $h_{\mathfrak{a}}$ is *integral* in H and generates the ideal $\mathfrak{a}$. The same is true for $h_{\mathfrak{a}}^{\sigma}$ for any $\sigma \in \text{Gal}(H/K)$.

LEMMA 4.1: For all $\sigma \in \text{Gal}(H/K)$, $h_{\mathfrak{a}}^{\sigma^{-1}} \equiv 1 \pmod{H^{*2}}$.

PROOF: First note that this identity makes sense, independent of the choice of sign for $h_{\mathfrak{a}}$. We have seen, in the proof of Lemma 3.1, that $\Delta(\mathcal{O})/\Delta(\mathfrak{b}^2) = h_{\mathfrak{b}^2}^{12}$ is a 24^{th} power in H^* . Hence $h_{\mathfrak{b}^2} \equiv \pm 1 \pmod{H^{*2}}$. Since we may find $\mathfrak{b}$ such that $\mathfrak{a} = (\alpha)\mathfrak{b}^2$, we find from (4.2) that $h_{\mathfrak{a}} \equiv \pm \alpha \pmod{H^{*2}}$. Hence $h_{\mathfrak{a}}^{\sigma^{-1}} \equiv (\text{mod } H^{*2})$ for any $\sigma \in \text{Gal}(H/K)$.

LEMMA 4.2: Let K' be a quadratic extension of K with conductor $\mathfrak{a}$. Then we may choose the sign of $h_{\mathfrak{a}}$ so that $HK' = H(\sqrt{h_{\mathfrak{a}}})$.

PROOF: Write $K' = K(\sqrt{\alpha})$. Since $\mathfrak{a}$ is the discriminant ideal of K'/K and α is the discriminant of the specific K -basis $\langle 1, \sqrt{\alpha}/2 \rangle$ we

find $(\alpha)\mathfrak{b}^2 = \mathfrak{a}$ with $\mathfrak{b}$ an ideal of K . Raising this identity to the h^{th} power and writing $(\beta) = \mathfrak{b}^h$ we find $(\alpha^h \beta^2) = \mathfrak{a}^h = (\mathbb{N}_{H/K} h_{\mathfrak{a}})$. Since h is odd and $\mathcal{O}_K^* = \langle \pm 1 \rangle$, we may choose the sign of $h_{\mathfrak{a}}$ so that $\alpha \equiv \mathbb{N}_{H/K} h_{\mathfrak{a}} \pmod{K^{*2}}$. Then $K' = K(\sqrt{\mathbb{N}_{H/K} h_{\mathfrak{a}}})$ and $HK' = H(\sqrt{\mathbb{N}_{H/K} h_{\mathfrak{a}}})$.

By Lemma 4.1, $h_{\mathfrak{a}} \equiv h_{\mathfrak{a}}^{\sigma} \pmod{H^{*2}}$ so multiplying over the entire Galois group we find $h_{\mathfrak{a}}^h \equiv \mathbb{N}_{H/K} h_{\mathfrak{a}} \pmod{H^{*2}}$. Since h is odd, $h_{\mathfrak{a}} \equiv h_{\mathfrak{a}}^h \equiv \mathbb{N}_{H/K} h_{\mathfrak{a}} \pmod{H^{*2}}$ and $HK' = H(\sqrt{h_{\mathfrak{a}}})$ as claimed.

Now let A be an elliptic curve over H such that χ_A is $\text{Gal}(H/K)$ equivariant. By [2, 12.3.1] we may write $A = A(p)^{\psi}$ with

$$\psi \in \text{Hom}(\text{Gal}(\bar{H}/H), \pm 1)^{\text{Gal}(H/K)} \simeq \text{Hom}(\text{Gal}(\bar{K}/K), \pm 1).$$

Let $\mathfrak{a}$ be the conductor of ψ and write the associated quadratic extension $H' = H(\sqrt{h_{\mathfrak{a}}})$ as permitted by Lemma 4.2. For simplicity, assume that $\mathfrak{a}$ is prime to p . Let ρ be a generator of $\text{Gal}(H'/H)$; we then have the identification

$$\underline{\omega}_{A/H} = \{\omega \in \underline{\omega}_{A(p)/H'} : \omega^{\rho} = -\omega\}.$$

Hence the differential $\omega_A = (1/\sqrt{h_{\mathfrak{a}}}) \cdot \omega(p)$ descends to A over H .

PROPOSITION 4.3: *Either ω_A or $2\omega_A$ is a global minimal differential on A/H .*

PROOF: We clearly have $\Delta_{\omega_A} = -p^3 h_{\mathfrak{a}}^6$ so $(\Delta_{\omega_A}) = (-p^3) \mathfrak{a}^6$. This is equal to $\mathcal{D}_A$ except in the case when $\left(\frac{2}{p}\right) = -1$ and $8 \mid \mathfrak{a}$ [2, 14.1.1]. In that case it is equal to $(2^{12})\mathcal{D}_A$.

COROLLARY 4.4: *If K has prime discriminant and the Hecke character χ_A of A is $\text{Gal}(H/K)$ equivariant, then $\delta_A \sim c_A \sim 1$ in $\text{Pic}(R)$.*

Indeed, the minimal differential given in Proposition 4.3 is determined up to sign.

§5. The sign of $h_{\mathfrak{a}}$

When the ideal $\mathfrak{a}$ of K is prime to (p) , we may normalize the sign of $h_{\mathfrak{a}}$ by insisting that $\mathbb{N}_{H/K} h_{\mathfrak{a}}$ is a square $\pmod{\sqrt{-p}}$. Then the following identities hold in H^* :

$$\begin{aligned} h_{\mathfrak{a}\mathfrak{b}} &= h_{\mathfrak{a}}^{\sigma_{\mathfrak{b}}^{-1}} h_{\mathfrak{b}} \\ (5.1) \quad h_{\mathfrak{a}^{\tau}} &= h_{\mathfrak{a}}^{\tau} \\ h_{(\alpha)} &= \alpha \quad \text{if } \alpha \equiv 1 \pmod{\sqrt{-p}}. \end{aligned}$$

Hence there is a unique continuous 1-cocycle

$$\phi : I_K \rightarrow H^*$$

which is the identity on principal idèles and satisfies $\phi(a) = \prod_{v \nmid p, \infty} h_{a_v}^{v(a)}$ for all idèles which are trivial at ∞ and congruent to 1 (mod $\sqrt{-p}$). (The group I_K acts on H^* via its quotient $I_K/K^* \cdot (\mathbb{C}^* \times \prod_v \mathcal{O}_v^*) = \text{Gal}(H/K)$, and the cocycle ϕ is τ -equivariant.)

Recall the elements t_a in $T^*/\pm 1$ defined in [2, 15.2.5]. Again, when a is prime to (p) we may normalize the sign of t_a by insisting that t_a^h is a square (mod $\sqrt{-p}$). We then have the identities in T^* :

$$(5.2) \quad \begin{aligned} t_{ab} &= t_a t_b \\ t_{a^\tau} &= t_a^\tau \\ t_{(\alpha)} &= \alpha \quad \text{if } \alpha \equiv 1 \pmod{\sqrt{-p}}. \end{aligned}$$

Since $(t_a) = a$ we find:

PROPOSITION 5.3: *The elements $u_a = t_a/h_a^{\sigma_a}$ are units in the field HT which satisfy the identities*

$$\begin{aligned} u_{ab} &= u_a \cdot u_b^{\sigma_a} \\ u_{a^\tau} &= u_a^\tau \\ u_{(\alpha)} &= 1. \end{aligned}$$

Since u_a depends only on the class of a in $\text{Pic}(\mathcal{O})$ it is convenient to write u_{σ_a} for the unit u_a . By Proposition 5.3 the assignment

$$\begin{aligned} \sigma &\rightarrow u_\sigma \\ \tau &\rightarrow 1 \end{aligned}$$

gives a 1-cocycle f on $\text{Gal}(HT/T^+) \simeq \text{Gal}(H/\mathbb{Q})$ with values in the units U of $(HT)^*$.

QUESTION 5.4: *Is $f \sim 1$ in $H^1(\text{Gal}(HT/T^+), U)$?*

As a stronger question, one can ask if $\epsilon = \sum_\sigma u_\sigma$ is a *unit* of HT .

REFERENCES

- [1] M. DEURING: Die Klassenkörper der Komplexen Multiplication. *Ency. der Math. Wiss. Band I, 2. Teil, Heft 10, Teil II* (1958).
- [2] B. GROSS: Arithmetic on elliptic curves with complex multiplication. Springer Lecture Notes 776 (1980).
- [3] S. LANG: *Elliptic functions*. Reading: Addison-Wesley (1973).
- [4] A. NÉRON: Modèles minimaux des variétés abéliennes sur les corps locaux et globaux. *IHES Publ. Math. No. 21 (1964)* 361–483.
- [5] J.T. TATE: Algorithm for determining the type of singular fiber in an elliptic pencil. *Springer Lecture Notes* 476 (1975) 33–52.

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