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CHERN CLASSES OF GROUP REPRESENTATIONS OVER A NUMBER FIELD

Beno Eckmann and Guido Mislin

Introduction

We consider representations $\rho: G \rightarrow GL_m(K)$ of finite groups G over an algebraic number field $K \subset \mathbb{C}$ and their Chern classes $c_i(\rho) \in H^{2i}(G; \mathbb{Z})$, $i = 1, 2, \dots, m$; i.e., the Chern classes of the associated (flat) complex vector bundles over the classifying space (Eilenberg-MacLane complex) of G . Our main objective is to determine the *best universal bound* $E_K(i)$ for the order of $c_i(\rho)$ when ρ ranges over all K -representations of all finite groups G . We then apply the result to $GL(\mathcal{O}(K))$, the stable linear group over the integers of K with its inclusion representation in $GL(K)$, and obtain information on the order of the corresponding Chern classes $c_i \in H^{2i}(GL(\mathcal{O}(K)); \mathbb{Z})$. Finally some number-theoretic properties of the $E_K(i)$ are discussed.

1: The existence of the universal bound $E_K(i)$ above is known from the work of Grothendieck [7]. We will denote by $\bar{E}_K(i)$ the upper bound for $E_K(i)$ resulting from his approach. However, we determine $\bar{E}_K(i)$ without appealing to Grothendieck's method: We describe, by means of elementary character theory of finite groups, the relationship between Galois operations and Adams' ψ -operations on representations; this yields directly the formula $c_i(\rho^\sigma) = k^i c_i(\rho)$ for any automorphism σ of \mathbb{C} which acts on the n -th roots of unity by the k -th power map ($n = \text{exponent of } G$). From this the prime factorization of $\bar{E}_K(i)$ is obtained in terms of two invariants of K (cf. [4] where these were introduced in connection with the order of the Euler class). In particular, for $K = \mathbb{Q}$ and i even $= 2\ell$, $\bar{E}_{\mathbb{Q}}(2\ell)$ is the denominator of $B_{2\ell}/4\ell$ where $B_{2\ell}$ is the 2ℓ -th Bernoulli number

($B_2 = \frac{1}{6}$, $B_4 = \frac{1}{30}$, $B_6 = \frac{1}{42}$, \dots); the actual value of $E_{\mathbb{Q}}(2\ell)$ is one half of this integer, see below.

2: A lower bound $\underline{E}_K(i)$ for $E_K(i)$ is easily obtained from the order of Chern classes of K -representations of cyclic groups (cf. also Soulé [12]); namely $\underline{E}_K(i) = \ell cm\{m | \phi_K(m) \text{ divides } i\}$ where $\phi_K(m)$ is the degree of the m -th cyclotomic extension $K(\zeta_m)$ of K . It turns out that $\bar{E}_K(i)$ and $\underline{E}_K(i)$ differ at most by a factor 2. They always agree if i is odd. If i is even $= 2\ell$, one has $\bar{E}_K(2\ell) = 2\underline{E}_K(2\ell)$ except for those K for which $\text{Gal}(K(\zeta_{2^\beta})/K)$ is cyclic for all β ; such number fields will be called *exceptional*. Thus

$$(*) \quad E_K(i) = \bar{E}_K(i) = \underline{E}_K(i) = \ell cm\{m | \phi_K(m) \text{ divides } i\} \\ \text{if } i \text{ is odd, or if } i \text{ is even and } K \text{ exceptional.}$$

3: There remains the task to find out, for i even $= 2\ell$ and K not exceptional, which of the two possibilities is $E_K(2\ell)$. The result depends on whether K is formally real or not (formally real fields are not exceptional, but the converse is not true):

$$(**) \quad E_K(2\ell) = \underline{E}_K(2\ell) \text{ if } K \text{ is formally real, and} \\ E_K(2\ell) = \bar{E}_K(2\ell) \text{ if } K \text{ is not formally real.}$$

For $K = \mathbb{Q}$ this yields $E_{\mathbb{Q}}(2\ell) = \text{denominator of } B_{2\ell}/2\ell$ (as shown already in [3]).

The precise result (**) depends on a careful analysis of representations over the respective fields K of 2-groups, in particular of the generalized quaternion group of order 2^β , $\beta \geq 3$. We summarize:

THEOREM: *Let $K \subset \mathbb{C}$ be a number field. The best universal bound for the order of the Chern classes $c_i(\rho)$, when ρ ranges over all K -representations of all finite groups is*

$$E_K(i) = \bar{E}_K(i) \text{ if } i \text{ is odd or if } K \text{ is not formally real,} \\ = \underline{E}_K(i) \text{ if } i \text{ is even and } K \text{ formally real,}$$

where $\underline{E}_K(i) = \ell cm\{m | \phi_K(m) \text{ divides } i\}$; and $\bar{E}_K(i) = \underline{E}_K(i)$ if i is odd, or if i is even and K exceptional, $\bar{E}_K(i) = 2\underline{E}_K(i)$ otherwise.

4: These results can, of course, be applied to the Chern class of the inclusion of the arithmetic group $GL_m(\mathcal{O}(K))$ into $GL_m(K)$. We

restrict ourselves to discussing the precise order of the corresponding *stable* class $c_i \in H^{2i}(GL(\mathcal{O}(K)); \mathbb{Z})$; namely, if i is odd or if K is not formally real, this order is $\bar{E}_K(i) = E_K(i)$. In the remaining cases it is either $E_K(i)$ or $\bar{E}_K(i) = 2E_K(i)$, but we do not know which is correct.

5: We further investigate to what extent $E_K(i)$ differs from $E_Q(i)$ [=2 if i is odd, =denominator of B_i/i if i is even]. Clearly $E_K(i)$ is a multiple of $E_Q(i)$; their p -primary parts agree in several cases, e.g., if p does not divide the discriminant of K . For quadratic number fields K different from $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{2})$, and $\mathbb{Q}(\sqrt{-1})^{q-1/2}q$, q an odd prime, we get

$$\begin{aligned} E_K(i) &= E_Q(i) \text{ if } i \text{ is odd or } K \text{ real,} \\ &= 2E_Q(i) \text{ if } i \text{ is even and } K \text{ imaginary.} \end{aligned}$$

6: The plan of the paper is as follows. In Section 1 we outline the relation between Galois operations and ψ^k -operations in the representation ring of a finite group G . Section 2 deals with the Galois group of cyclotomic extensions of K (exceptional and not exceptional fields); in particular the exponent of $\text{Gal}(K(\zeta_p)/K)$ is compared with its order – they are equal if p is an odd prime, and may differ by a factor 2 if $p = 2$. The upper bound $\bar{E}_K(i)$ is discussed in Section 3, first for p -groups and then (by a standard p -Sylow subgroup argument) for arbitrary finite groups. Then the lower bound $\underline{E}_K(i)$ is introduced and compared with $\bar{E}_K(i)$. In Section 4 the best bound $E_K(i)$ is determined (see Theorem above). Application to arithmetic groups is dealt with in Section 5, and number-theoretic results concerning the $E_K(i)$ are established in Section 6.

1. Operations in the representation ring

1.1: Let G be a finite group, and $R(G)$ its complex representation ring; i.e., its character ring. In this section we describe the relations between operations on $R(G)$ arising from automorphisms of \mathbb{C} , on the one hand, and the ψ -operations on the other hand (cf. also [5], [6], [9], and [10]).

Let $\rho: G \rightarrow GL_m(\mathbb{C})$ be a representation, χ_ρ its character. Any $\sigma \in \text{Aut } \mathbb{C}$, the automorphism group of the field \mathbb{C} , induces an automorphism σ_* of $GL_m(\mathbb{C})$ by applying σ to the matrix entries. The composition $\sigma_*\rho$ is a representation ρ^σ with character $\chi_{\rho^\sigma} = \sigma\chi_\rho$. Applying σ to the values of a (virtual or actual) character of G thus

defines a ring automorphism of $R(G)$, also called “Galois action” on $R(G)$.

For any subfield $K \subset \mathbb{C}$ we consider the subgroup $\text{Gal}(\mathbb{C}/K)$ of $\text{Aut } \mathbb{C}$ consisting of automorphisms over K . Let n be a multiple of the exponent of G , and $K(\zeta_n)$ the n -th cyclotomic extension of K (the field obtained by adjoining to K a primitive n -th root of unity ζ_n); it is a finite Galois extension with Abelian Galois group $\text{Gal}(K(\zeta_n)/K)$. Throughout Sections 1–3 we will also write K_n for $K(\zeta_n)$, to simplify notations. Since all $\chi \in R(G)$ have values in K_n the action of $\text{Gal}(\mathbb{C}/K)$ on $R(G)$ factors through the restriction map $\text{Gal}(\mathbb{C}/K) \rightarrow \text{Gal}(K_n/K)$, which is surjective by virtue of the “isomorphism extension theorem” ([1]). Thus the (virtual) characters $\chi \in R(G)$ which take their values in K are precisely those which are fixed under the action of $\text{Gal}(\mathbb{C}/K)$.

1.2: We recall that the ψ -operations are defined by means of exterior powers and “Newton polynomials” (cf. [5]): Define $N_k(x_1, \dots, x_k) \in \mathbb{Z}[x_1, \dots, x_k]$ by $N_1(x_1) = x_1$ and, for $k > 1$, $N_k(x_1, \dots, x_k) = x_1 N_{k-1} - x_2 N_{k-2} + \dots + (-1)^{k-2} x_{k-1} N_1 + (-1)^{k-1} k x_k$. Writing $s_j(y_1, \dots, y_n)$ for the j -th elementary symmetric function in the indeterminates y_1, \dots, y_n ($s_j(y_1, \dots, y_n) = 0$ for $j > n$), one has

$$N_k(s_1, \dots, s_k) = y_1^k + \dots + y_n^k.$$

Given a representation ρ of G of degree m , let $\Lambda^j \chi_\rho$ be the character of the j -th exterior power of ρ . Then one defines, for $k \geq 1$,

$$\psi^k \chi_\rho = N_k(\Lambda^1 \chi_\rho, \dots, \Lambda^k \chi_\rho) \in R(G).$$

A simple computation now yields, for any $g \in G$, and $j \geq 1$,

$$\Lambda^j \chi_\rho(g) = s_j(\omega_1, \dots, \omega_m)$$

where $\omega_1, \dots, \omega_m \in \mathbb{C}$ are the eigenvalues of $\rho(g)$; and thus

$$\psi^k \chi_\rho(g) = \omega_1^k + \dots + \omega_m^k = \chi_\rho(g^k), \quad k \geq 1.$$

This operation $\chi_\rho \mapsto \psi^k \chi_\rho$ from actual to virtual characters is obviously additive and multiplicative and can thus be extended to all of $R(G)$ to yield ring operations $\psi^k: R(G) \rightarrow R(G)$ for all $k \geq 1$.

The relation of these ψ^k with the Galois action on $R(G)$ is now easily obtained (cf. [9] for a different proof):

PROPOSITION 1.1: *Let G be a finite group, and n a multiple of the exponent of G . For $\sigma \in \text{Aut } \mathbb{C}$ choose a positive integer k such that σ*

acts on any n -th root of unity ω by $\sigma(\omega) = \omega^k$. Then

$$\sigma\chi = \psi^k\chi$$

for all $\chi \in R(G)$.

PROOF: Both σ and ψ^k being additive it suffices to consider an actual character χ_ρ of a representation $\rho: G \rightarrow GL_m(\mathbb{C})$. Let $\omega_1, \dots, \omega_m \in \mathbb{C}$ be the eigenvalues of $\rho(g)$, for $g \in G$. One has $\sigma\chi_\rho(g) = \sigma(\omega_1 + \dots + \omega_m) = \omega_1^k + \dots + \omega_m^k = \psi^k\chi_\rho(g)$ as noted above.

1.3: The positive integer k associated in Proposition 1.1 with n and $\sigma \in \text{Aut } \mathbb{C}$ is, of course, prime to n and determined by σ modulo n ; and each k prime to n occurs for some $\sigma \in \text{Aut } \mathbb{C}$.

Let K be a subfield of \mathbb{C} . We look for those k which belong to automorphisms $\sigma \in \text{Gal}(\mathbb{C}/K) \subset \text{Aut } \mathbb{C}$, for a fixed n . We write $\mu_K(n)$ for the homomorphism $\text{Gal}(\mathbb{C}/K) \rightarrow (\mathbb{Z}/n\mathbb{Z})^*$ which assigns to σ the class modulo n of integers k with $\sigma(\omega) = \omega^k$ on the n -th roots of unity ω . Then $\mu_K(n)$ factors as $\mu_K(n) = \lambda_K(n) \cdot \text{res}$, where res is the surjective restriction map $\text{Gal}(\mathbb{C}/K) \rightarrow \text{Gal}(K_n/K)$ and $\lambda_K(n)$ is an isomorphism of $\text{Gal}(K_n/K)$ onto a subgroup $\text{im } \lambda_K(n)$ of $(\mathbb{Z}/n\mathbb{Z})^*$. If the residue class of an integer k is in $\text{im } \lambda_K(n)$ we will say, for simplicity, that “ k is in the image of $\lambda_K(n)$ ” and write $k \in \text{im } \lambda_K(n)$.

Since a character $\chi \in R(G)$ takes its values in K if and only if it is fixed under $\text{Gal}(\mathbb{C}/K)$, Proposition 1.1. now yields the following result stated for actual characters.

THEOREM 1.2: Let G be a finite group, n a multiple of its exponent, and $K \subset \mathbb{C}$ a subfield. The character χ_ρ of $\rho: G \rightarrow GL_m(\mathbb{C})$ takes its values in K if and only if

$$\psi^k\chi_\rho = \chi_\rho$$

for all positive $k \in \text{im } \lambda_K(n)$.

We may take, in particular, for K the field $\mathbb{Q}(\chi_\rho)$ obtained by adjoining to \mathbb{Q} the values of a given character χ_ρ of G . We then get

COROLLARY 1.3: Let G be a finite group of exponent n , and χ_ρ a character of G . Then

$$\psi^k\chi_\rho = \chi_\rho$$

for all positive $k \in \text{im } \lambda_{\mathbb{Q}(\chi_\rho)}(n)$.

REMARK: If χ_ρ takes its values in \mathbb{Q} then $\psi^k \chi_\rho = \chi_\rho$ for all positive k prime to the exponent of G ; in other words, one has the well-known fact that $\chi_\rho(g^k) = \chi_\rho(g)$ for all $g \in G$ and all k prime to the exponent of G .

2. The Galois group of the cyclotomic extensions

2.1: In order to apply Theorem 1.2 to Chern classes of representations with character values in the subfield $K \subset \mathbb{C}$ we need more precise information on $\text{im } \lambda_K(n)$; i.e., on the Galois group $\text{Gal}(K_n/K)$. It will be sufficient to provide such information for prime powers $n = p^\beta$ only.

It is convenient to use the following notation, already introduced in [4]:

$$\gamma_K(2) = \sup\{\alpha \mid K_4 = K_{2^{\alpha+1}}\}$$

and, for an odd prime p ,

$$\gamma_K(p) = \sup\{\alpha \mid K_p = K_{p^\alpha}\}.$$

Moreover we write, for any integer $n > 0$, $\phi_K(n)$ for the degree of K_n over K ($\phi_\mathbb{Q}(n) = \phi(n)$, the Euler function).

For an odd prime p , $\text{Gal}(K_{p^\beta}/K)$ being isomorphic to a subgroup of the cyclic group $(\mathbb{Z}/p^\beta\mathbb{Z})^*$ is itself cyclic. If $\beta \leq \gamma_K(p)$, then $K_{p^\beta} = K_p$ and $\text{Gal}(K_{p^\beta}/K)$ is cyclic of order $\phi_K(p)$. If $\beta \geq \gamma_K(p) = \gamma$, the order of $\text{Gal}(K_{p^\beta}/K)$ is $\phi_K(p)p^{\beta-\gamma}$. We summarize, using the symbol C_m for the cyclic group of order m .

THEOREM 2.1: *Let $K \subset \mathbb{C}$ be a subfield, p an odd prime, and write $\gamma = \gamma_K(p)$, $\phi = \phi_K(p)$. Then*

- (a) *for $\beta \leq \gamma$, $\text{Gal}(K_{p^\beta}/K) \cong C_\phi$,*
- (b) *for $\beta \geq \gamma$, $\text{Gal}(K_{p^\beta}/K) \cong C_{\phi \cdot p^{\beta-\gamma}}$.*

2.2: For $p = 2$ the situation is more complicated due to the fact that $\text{Gal}(\mathbb{Q}_{2^\beta}/\mathbb{Q}) = (\mathbb{Z}/2^\beta\mathbb{Z})^*$ is not cyclic for $\beta > 2$.

The restriction map $\text{Gal}(K_{2^\beta}/K) \rightarrow \text{Gal}(\mathbb{Q}_{2^\beta}/\mathbb{Q})$ is injective, the image being $\text{Gal}(\mathbb{Q}_{2^\beta}/K \cap \mathbb{Q}_{2^\beta})$. We thus have to look closer at the subgroups of $\text{Gal}(\mathbb{Q}_{2^\beta}/\mathbb{Q}) \cong C_2 \times C_{2^{\beta-2}}$ for $\beta \geq 3$, and at the corresponding fixed fields $L \subset \mathbb{Q}_{2^\beta}$.

LEMMA 2.2: *Let L be a subfield of \mathbb{Q}_{2^β} . Then $\text{Gal}(\mathbb{Q}_{2^\beta}/L)$ is cyclic if and only if L is not contained in the maximal real subfield $\mathbb{Q}(\zeta^2 + \zeta^{-2}) = \mathbb{Q}(\cos \pi/2^{\beta-2})$ of $\mathbb{Q}_{2^{\beta-1}}$ where $\zeta = \zeta_{2^\beta}$.*

PROOF: We consider $\beta \geq 3$. In the diagram of field extensions

$$\begin{array}{ccccccc} \mathbb{Q}_{2^\beta} & \supset & \mathbb{Q}_{2^{\beta-1}} & \supset & \mathbb{Q}_{2^{\beta-2}} & \supset & \cdots \supset \mathbb{Q} \\ \parallel & & \cup & & \cup & & \parallel \\ \mathbb{Q}_{2^\beta} & \supset & \mathbb{Q}(\zeta^2 + \zeta^{-2}) & \supset & \mathbb{Q}(\zeta^4 + \zeta^{-4}) & \supset & \cdots \supset \mathbb{Q} \end{array}$$

the group $\text{Gal}(\mathbb{Q}_{2^\beta}/\mathbb{Q}(\zeta^2 + \zeta^{-2}))$ is $\cong C_2 \times C_2$, generated by $\zeta \mapsto -\zeta$ and $\zeta \mapsto \zeta^{-1}$; it is a subgroup of all further $\text{Gal}(\mathbb{Q}_{2^\beta}/\mathbb{Q}(\zeta^{2^j} + \zeta^{-2^j}))$ which thus are non-cyclic. Conversely, any non-cyclic subgroup of $\text{Gal}(\mathbb{Q}_{2^\beta}/\mathbb{Q})$ has one of the $\mathbb{Q}(\zeta^{2^j} + \zeta^{-2^j})$ as fixed field, contained in $\mathbb{Q}(\zeta^2 + \zeta^{-2}) = \mathbb{Q}(\cos \pi/2^{\beta-2})$.

COROLLARY 2.3: *$\text{Gal}(\mathbb{Q}_{2^\beta}/L)$ is cyclic if and only if $\text{Gal}(\mathbb{Q}_{2^{\beta+1}}/L)$ is cyclic or $\cong C_2 \times C_2$.*

PROOF: The if-part is obvious, $\text{Gal}(\mathbb{Q}_{2^\beta}/L)$ being a factor group of $\text{Gal}(\mathbb{Q}_{2^{\beta+1}}/L)$.

For the converse, we may take $\beta \geq 3$. Assume that $\text{Gal}(\mathbb{Q}_{2^\beta}/L)$ is cyclic, hence L is none of the $\mathbb{Q}(\zeta^{2^j} + \zeta^{-2^j})$; and that $\text{Gal}(\mathbb{Q}_{2^{\beta+1}}/L)$ is *not* cyclic and thus L is contained in $\mathbb{Q}(\zeta + \zeta^{-1})$ – note that $\zeta = \zeta_{2^\beta} = \zeta_{2^{\beta+1}}^2$. Then L must be $= \mathbb{Q}(\zeta + \zeta^{-1})$, whence $\text{Gal}(\mathbb{Q}_{2^{\beta+1}}/L) \cong C_2 \times C_2$.

2.3: Let now $K \subset \mathbb{C}$ be a subfield with $\gamma = \gamma_K(2) < \infty$. By definition $K_{2^{\gamma+1}} = K_4$ and $K_{2^{\gamma+2}} = K_8$. Clearly $|\text{Gal}(K_{\gamma+1}/K)| = 1$ if $\sqrt{-1} \in K$, $= 2$ if $\sqrt{-1} \notin K$; and $|\text{Gal}(K_{\gamma+2}/K)| = 2$ if $\sqrt{-1} \in K$, $= 4$ if $\sqrt{-1} \notin K$. Moreover $|\text{Gal}(K_{2^\beta}/K)|$ for $\beta \geq \gamma + 2$ is $= 2^{\beta-\gamma-1}$ if $\sqrt{-1} \in K$, $= 2^{\beta-\gamma}$ if $\sqrt{-1} \notin K$. If $\sqrt{-1} \in K$ these groups are all cyclic; if $\sqrt{-1} \notin K$ they may be cyclic or not, as follows.

If $\beta \geq \gamma + 2$, $K_{2^{\beta+1}}$ has degree 2 over K_{2^β} . We assume that $\text{Gal}(K_{2^\beta}/K) \cong \text{Gal}(\mathbb{Q}_{2^\beta}/K \cap \mathbb{Q}_{2^\beta})$ is cyclic and consider the inclusion of index 2

$$\text{Gal}(\mathbb{Q}_{2^{\beta+1}}/K \cap \mathbb{Q}_{2^{\beta+1}}) \subset \text{Gal}(\mathbb{Q}_{2^{\beta+1}}/K \cap \mathbb{Q}_{2^\beta}),$$

the latter group being cyclic or $\cong C_2 \times C_2$. Hence the first group, isomorphic to $\text{Gal}(K_{2^{\beta+1}}/K)$, is cyclic. Thus $\text{Gal}(K_{2^{\beta+1}}/K)$ is cyclic if

$\text{Gal}(K_{2^\beta}/K)$ is and, of course, vice-versa. It follows that $\text{Gal}(K_{2^\beta}/K)$ is cyclic for all $\beta \geq 1$ if and only if $\text{Gal}(K_{2^{\gamma+2}}/K) = \text{Gal}(K_8/K)$ is C_4 , non-cyclic for all $\beta \geq \gamma + 2$ if and only if $\text{Gal}(K_{2^{\gamma+2}}/K) = \text{Gal}(K_8/K)$ is $C_2 \times C_2$. By Lemma 2.2 the alternative depends on $K \cap \mathbb{Q}_{2^{\gamma+2}}$ not being contained, or being contained in $\mathbb{Q}(\cos \pi/2^\gamma)$. We summarize:

THEOREM 2.4: *Let $K \subset \mathbb{C}$ be a subfield with $\gamma = \gamma_K(2) < \infty$, and $\beta \geq \gamma + 2$. The Galois group $\text{Gal}(K_{2^\beta}/K)$ is*

- (a) $\cong C_{2^{\beta-\gamma-1}}$ if $\sqrt{-1} \in K$
- (b) $\cong C_{2^{\beta-\gamma}}$ if $\sqrt{-1} \notin K$ and $K \cap \mathbb{Q}_{2^{\gamma+2}} \not\subset \mathbb{Q}(\cos \pi/2^\gamma)$
- (c) $\cong C_2 \times C_{2^{\beta-\gamma-1}}$ if $\sqrt{-1} \notin K$ and $K \cap \mathbb{Q}_{2^{\gamma+2}} \subset \mathbb{Q}(\cos \pi/2^\gamma)$

Thus the exponent of $\text{Gal}(K_{2^\beta}/K)$ is $2^{\beta-\gamma-1}$ in case (a) and (c), $2^{\beta-\gamma}$ in case (b).

We introduce the following terminology.

DEFINITION 2.5: A subfield $K \subset \mathbb{C}$ is *exceptional*^{*)} if $\gamma = \gamma_K(2) < \infty$ and $\text{Gal}(K_{2^\beta}/K)$ is cyclic for all $\beta \geq 1$.

Clearly any subfield $L \subset \mathbb{C}$ with $\gamma < \infty$ containing $\sqrt{-1}$ is exceptional. If $\sqrt{-1} \notin K$ and $\gamma = \gamma_K(2) < \infty$ then K is exceptional if and only if $\text{Gal}(K_{2^{\gamma+2}}/K)$ is cyclic (i.e., $K \cap \mathbb{Q}_{2^{\gamma+2}} \not\subset \mathbb{Q}(\cos \pi/2^\gamma)$).

COROLLARY 2.6: *If $K \subset \mathbb{C}$ is a formally real field (i.e., -1 is not a sum of squares in K) then K is not exceptional.*

PROOF: We may assume $\gamma = \gamma_K(2) < \infty$. Let $j: K \rightarrow \mathbb{R}$ be an embedding. Since $\sqrt{-1} \notin j(K)$, $\text{Gal}(\mathbb{Q}_{2^{\gamma+2}}/j(K) \cap \mathbb{Q}_{2^{\gamma+2}})$ is of order 4, and hence $j(K) \cap \mathbb{Q}_{2^{\gamma+2}}$ is a real subfield of codimension 4 of $\mathbb{Q}_{2^{\gamma+2}}$; the only such subfield is $\mathbb{Q}(\cos \pi/2^\gamma)$. Thus $\text{Gal}(K_{2^{\gamma+2}}/K) \cong \text{Gal}(\mathbb{Q}_{2^{\gamma+2}}/j(K) \cap \mathbb{Q}_{2^{\gamma+2}})$ is $C_2 \times C_2$.

COROLLARY 2.7: *Among the quadratic number fields only $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-2})$ are exceptional.*

PROOF: Let K be a quadratic number field and $\beta > 2$. Then $K \cap \mathbb{Q}_{2^\beta}$ is either \mathbb{Q} or one of the quadratic subfields of \mathbb{Q}_{2^β} , i.e., $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-2})$, in which case K is equal to one of these three

^{*)}Our terminology differs from the one used in [12].

fields. If $K \cap \mathbb{Q}_{2^\beta}$ is \mathbb{Q} or $\mathbb{Q}(\sqrt{2})$ then K is not exceptional (note that $\mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\cos \pi/2^{\beta-2})$ for $\beta > 3$), while the imaginary cases $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-2})$ are exceptional.

2.4: From the above results on $\text{Gal}(K_n/K)$ for $n = p^\alpha$ the values of the integers $k \in \text{im } \lambda_K(p^\beta)$ can easily be described.

Case $p = \text{odd prime}$: We write $\gamma = \gamma_K(p)$, $\phi = \phi_K(p)$. If $\beta \leq \gamma$, then $\text{Gal}(K_{p^\beta}/K)$ is cyclic of order ϕ and so is its image under $\lambda_K(p^\beta)$ in the cyclic group $(\mathbb{Z}/p^\beta\mathbb{Z})^*$; thus $\text{im } \lambda_K(p^\beta)$ consists of those elements of $(\mathbb{Z}/p^\beta\mathbb{Z})^*$ whose order divides ϕ .

If $\beta \geq \gamma$, $\text{Gal}(K_{p^\beta}/K)$ is cyclic of order $\phi \cdot p^{\beta-\gamma}$, and $\text{im } \lambda_K(p^\beta)$ consists of those elements of $(\mathbb{Z}/p^\beta\mathbb{Z})^*$ whose order divides $\phi p^{\beta-\gamma}$. In other words, $k \in \text{im } \lambda_K(p^\beta)$ if and only if $k^{\phi p^{\beta-\gamma}} \equiv 1 \pmod{p^\beta}$, i.e., $k^\phi \equiv 1 \pmod{p^\gamma}$. We thus have.

THEOREM 2.1': *Let $K \subset \mathbb{C}$ be a subfield, p an odd prime, and write $\gamma = \gamma_K(p)$, $\phi = \phi_K(p)$. The integers $k \in \text{im } \lambda_K(p^\beta)$ are characterized*

(a) *for $\beta \leq \gamma$, by $k^\phi \equiv 1 \pmod{p^\beta}$,*

(b) *for $\beta \geq \gamma$, by $k^\phi \equiv 1 \pmod{p^\gamma}$.*

Case $p = 2$: If $2 \leq \beta \leq \gamma + 1$ then $|\text{Gal}(K_{2^\beta}/K)| = 1$ or 2 according to whether $\sqrt{-1} \in K$ or $\sqrt{-1} \notin K$. In the first case $\text{im } \lambda_K(2^\beta)$ is trivial in $(\mathbb{Z}/2^\beta\mathbb{Z})^*$, i.e. $k \in \text{im } \lambda_K(2^\beta)$ if and only if $k \equiv 1 \pmod{2^\beta}$. In the second case $\text{im } \lambda_K(2^\beta)$ consists of the classes $\pm 1 \pmod{2^\beta}$, i.e., $k \in \text{im } \lambda_K(2^\beta)$ if and only if $k \equiv \pm 1 \pmod{2^\beta}$.

If $\beta \geq \gamma + 2$ and $\sqrt{-1} \in K$ then $\text{im } \lambda_K(2^\beta)$ is a subgroup of order $2^{\beta-\gamma-1}$ of the cyclic factor of $(\mathbb{Z}/2^\beta\mathbb{Z})^*$ generated by $5 \pmod{2^\beta}$. Hence $k \in \text{im } \lambda_K(2^\beta)$ if and only if $k^{2^{\beta-\gamma-1}} \equiv 1 \pmod{2^\beta}$, i.e., $k \equiv 1 \pmod{2^{\gamma+1}}$. If $\beta \geq \gamma + 2$ and $\sqrt{-1} \notin K$, we have to deal separately with the cases K exceptional or not:

If K is exceptional, $\sqrt{-1} \notin K$, then $\text{im } \lambda_K(2^\beta)$ is a cyclic subgroup of order $2^{\beta-\gamma}$ of $(\mathbb{Z}/2^\beta\mathbb{Z})^*$ and must contain classes $\pmod{2^\beta}$ which are $\equiv -1 \pmod{4}$. Hence $k \in \text{im } \lambda_K(2^\beta)$ if and only if $k^{2^{\beta-\gamma-1}} \equiv 1$ or $-1 + 2^{\beta-1}$ (note that $\beta \geq \gamma + 2 \geq 3$) $\pmod{2^\beta}$; i.e., $k \equiv 1$ or $-1 + 2^\gamma \pmod{2^{\gamma+1}}$.

If K is not exceptional, then $\text{im } \lambda_K(2^\beta)$ is a subgroup $\cong C_2 \times C_{2^{\beta-\gamma-1}}$ of $(\mathbb{Z}/2^\beta\mathbb{Z})^*$; hence $k \in \text{im } \lambda_K(2^\beta)$ if and only if $k^{2^{\beta-\gamma-1}} \equiv \pm 1 \pmod{2^\beta}$, i.e., $k \equiv \pm 1 \pmod{2^{\gamma+1}}$.

We thus have

THEOREM 2.4': *Let $K \subset \mathbb{C}$ be a subfield and write $\gamma = \gamma_K(2)$. The integers $k \in \text{im } \lambda_K(2^\beta)$ are characterized*

(a) *for $\beta \leq \gamma + 1$*

- (a₁) *by $k \equiv 1 \pmod{2^\beta}$ if $\sqrt{-1} \in K$,*
- (a₂) *by $k \equiv \pm 1 \pmod{2^\beta}$ if $\sqrt{-1} \notin K$.*

(b) *for $\beta \geq \gamma + 1$*

- (b₁) *by $k \equiv \pm 1 \pmod{2^{\gamma+1}}$ if K is not exceptional,*
- (b₂) *by $k \equiv 1 \pmod{2^{\gamma+1}}$ if K is exceptional and $\sqrt{-1} \in K$,*
- (b₃) *by $k \equiv 1$ or $-1 + 2^\gamma \pmod{2^{\gamma+1}}$ if K is exceptional and $\sqrt{-1} \notin K$.*

3. Upper bounds for the order of Chern classes

3.1: Given a representation $\rho: G \rightarrow GL(C) = \bigcup_m GL_m(\mathbb{C})$ of the finite group G , the Chern classes

$$c_i(\rho) \in H^{2i}(G; \mathbb{Z})$$

are defined as the Chern classes of the associated flat complex vector bundle $\xi(\rho)$ over the classifying space $K(G, 1)$. We write $\psi^k \rho$ for the (equivalence class of) the virtual representation with character $\psi^k \chi_\rho \in R(G)$. We recall (Proposition 1.1) that if k is prime to the order of G then $\psi^k \rho$ can be represented by an actual representation whose character is Galois conjugate to χ_ρ . The canonical map from $R(G)$ to $K^0(K(G, 1))$, the group of stable complex vector bundles over $K(G, 1)$, is compatible with the ψ^k -operations on $R(G)$ on one side, and on the bundles on the other.

We now apply to Chern classes the results of the preceding sections, as follows.

THEOREM 3.1: *Let G be a finite group, n a multiple of the exponent of G .*

(a) *If σ is any automorphism of \mathbb{C} and k a positive integer such that σ acts on the n -th roots of unity by the k -power map, then, for all $i \geq 0$,*

$$c_i(\sigma_* \rho) = k^i c_i(\rho).$$

(b) If $K \subset \mathbb{C}$ is a subfield containing $\chi_\rho(G)$, then the Chern classes $c_i(\rho)$, $i \geq 0$, satisfy

$$(1 - k^i)c_i(\rho) = 0$$

for all positive integers $k \in \text{im } \lambda_K(n)$.

PROOF: (a) By Proposition 1.1 $\sigma\chi_\rho = \chi_{\sigma_*\rho} = \psi^k\chi_\rho$ and hence $\sigma_*\rho = \psi^k\rho$. By a well-known property of the Chern classes, applied to the associated bundles, one has

$$c_i(\psi^k\rho) = k^i c_i(\rho)$$

for all $i \geq 0$, whence the assertion. (b) If $\chi_\rho(G) \subset K$, then $\psi^k\chi_\rho = \chi_\rho$ for all positive integers $k \in \text{im } \lambda_K(n)$ (Theorem 1.2), hence $c_i(\rho) = k^i c_i(\rho)$.

From this one obtains bounds for the order of the Chern classes. We proceed by doing so first for p -groups.

COROLLARY 3.2: Let G be a finite p -group, and $\rho: G \rightarrow GL_m(\mathbb{C})$ a representation whose character χ_ρ takes its values in the subfield $K \subset \mathbb{C}$. Then the Chern classes $c_i(\rho)$ have the following properties.

Case p odd:

(a) If $m \not\equiv 0 \pmod{\phi_K(p)}$ then $c_i(\rho) = 0$.

(b) If $\gamma = \gamma_K(p) < \infty$ then $p^\gamma c_i(\rho) = 0$ for all $i \geq 0$.

Case $p = 2$:

(c) If $\sqrt{-1} \notin K$ then $2c_i(\rho) = 0$ for all odd i .

(d) If $\gamma = \gamma_K(2) < \infty$ then $2^{\gamma+1}c_i(\rho) = 0$ for all $i \geq 0$; and if K is exceptional with $\sqrt{-1} \notin K$, one even has $2^\gamma c_i(\rho) = 0$ for all $i \geq 0$.

PROOF: Let p^β be the order of G .

Case p odd: (a) We choose an integer $k > 0$ such that its class modulo p generates a subgroup of order $\phi = \phi_K(p)$ of $(\mathbb{Z}/p^\beta\mathbb{Z})^*$. By Theorem 2.1' we know that $k \in \text{im } \lambda_K(p^\beta)$ and thus, by Theorem 3.1, $(k^i - 1)c_i(\rho) = 0$ for all i . If $i \not\equiv 0 \pmod{\phi}$ then $k^i - 1$ is not divisible by p ; since $c_i(\rho)$ is a p -torsion element, it follows that $c_i(\rho) = 0$.

For proving (b) we may assume $i \equiv 0 \pmod{\phi}$ and $\beta \geq \gamma$. We choose k such that $k^\phi \equiv 1 \pmod{p^\gamma}$ but $\not\equiv 1 \pmod{p^{\gamma+1}}$. Again $k \in \text{im } \lambda_K(p^\beta)$ and thus $(k^i - 1)c_i(\rho) = 0$. Let p^μ be the highest power of p in i . Then $k^{\phi p^\mu} \equiv 1 \pmod{p^{\gamma+\mu}}$ but not $\pmod{p^{\gamma+\mu+1}}$; i.e., the highest power of p which divides $k^i - 1$ is precisely $p^{\gamma+\mu}$, and hence $p^\gamma c_i(\rho) = 0$.

Case $p = 2$: (c) If $\sqrt{-1} \notin K$ there exists a $k \in \text{im } \lambda_K(2^\beta)$ such that $k \equiv -1 \pmod{4}$. Then, for odd i , $k^i \equiv -1 \pmod{4}$. Since $k^i c_i(\rho) = c_i(\rho)$, $2c_i(\rho)$ is a multiple of $4c_i(\rho)$ and thus $= 0$, $c_i(\rho)$ being 2-torsion.

To prove (d) we may assume $\beta \geq \gamma + 1$. We choose $k \equiv 1 \pmod{2^{\gamma+1}}$, but $\not\equiv 1 \pmod{2^{\gamma+2}}$. Then $k \in \text{im } \lambda_K(2^\beta)$ and $(k^i - 1)c_i(\rho) = 0$. Since the highest power of 2 dividing $k^i - 1$ is the same as that dividing $2^{\gamma+1}$, we have $2^{\gamma+1}ic_i(\rho) = 0$. If K is exceptional but $\sqrt{-1} \notin K$, then there exists $k \equiv -1 + 2^\gamma \pmod{2^{\gamma+1}}$ with $k \in \text{im } \lambda_K(2^\beta)$. Note that necessarily $\gamma \geq 2$ since $\gamma = 1$ and $\sqrt{-1} \notin K$ imply $K \neq K(4) \neq K(8)$, thus $\text{Gal}(K_8/K) \cong \text{Gal}(\mathbb{Q}_8/\mathbb{Q})$ which is not cyclic, contradicting the assumption that K be exceptional. For $\gamma \geq 2$ and i even, the highest power of 2 dividing $k^i - 1 = (-1 + 2^\gamma + s \cdot 2^{\gamma+1})^i - 1$ is the same as that dividing $2^\gamma i$. Therefore, in that case, $2^\gamma ic_i(\rho) = 0$.

3.2: In order to deal with the case of arbitrary finite groups and number fields K we need the following notation.

DEFINITION 3.3: Let K be a number field. We put, for any integer $m > 0$,

$$\bar{E}_K(m) = \ell c m \{n \geq 1 \mid m \equiv 0 \pmod{\exp(\text{Gal } K_n/K)}\}.$$

For any natural number n we will write n_p for its p -primary factor. Note that the p -primary part of $\bar{E}_K(m)$ is

$$(*) \quad \bar{E}_K(m)_p = \max\{p^\alpha \mid m \equiv 0 \pmod{\exp \text{Gal}(K_{p^\alpha}/K)}\}.$$

Indeed, $\bar{E}_K(m)$ is clearly a multiple of the right-hand side. Conversely, if p^α divides $\bar{E}_K(m)$ then there is an $n = p^\alpha q$ such that $m \equiv 0 \pmod{\exp \text{Gal}(K_{p^\alpha q}/K)}$ which is a multiple of $\exp \text{Gal}(K_{p^\alpha}/K)$, and hence $m \equiv 0 \pmod{\exp \text{Gal}(K_{p^\alpha}/K)}$.

PROPOSITION 3.4: The p -primary part of $\bar{E}_K(m)$ is as follows, in terms of the invariants $\gamma_K(p)$ and $\phi_K(p)$:

- (a) If p is odd and $m \equiv 0 \pmod{\phi_K(p)}$, then $\bar{E}_K(m)_p = p^{\gamma_K(p)} m_p$.
- (b) If p is odd and $m \not\equiv 0 \pmod{\phi_K(p)}$, then $\bar{E}_K(m)_p = 1$.
- (c) $p = 2$: For odd m , $\bar{E}_K(m)_2 = 2$ if $\sqrt{-1} \notin K$ and $\bar{E}_K(m)_2 = 2^{\gamma_K(2)+1}$ if $\sqrt{-1} \in K$.
- (d) $p = 2$: For even m , $\bar{E}_K(m)_2 = 2^{\gamma_K(2)+1} m_2$ if $\sqrt{-1} \in K$ or if K is not exceptional; and $\bar{E}_K(m)_2 = 2^{\gamma_K(2)} m_2$ if $\sqrt{-1} \notin K$ and K exceptional.

PROOF: *Case p odd.* Here $\exp \text{Gal}(K_{p^\alpha}/K) = |\text{Gal}(K_{p^\alpha}/K)| = \phi_K(p)p^{\alpha-\gamma_K(p)}$ for $\alpha \geq \gamma_K(p)$, and $= \phi_K(p)$ for $\alpha \leq \gamma_K(p)$. If $m \equiv 0 \pmod{\phi_K(p)}$ this integer divides m if $p^\alpha = p^{\gamma_K(p)}m_p$, but not for higher powers α of p , hence (a). If $m \not\equiv 0 \pmod{\phi_K(p)}$ then only $\alpha = 0$ will do and $\bar{E}_K(m)_p = 1$.

Case $p = 2$: Since $\text{Gal}(K_2/K) = 1$, $\bar{E}_K(m)$ is even for all m . If m is odd and $\sqrt{-1} \notin K$, $|\text{Gal}(K_4/K)| = 2$ does not divide m nor does $\exp \text{Gal}(K_{2^\alpha}/K)$ for any $\alpha \geq 2$, hence $\bar{E}_K(m)_2 = 2$. If m is odd and $\sqrt{-1} \in K$, then $\exp \text{Gal}(K_{2^\alpha}/K) = 1$ for $\alpha \leq \gamma_K(2) + 1$ and thus divides m , but not for higher powers α of 2; hence $\bar{E}_K(m)_2 = 2^{\gamma_K(2)+1}$.

If m is even and $\sqrt{-1} \in K$ or K not exceptional, then $\exp \text{Gal}(K_{2^\alpha}/K) = 2^{\alpha-\gamma-1}$ for $\alpha \geq \gamma + 2$ (cf. Theorem 2.4) divides m for $2^\alpha = 2^{\gamma+1}m_2$ but not for higher powers α of 2, hence $\bar{E}_K(m)_2 = 2^{\gamma+1}m_2$ ($\gamma = \gamma_K(2)$). Finally, if m is even and $\sqrt{-1} \notin K$, K exceptional, we have $\exp \text{Gal}(K_{2^\alpha}/K) = 2^{\alpha-\gamma}$ for $\alpha \geq \gamma + 2$ and obtain $\bar{E}_K(m)_2 = 2^\gamma m_2$.

REMARK 3.5: If $K \subset \mathbb{C}$ is not assumed to be a number field, $\bar{E}_K(m)_p$ can be *defined* by the above formula (*), and Proposition 3.4 still holds (with $\bar{E}_K(m)_p = \infty$ if $m \equiv 0 \pmod{\phi_K(p)}$ and $\gamma_K(p) = \infty$).

3.3: Let now G be an arbitrary finite group, $\rho: G \rightarrow GL(\mathbb{C})$ a representation with $\chi_\rho(G) \subset K \subset \mathbb{C}$. By naturality of the Chern classes, and since the restriction of $H^{2i}(G; \mathbb{Z})$ to any p -Sylow subgroup of G is injective on the p -torsion subgroup of $H^{2i}(G; \mathbb{Z})$, the p -primary part of $c_i(\rho)$ fulfills the statements of Corollary 3.2 for any prime ($\gamma = \gamma_K(p)$). By virtue of Proposition 3.4 this means that it is annihilated by $\bar{E}_K(i)_p$. If K is a number field, $\bar{E}_K(i)$ is defined, and it follows that $\bar{E}_K(i)c_i(\rho) = 0$, for all $i \geq 1$:

THEOREM 3.6: *Let $K \subset \mathbb{C}$ be a number field, G a finite group, and $\rho: G \rightarrow GL(\mathbb{C})$ a representation whose character χ_ρ takes its values in K . Then the Chern classes $c_i(\rho)$, $i \geq 1$, fulfill*

$$\bar{E}_K(i)c_i(\rho) = 0.$$

We note that $\bar{E}_K(i)_p$ agrees with the bound for the order of the p -torsion Grothendieck-Chern classes [7] obtained from Galois invariance – as it must be the case in view of Grothendieck’s comparison theorem.

3.4: According to Theorem 3.6 $\bar{E}_K(i)$ is a universal bound for the order of the i -th Chern class $c_i(\rho)$, for all finite groups G and all

representations $\rho: G \rightarrow GL(\mathbb{C})$ which are defined over the number field K . We will denote by $E_K(i)$ the “best universal bound” in this sense; i.e., $E_K(i)$ is the minimum of all positive integers n such that $nc_i(\rho) = 0$ for all finite groups and all K -representations. Obviously $E_K(i)$ divides $\bar{E}_K(i)$. A lower bound for $E_K(i)$ is easily obtained by looking at finite cyclic groups as follows (the argument essentially occurs in Soulé [12] or in Eckmann–Mislin [4] and is given here for completeness). We write C_n for the cyclic group of order n .

PROPOSITION 3.7: *Let $K \subset \mathbb{C}$ be a subfield and $i \equiv 0 \pmod{\phi_K(n)}$, $i > 0$. There exists a representation ρ of C_n over K whose Chern class $c_i(\rho)$ has order n .*

PROOF: Let χ be a faithful \mathbb{C} -irreducible character of C_n . Then $\sum_{\sigma} \sigma\chi$, the sum extended over all $\sigma \in \text{Gal}(K(n)/K)$, is the character of a representation of C_n over K , of degree $\phi_K(n) = j$. Its Chern class $c_i(\sum \sigma\chi) = \prod c_i(\sigma\chi) \in H^{2i}(C_n; \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ has order n , since all $\sigma\chi$ are faithful and thus have Chern class $c_1(\sigma\chi) \in H^2(C_n; \mathbb{Z})$ of order n . By assumption $i = kj$; the representation ρ with character $k\sum \sigma\chi$ is realizable over K , and its top Chern class $c_i(\rho) \in H^{2i}(C_n; \mathbb{Z})$ has order n .

If, for a prime p , $\gamma_K(p) = \infty$ and $i \equiv 0 \pmod{\phi_K(p)}$ it follows that for any $\alpha \geq 1$ there exists a representation ρ of C_{p^α} over K such that the Chern class $c_i(\rho)$ has order p^α (since $\phi_K(p^\alpha) = \phi_K(p)$). Thus there is, in that case, *no universal bound* for the order of the p -primary part of $c_i(\rho)$. If $\gamma_K(p) < \infty$, in particular for number fields, we have seen that such universal bounds do exist, namely $\bar{E}_K(i)_p$ for any i . From Proposition 3.7 one immediately gets a lower estimate for the p -primary part of the best universal bound $E_K(i)$. We use the following notation.

DEFINITION 3.8: For a number field $K \subset \mathbb{C}$ and an integer $m > 0$ we put

$$\underline{E}_K(m) = \ell cm\{n \geq 1 \mid m \equiv 0 \pmod{\phi_K(n)}\}.$$

The p -primary part is $\underline{E}_K(m)_p = \max\{p^\alpha \mid m \equiv 0 \pmod{\phi_K(p^\alpha)}\}$, and this may serve as a definition if $K \subset \mathbb{C}$ is an arbitrary field with $\gamma_K(p) < \infty$. We obviously have

$$\underline{E}_K(m)_p \leq E_K(m)_p \leq \bar{E}_K(m)_p$$

and

$$\underline{E}_K(m) \leq E_K(m) \leq \bar{E}_K(m).$$

If p is odd, $\phi_K(p^\alpha) = |\text{Gal}(K_{p^\alpha}/K)| = \exp \text{Gal}(K_{p^\alpha}/K)$ and hence $E_K(m)_p = \bar{E}_K(m)_p$. If $p = 2$, $E_K(m)_2$ and $\bar{E}_K(m)_2$ differ at most by a factor 2; more precisely

PROPOSITION 3.9: *Let $K \subset \mathbb{C}$ be a subfield with $\gamma_K(2) < \infty$. Then*

- (a) *If K is exceptional or m odd, $E_K(m)_2 = \bar{E}_K(m)_2$,*
- (b) *If K is not exceptional and m even, $E_K(m)_2 = \frac{1}{2}\bar{E}_K(m)_2$.*

PROOF: If K is exceptional, $\text{Gal}(K_{2^\alpha}/K)$ is cyclic. If m is odd and K not exceptional, then $\sqrt{-1} \notin K$ and therefore $E_K(m)_2 = \bar{E}_K(m)_2 = 2$. If m is even and K not exceptional, then $E_K(m)_2 = 2^{\gamma_K(2)}m_2 = \frac{1}{2}\bar{E}_K(m)_2$, cf. Proposition 3.4.

For number fields we thus have the result

THEOREM 3.10: *Let $K \subset \mathbb{C}$ be a number field. The best universal bound $E_K(i)$ for the order of the Chern classes $c_i(\rho)$ of representations of finite groups over K is either $E_K(i)$ or $\bar{E}_K(i)$. If K is exceptional or if i is odd, these integers coincide; if K is not exceptional and i even, they differ by a factor 2.*

In the next section we show that in the case $E_K(2) = \frac{1}{2}\bar{E}_K(2)$ we have $E_K(i) = \bar{E}_K(i)$ if K is formally real, and $= \bar{E}_K(i)$ if K is not formally real.

4. The best universal bound

4.1: We will first discuss Chern classes of representations of groups of special type, namely the groups C_{2^α} (cyclic of order 2^α), D_{2^α} (dihedral of order 2^α), SD_{2^α} (semidihedral of order 2^α) and Q_{2^α} (generalized quaternion group of order 2^α); for the notation and the representations of these groups see [3] and [4].

In this section, we write ω_α for ζ_{2^α} (a primitive 2^α -th root of 1). The following lemma will play an important rôle.

LEMMA 4.1: *Let $K \subset \mathbb{C}$ be a subfield. If $\alpha > \gamma_K(2)$, then both the fields $K(\omega_{\alpha+1} + \omega_{\alpha+1}^{-1})$ and $K(\omega_{\alpha+1} - \omega_{\alpha+1}^{-1})$ have degree 2 over $K(\omega_\alpha + \omega_\alpha^{-1})$.*

PROOF: We deal with the case of the field $K(\omega_{\alpha+1} + \omega_{\alpha+1}^{-1})$, the case of $K(\omega_{\alpha+1} - \omega_{\alpha+1}^{-1})$ being entirely analogous.

We consider the inclusion diagram

$$\begin{array}{ccc} K(\omega_\alpha) & \subset & K(\omega_{\alpha+1}) \\ \cup & & \cup \\ K(\omega_\alpha + \omega_\alpha^{-1}) & \subset & K(\omega_{\alpha+1} + \omega_{\alpha+1}^{-1}) \end{array}$$

The degree of $K(\omega_{\alpha+1} + \omega_{\alpha+1}^{-1})$ over $K(\omega_\alpha + \omega_\alpha^{-1})$ is at most 2 and $K(\omega_{\alpha+1})$ has degree 2 over $K(\omega_\alpha)$ for $\alpha > \gamma_K(2)$ (cf. [4]). It thus suffices to show that $K(\omega_\alpha) = K(\omega_\alpha + \omega_\alpha^{-1})$ implies $K(\omega_{\alpha+1}) = K(\omega_{\alpha+1} + \omega_{\alpha+1}^{-1})$. But, since $\sqrt{-1} \in K(\omega_\alpha)$, $K(\omega_\alpha) = K(\omega_\alpha + \omega_\alpha^{-1})$ implies $K(\omega_{\alpha+1} + \omega_{\alpha+1}^{-1}) = K(\omega_{\alpha+1} + \omega_{\alpha+1}^{-1})(\sqrt{-1}) = K(\omega_{\alpha+1})$.

PROPOSITION 4.2: *Let G denote one of the groups C_{2^α} , D_{2^α} or SD_{2^α} for some α , and let $K \subset \mathbb{C}$ be a number field. If $\rho: G \rightarrow GL(\mathbb{C})$ is a representation with $\chi_\rho(G) \subset K$, then the Chern classes of ρ satisfy, for all $i > 0$,*

$$\underline{E}_K(i)c_i(\rho) = 0$$

PROOF: Since the Schur index $s_K(\chi_\rho)$ is $= 1$ for any representation of a group of type C_{2^α} , D_{2^α} or SD_{2^α} , the assumption $\chi_\rho(G) \subset K$ implies that ρ is equivalent to a K -representation. We will first consider the case where χ_ρ is K -irreducible, i.e. $\chi_\rho = \sum \sigma \chi$ for some \mathbb{C} -irreducible character χ , σ ranging over $\text{Gal}(K(\chi)/K)$. It suffices to consider the case K not exceptional and i even, since in all other cases $\underline{E}_K(i) = \bar{E}_K(i)$ (cf. Theorem 3.10). Hence, in view of Proposition 3.9, the assertion of the proposition will follow if we can show that $2^{\gamma}c_i(\rho) = 0$ ($\gamma = \gamma_K(2)$). We may furthermore assume ρ to be faithful, since any quotient of one of the groups in question is again of type C_{2^β} , D_{2^β} or SD_{2^β} .

(1) The case $G = C_{2^\alpha}$. If $\alpha \leq \gamma + 1$ then $2^\gamma m c_m(\rho) = 0$ for m even since $H^{2m}(C_{2^\alpha}; \mathbb{Z})$ has order 2^α . If $\alpha > \gamma + 1$, we restrict ρ to $H \subset G$, the subgroup of index 2. We may assume inductively that $2^{\gamma}j c_j(\rho|H) = 0$ for all $j > 0$. Let $\text{tr}: H^{2m}(H; \mathbb{Z}) \rightarrow H^{2m}(G; \mathbb{Z})$ denote the transfer map. Then $2c_m(\rho) = \text{tr } c_m(\rho|H)$. We claim that $\rho|H$, is of the form $\tau \oplus \tau$ for some K -representation τ of H . Indeed, if $\chi_\rho = \sum \sigma \chi$, $\sigma \in \text{Gal}(K(\chi)/K)$, then $\chi_\rho|H = \sum \sigma \chi|H$, and since $K(\chi) = K(\omega_\alpha)$ has degree 2 over $K(\chi|H) = K(\omega_{\alpha-1})$ for $\alpha > \gamma + 1$ we see that the kernel of the restriction map $\text{Gal}(K(\chi)/K) \rightarrow \text{Gal}(K(\chi|H)/K)$ has order 2; it follows that $\chi_\rho|H$ is of the form $\chi_\tau + \chi_\tau$ for some K -representation τ .

For an even m we thus have

$$c_m(\rho|H) = c_m(\tau \oplus \tau) = 2 \sum_{\substack{i < j \\ i+j=m}} c_i(\tau)c_j(\tau) + (c_{m/2}(\tau))^2$$

By induction we may assume

$$2^\gamma(i+j)c_i(\tau)c_j(\tau) = 2^\gamma i c_i(\tau)c_j(\tau) + 2^\gamma j c_i(\tau)c_j(\tau) = 0$$

and $2^{\gamma-1}mc_{m/2}(\tau) = 0$ (note that for i odd, $2c_i(\tau) = 0$ since $\sqrt{-1} \notin K$). Thus

$$\begin{aligned} 2^\gamma mc_m(\rho) &= 2^{\gamma-1}m \operatorname{tr} c_m(\rho|H) = \\ &= \operatorname{tr} \left(2^\gamma \sum_{\substack{i \leq j \\ i+j=m}} (i+j)c_i(\tau)c_j(\tau) + 2^{\gamma-1}mc_{m/2}(\tau)^2 \right) = 0 \end{aligned}$$

(2) The case $G = D_{2^\alpha}$. Let $\chi_\rho = \Sigma \sigma \chi$, where χ is \mathbb{C} -irreducible and faithful, $\sigma \in \operatorname{Gal}(K(\chi)/K) = \operatorname{Gal}(K(\omega_{\alpha-1} + \omega_{\alpha-1}^{-1})/K)$. It suffices to consider the case m even. If $\alpha \leq \gamma + 2$, then $2^\gamma mc_m(\rho) = 0$ since $2^{\alpha-1}H^{2m}(D_{2^\alpha}; \mathbb{Z}) = 0$ for $m > 0$. If $\alpha > \gamma + 2$, the kernel of the restriction map $\operatorname{Gal}(K(\omega_{\alpha-1} + \omega_{\alpha-1}^{-1})/K) \rightarrow \operatorname{Gal}(K(\omega_{\alpha-2} + \omega_{\alpha-2}^{-1})/K)$ is of order 2 (cf. Lemma 4.1). Hence, if we restrict χ_ρ to $D_{2^{\alpha-1}} \subset D_{2^\alpha}$, we infer that $\chi_\rho|_{D_{2^{\alpha-1}}}$ is of the form $\chi_\tau + \chi_\tau$ for some K -representation τ of $D_{2^{\alpha-1}}$. By the argument used in 1), this implies that $2^\gamma mc_m(\rho) = 0$.

(3) The case $G = SD_{2^\alpha}$. The argument is very similar. One restricts χ_ρ to $D_{2^{\alpha-1}} \subset SD_{2^\alpha}$ and notes that the kernel of the restriction map $\operatorname{Gal}(K(\omega_{\alpha-1} - \omega_{\alpha-1}^{-1})/K) \rightarrow \operatorname{Gal}(K(\omega_{\alpha-2} + \omega_{\alpha-2}^{-1})/K)$ has order 2 if $\alpha > \gamma + 2$.

This settles the case of an irreducible ρ . If $\rho = \lambda \oplus \tau$ with λ and τ K -representations of degree smaller than the degree of ρ , then we assume by induction that the proposition holds for λ and τ . Hence

$$2^\gamma mc_m(\rho) = \sum_{i+j=m} (2^\gamma i c_i(\lambda)c_j(\tau) + 2^\gamma j c_i(\lambda)c_j(\tau)) = 0$$

and the proof of the proposition is completed.

4.2: The Chern classes of representations χ of Q_{2^α} are harder to compute, due to the fact that the Schur index $s_K(\chi)$ may be $=1$ or 2 . The following two lemmas are well known (cf. [8, Chapter 10]).

LEMMA 4.3: *Let $K \subset \mathbb{C}$ be a subfield and let χ be the faithful \mathbb{C} -irreducible character of Q_8 . Then $s_K(\chi) = 1$ if and only if -1 is a sum of two squares in K .*

LEMMA 4.4: *Let $K \subset \mathbb{C}$ be a subfield and χ a \mathbb{C} -irreducible character of the finite group G . If $H \subset G$ is a subgroup such that $\chi|_H$ is \mathbb{C} -irreducible, then $s_K(\chi)$ divides $s_K(\chi|_H)$.*

The following corollary is a straightforward consequence.

COROLLARY 4.5: *Under the hypotheses of Lemma 4.4 one has always $s_{K(\chi)}(\chi) = s_{K(\chi)}(\chi|_H)$.*

PROOF: Since $s_K(\chi) \cdot \chi$ and thus $s_K(\chi) \cdot \chi|_H$ are characters of representations defined over $K(\chi)$, we infer that $s_{K(\chi)}(\chi|_H)$ must divide $s_K(\chi) = s_{K(\chi)}(\chi)$. Therefore, $s_{K(\chi)}(\chi) = s_{K(\chi)}(\chi|_H)$ by the previous Lemma.

PROPOSITION 4.6: *Let χ denote a \mathbb{C} -irreducible faithful character of Q_{2^α} for some $\alpha \geq 3$, and let $K \subset \mathbb{C}$ be a subfield. Then $s_K(\chi) = 1$ if and only if -1 is a sum of two squares in $K(\omega_{\alpha-1} + \omega_{\alpha-1}^{-1})$.*

PROOF: From the classification of characters of Q_{2^α} (see e.g. [3]) one sees that $\chi|_{Q_8}$ is \mathbb{C} -irreducible and faithful on $Q_8 \subset Q_{2^\alpha}$. Hence $s_{K(\chi)}(\chi) = s_{K(\chi)}(\chi|_{Q_8})$ and the assertion follows from Lemma 4.3, noting that $K(\chi) = K(\omega_{\alpha-1} + \omega_{\alpha-1}^{-1})$.

Let K be a number field. If K is formally real then -1 is not a sum of two squares in K . If K is not formally real then -1 is a sum of two or four squares in K by Siegel's Theorem; the following criterion permits to determine whether -1 is a sum of 2 squares or not (cf. [11, Chapter 11]).

We write $K_{(p)}$ for the completion of K at the prime ideal p .

LEMMA 4.7: *Let K be a number field which is not formally real. Then -1 is a sum of two squares in K if and only if for every prime ideal p of K dividing 2 the degree $(K_{(p)} : \mathbb{Q}_{(2)})$ is even.*

COROLLARY 4.8: *If K is a number field which is not formally real, then -1 is a sum of 2 squares in $K(\sqrt{2})$.*

PROOF: Let p be a prime ideal of $K(\sqrt{2})$ which divides 2. Then $(K(\sqrt{2})_{(p)} : \mathbb{Q}_{(2)})$ is even since it is divisible by $(\mathbb{Q}_{(2)}(\sqrt{2}) : \mathbb{Q}_{(2)}) = 2$.

COROLLARY 4.9: *If K is a number field which is not formally real,*

then $s_K(\chi) = 1$ for every \mathbb{C} -irreducible faithful character χ of Q_{2^α} , $\alpha \geq 4$.

PROOF: Note that $K(\chi) = K(\omega_{\alpha-1} + \omega_{\alpha-1}^{-1}) \supset K(\omega_3 + \omega_3^{-1}) = K(\sqrt{2})$ and -1 is a sum of two squares in $K(\sqrt{2})$. Therefore $s_K(\chi) = 1$ by Proposition 4.6.

4.3: We can now proceed to the discussion of the Chern classes of representations of Q_{2^α} . First we need

LEMMA 4.10: Suppose $K \subset \mathbb{C}$ is not exceptional and $\alpha > \gamma + 1$, ($\gamma = \gamma_K(2)$). Then $\text{Gal}(K(\omega_\alpha + \omega_\alpha^{-1})/K)$ is cyclic of order $2^{\alpha-\gamma-1}$.

PROOF: We have already seen that $\text{Gal}(K(\omega_\alpha)/K)$ is (non-cyclic) of order $2^{\alpha-\gamma}$ (Theorem 2.4). From the embedding $\text{Gal}(K(\omega_\alpha + \omega_\alpha^{-1})/K) \rightarrow \text{Gal}(\mathbb{Q}(\omega_\alpha + \omega_\alpha^{-1})/\mathbb{Q}) \cong C_{2^{\alpha-2}}$ we infer that $\text{Gal}(K(\omega_\alpha + \omega_\alpha^{-1})/K)$ is cyclic of order $2^{\alpha-\gamma-1}$.

PROPOSITION 4.11: Let $K \subset \mathbb{C}$ be a number field which is not exceptional and let $\gamma = \gamma_K(2)$. Then the following holds.

(a) If K is formally real, then for every K -representation $\rho: Q_{2^\alpha} \rightarrow GL(K)$ the Chern classes satisfy $\underline{E}_K(i)c_i(\rho) = 0$ for all $i > 0$.

(b) If K is not formally real, there exists for every even $i > 0$ an α and a K -representation $\rho: Q_{2^\alpha} \rightarrow GL(K)$ such that $c_i(\rho)$ has maximal possible order $\bar{E}_K(i)_2 = 2\underline{E}_K(i)_2 = 2^{\gamma+1}i_2$.

PROOF: (a) It suffices to consider the case of a faithful and irreducible ρ (a proper quotient of Q_{2^α} is of type D_{2^β} and the assertion follows from Proposition 4.2 if ρ factors through a dihedral group). Therefore $\chi_\rho = s_K(\chi)\Sigma\sigma\chi = 2\Sigma\sigma\chi = \theta + \theta$ where θ is a character of Q_{2^α} with $\theta(G) \subset K$. By Theorem 3.6 we infer $\bar{E}_K(j)c_j(\theta) = 0$ for $j > 0$. For j even, one has $\bar{E}_K(j) = 2\underline{E}_K(j)$ since K is not exceptional. Hence, for m even

$$\begin{aligned} \underline{E}_K(m)c_m(\rho) &= 2^\gamma m c_m(\rho) = \frac{1}{2} \bar{E}_K(m)c_m(\theta + \theta) \\ &= \frac{1}{2} 2 \sum_{\substack{i \leq j \\ i+j=m}} 2^{\gamma+1}(i+j)c_i(\theta)c_j(\theta) + 2^\gamma m c_{m/2}(\theta)^2 = 0. \end{aligned}$$

This settles the case for m even. For m odd, $\underline{E}_K(m) = \bar{E}_K(m)$ and therefore $\underline{E}_K(m)c_m(\rho) = 0$.

(b) Let χ denote a \mathbb{C} -irreducible and faithful character of $Q_{2^{\gamma+\delta+1}}$, $\delta > 1$. Then $\psi = \Sigma \sigma\chi$, $\sigma \in \text{Gal}(K(\omega_{\alpha+\delta} + \omega_{\alpha+\delta}^{-1})/K)$, is the character of a K -representation ($s_K(\chi) = 1$ by Corollary 4.9). By Lemma 4.10, $\text{Gal}(K(\omega_{\gamma+\delta} + \omega_{\gamma+\delta}^{-1})/K)$ has order $2^{\delta-1}$ and therefore ψ is of degree $\chi(1)2^{\delta-1} = 2^\delta$. Since $c_2(\sigma\chi) \in H^4(Q_{2^{\gamma+\delta+1}}; \mathbb{Z})$ is a generator for every σ , it follows that $c_2(\psi)$ has maximal order $2^{\gamma+\delta+1} = \bar{E}_K(2\delta)_2$. If $m = 2^\beta t$ with t odd and $\beta > 1$, we choose $\delta = \beta$ and we take for ρ the representation with character $t\psi$; the order of $c_m(\rho)$ is then $2^{\gamma+\beta+1} = \bar{E}_K(m)_2$. This settles (b) for all m with $m_2 > 2$. In case $m_2 = 2$ we first construct ψ as above with $\delta = 2$ and obtain a representation with $c_4(\psi)$ of order $\bar{E}_K(4)_2 = 2^{\gamma+3}$; we claim that $c_2(\psi)$ has (maximal) order $2^{\gamma+2} = \bar{E}_K(2)_2$: Since $\psi = \chi + \sigma\chi$ one has

$$c_2(\psi) = c_2(\chi) + c_2(\sigma\chi) + c_1(\chi)c_1(\sigma\chi).$$

Moreover $c_2(\chi)$ and $c_2(\sigma\chi)$ have both order $2^{\gamma+3} = |Q_{2^{\gamma+3}}|$ and $2c_1(\chi) = 0$ since $H^2(Q_{2^{\gamma+3}}; \mathbb{Z}) = C_2 \times C_2$. From Theorem 3.1 we infer that $c_2(\sigma\chi) = (2k+1)^2 c_2(\chi)$ for some integer k and hence $2^{\gamma+1}c_2(\sigma\chi) = 2^{\gamma+1}c_2(\chi)$. It follows that $2^{\gamma+1}c_2(\psi) = 2 \cdot 2^{\gamma+1}c_2(\chi) \neq 0$ and therefore $c_2(\psi)$ has order $2^{\gamma+2}$. More generally, if $m = 2t$ with t odd, then $c_m(t\psi)$ has order $\bar{E}_K(m)_2 = 2^{\gamma+2}$, which completes the proof of (b).

4.4: From the computations of Chern classes of representations of groups of special type one may now pass to arbitrary finite groups, by the method used to prove Theorem 4.2 of [3], making use of Theorem 1.3 of [3]. Then Theorem 3.10 together with Propositions 4.2 and 4.11 imply the following main theorem.

MAIN THEOREM 4.12: *Let $K \subset \mathbb{C}$ be a number field and $\rho: G \rightarrow GL(K)$ a K -representation of the finite group G . Then the best universal bound $E_K(i)$ for the order of the i -th Chern class $c_i(\rho) \in H^{2i}(G; \mathbb{Z})$, $i > 0$, is given by*

$$E_K(i) = \begin{cases} \bar{E}_K(i), & \text{if } K \text{ is formally real,} \\ \bar{E}_K(i), & \text{if } K \text{ is not formally real.} \end{cases}$$

Recall that $\bar{E}_K(m) = \bar{E}_K(m)$ if m is odd and $\bar{E}_K(m) = \frac{1}{2}\bar{E}_K(m)$ if m is even and K not exceptional (e.g., K formally real). We obtain thus

$$E_K(m) = \begin{cases} \frac{1}{2}\bar{E}_K(m), & \text{if } m \text{ is even and } K \text{ formally real} \\ \bar{E}_K(m), & \text{in all other cases.} \end{cases}$$

Using Proposition 3.4 we may deduce.

COROLLARY 4.13: *In terms of the invariants $\gamma_K(p)$ and $\phi_K(p)$ the p -primary component of $E_K(m)$ is as follows.*

- (a) *If p is odd and $m \equiv 0 \pmod{\phi_K(p)}$ then $E_K(m)_p = p^{\gamma_K(p)} m_p$.*
- (b) *If p is odd and $m \not\equiv 0 \pmod{\phi_K(p)}$ then $E_K(m)_p = 1$.*
- (c) *For odd m , $E_K(m)_2 = 2$ if $\sqrt{-1} \notin K$ and $E_K(m)_2 = 2^{\gamma_K(2)+1}$ if $\sqrt{-1} \in K$.*
- (d) *For even m , $E_K(m)_2 = 2^{\gamma_K(2)+1} m_2$ if $\sqrt{-1} \in K$ or if K is neither exceptional nor formally real; $E_K(m)_2 = 2^{\gamma_K(2)} m_2$ if K is formally real, or if $\sqrt{-1} \notin K$ and K exceptional.*

5. Applications to the cohomology of arithmetic groups

5.1: Let $K \subset \mathbb{C}$ be a number field with $\mathcal{O} = \mathcal{O}(K) \subset K$ its ring of algebraic integers. We consider the usual inclusion maps $GL_n(\mathcal{O}(K)) \subset GL(\mathcal{O}(K)) \subset GL(\mathbb{C})$ and write $c_i(\mathcal{O}) \in H^{2i}(GL(\mathcal{O}); \mathbb{Z})$ ($c_i(\mathcal{O}, n) \in H^{2i}(GL_n(\mathcal{O}); \mathbb{Z})$, respectively) for the universal Chern classes. By the results of Grothendieck [7], the classes $c_i(\mathcal{O}, n)$ are torsion classes and their p -primary orders are bounded by $\bar{E}_K(i)_p$ (see also Section 2).

Using the stability of the cohomology of $GL_n(\mathcal{O})$ (cf. [2]) one thus has

LEMMA 5.1: $\bar{E}_K(i)c_i(\mathcal{O}) = 0$ for all $m > 0$.

The universal bounds $E_K(i)$ for the orders of Chern classes of K -representations of finite groups may be used to obtain lower estimates for the order of $c_i(\mathcal{O})$. This will become clear from the following proposition.

PROPOSITION 5.2: *Let $\rho: G \rightarrow GL_n(K)$ be a K -representation of the finite group G , where $K \subset \mathbb{C}$ denotes a number field. If $\iota_n: GL_n(K) \subset GL_{n+1}(K)$ denotes the standard inclusion, then $\iota_n \rho$ is equivalent to a representation which factors through $GL_{n+1}(\mathcal{O}(K)) \subset GL_{n+1}(K)$; in particular*

$$c_i(\rho) = f^* c_i(\mathcal{O}) \in H^{2i}(G; \mathbb{Z})$$

for some $f: G \rightarrow GL(\mathcal{O})$.

PROOF: Let V denote the representation module of ρ and let $\{v_1, \dots, v_n\}$ be a K -basis for V . Then the $\mathcal{O}(K)$ -module $M \subset V$

generated by $\{gv_i | 1 \leq i \leq n, g \in G\}$ is a finitely generated $\mathcal{O}(K)$ -projective module (gl. $\dim \mathcal{O}(K) = 1$). Since $\mathcal{O}(K)$ is a Dedekind domain, we can find an $\mathcal{O}(K)$ -projective module N of rank one such that $M \oplus N$ is $\mathcal{O}(K)$ -free. If we consider N as a trivial G -module then $(M \oplus N) \otimes_{\mathcal{O}} K$ defines a representation of G equivalent to $\iota_n \rho$, and the constructed representation is by matrices with entries in $\mathcal{O}(K)$ with respect to a suitable basis.

Since $E_K(i)$ and $\bar{E}_K(i)$ differ by at most a factor two, we infer the following.

THEOREM 5.3: *The order of $c_i(\mathcal{O}) \in H^{2i}(GL(\mathcal{O}); \mathbb{Z})$, $i > 0$, is either $E_K(i)$ or $\bar{E}_K(i)$. If i is odd or if K is not formally real then the order is $\bar{E}_K(i)$.*

6. Remarks on the value of the bounds $E_K(m)$ for quadratic number fields

6.1: We have seen in [3] that one can express $E_{\mathcal{O}}(m)$ for even m as denominator of B_m/m , B_m the m -th Bernoulli number. The more general numbers $E_K(m)$ differ only little from $E_{\mathcal{O}}(m)$. In particular one has (we write \mathbb{Q}_{p^∞} for $\bigcup \mathbb{Q}_{p^n}$)

LEMMA 6.1: *If $K \subset \mathbb{C}$ is any number field and p a prime such that $K \cap \mathbb{Q}_{p^\infty} = \mathbb{Q}$, then for all m :*

- (a) $E_K(m)_p = E_{\mathcal{O}}(m)_p$ if p is odd.
- (b) In case $p = 2$, $E_K(m)_2 = E_{\mathcal{O}}(m)_2 (2E_{\mathcal{O}}(m)_2)$ if K is formally real or if m is odd (if K is not formally real and m even, respectively).

PROOF: If $K \cap \mathbb{Q}_{p^\infty} = \mathbb{Q}$ then $K \otimes_{\mathcal{O}} \mathbb{Q}_{p^\beta} = K_{p^\beta}$ for all β . Therefore $\gamma_K(p) = \gamma_{\mathcal{O}}(p)$ and $\phi_K(p) = \phi_{\mathcal{O}}(p)$. Hence (a) follows directly from the Main Theorem 4.12 and (b) follows by noting that $K \cap \mathbb{Q}_{2^\infty} = \mathbb{Q}$ implies that K is not exceptional.

REMARK 6.2: The hypotheses of Lemma 6.1 are fulfilled if p is a prime which does not divide the discriminant $d(K)$ of K . This follows from the fact that p is the only prime which divides the discriminant $d(\mathbb{Q}_{p^\beta})$, and that K and \mathbb{Q}_{p^β} are linearly disjoint if $(p, d(K)) = 1$.

6.2: If K is a quadratic number field, then $K \cap \mathbb{Q}_{p^\infty}$ is either K or \mathbb{Q} . We recall the structure of the quadratic subfields of \mathbb{Q}_{p^β} .

LEMMA 6.3: *Let p denote a prime number. Then*

(a) *If p is odd, \mathbb{Q}_{p^∞} has a unique quadratic subfield, which is $\mathbb{Q}(\sqrt{p})$ if $p \equiv 1(4)$, and $\mathbb{Q}(\sqrt{-p})$ if $p \equiv 3(4)$.*

(b) *If $p = 2$ then \mathbb{Q}_{p^∞} has precisely three quadratic subfields, namely $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{-2})$.*

PROOF: Let p be an odd prime. Then $\text{Gal}(\mathbb{Q}_{p^\infty}/\mathbb{Q})$, $\beta \geq 1$, has a unique subgroup of index 2 and therefore \mathbb{Q}_{p^∞} has a unique quadratic subfield. It is well known that $\sqrt{(-1)^{p-1/2}p} \in \mathbb{Q}_p$ and hence (a). The proof of (b) is similar. One notes that $\text{Gal}(\mathbb{Q}_{2^\infty}/\mathbb{Q})$ has precisely 3 subgroups of index 2 if $\beta \geq 3$.

If K is a quadratic number field, then $K = \mathbb{Q}(\sqrt{D})$ with $D \neq 1$ a square-free integer; if $D > 0$ then K is formally real (K is a “real quadratic number field”) and if $D < 0$ then K is not formally real (K is an “imaginary quadratic number field”). Note also that the only quadratic number fields which are exceptional are $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-2})$ (Corollary 2.7).

THEOREM 6.4: *Let K be a quadratic number field which is different from any of the following list:*

$$\mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{(-1)^{p-1/2} \cdot p}) \text{ with } p \text{ an odd prime.}$$

Then

$$E_K(m) = \begin{cases} E_{\mathbb{Q}}(m), & \text{if } K \text{ is real quadratic or } m \text{ odd,} \\ 2E_{\mathbb{Q}}(m), & \text{if } K \text{ is imaginary quadratic and } m \text{ even.} \end{cases}$$

PROOF: We have $K \cap \mathbb{Q}_{p^\infty} = \mathbb{Q}$ for all primes p . The assertion thus follows from Lemma 6.1.

It is easy to compute $E_K(m)$ for the quadratic fields not included in Theorem 6.4. One has only to notice that $\phi_K(p) = (p-1)/2$ and $\gamma_K(p) = 1$ if $K = \mathbb{Q}(\sqrt{(-1)^{p-1/2}p})$, p an odd prime; and that for $\mathbb{Q}(\sqrt{-1})$ ($\mathbb{Q}(\sqrt{2})$ or $\mathbb{Q}(\sqrt{-2})$) one has $\gamma_K(2) = 1$ ($\gamma_K(2) = 2$, respectively). One thus obtains, using Corollary 4.13

Case $K = \mathbb{Q}(\sqrt{-1})$: (K exceptional)

$$E_K(m) = \begin{cases} 4, & \text{if } m \text{ is odd,} \\ 2E_{\mathbb{Q}}(m), & \text{if } m \text{ is even.} \end{cases}$$

Case $K = \mathbb{Q}(\sqrt{-2})$: (K exceptional)

$$E_K(m) = \begin{cases} 2 & , \text{ if } m \text{ is odd,} \\ 2E_Q(m), & \text{ if } m \text{ is even.} \end{cases}$$

Case $K = \mathbb{Q}(\sqrt{2})$: (K not exceptional)

$$E_K(m) = \begin{cases} 2 & , \text{ if } m \text{ is odd,} \\ 2E_Q(m), & \text{ if } m \text{ is even.} \end{cases}$$

Case $K = \mathbb{Q}(\sqrt{-p})$, $p \equiv 3(4)$: ($d(K) = -p$)

$$E_K(m) = \begin{cases} 2 & , \text{ if } m \text{ is odd and } 2m \not\equiv 0 \pmod{p-1}, \\ 2pm_p & , \text{ if } m \text{ is odd and } 2m \equiv 0 \pmod{p-1}, \\ 2E_Q(m), & \text{ if } m \text{ is even.} \end{cases}$$

Case $K = \mathbb{Q}(\sqrt{p})$, $p \equiv 1(4)$: ($d(K) = p$)

$$E_K(m) = \begin{cases} 2 & , \text{ if } m \text{ is odd,} \\ E_Q(m) & , \text{ if } m \equiv 0 \pmod{p-1}, \\ E_Q(m)pm_p, & \text{ if } 2m \equiv 0 \pmod{p-1} \text{ and } m \not\equiv 0 \pmod{p-1}. \end{cases}$$

We can combine these results with Theorem 5.3 to obtain precise orders of universal Chern classes for certain fields, including all imaginary quadratic number fields.

These precise orders improve those given by Ch. Thomas [13].

THEOREM 6.5: *Let K be a quadratic number field with ring of integers $\mathcal{O} \subset K$.*

Then

(a) *If m is odd, the order of $c_m(\mathcal{O})$ is 2, with the following exceptions*

(a)₁ *If $K = \mathbb{Q}(\sqrt{-1})$, $c_{2\ell+1}(\mathcal{O})$ has order 4 for all ℓ .*

(a)₂ *If $K = \mathbb{Q}(\sqrt{-p})$ p a prime $\equiv 3(4)$ and $2m \equiv 0 \pmod{p-1}$, then $c_m(\mathcal{O})$ has order $2pm_p$.*

(b) *If m is even $\neq 0$ and K is imaginary quadratic, then $c_m(\mathcal{O})$ has order $2E_Q(m) = \text{den}(B_m/2m)$.*

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