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THE POINCARÉ-LELONG EQUATION ON COMPLETE KÄHLER MANIFOLDS

Ngaiming Mok, Yum-Tong Siu and Shing-Tung Yau

Dedicated to the memory of the late Professor Aldo Andreotti

In [11] Lelong studied the Poincaré-Lelong equation $i\partial\bar{\partial}u = \rho$, where $\rho = i\,\Sigma_{i,j}\,\rho_{i\bar{j}}\,dz^i \wedge d\bar{z}^j$ is a positive d-closed (1, 1) current defined on $\mathbb{C}^n$. He showed that under suitable growth conditions on ρ, the equation can be solved by reducing it to $\frac{1}{2}\Delta u = \Sigma_i\,\rho_{i\bar{i}}$, ($\rho_{i\bar{i}}$ being measures on $\mathbb{C}^n$).

In this paper we are interested in the Poincaré-Lelong equation on complete Kähler manifolds. We present here two different techniques. The first one is a Bochner identity obtained by considering $\Delta\|i\partial\bar{\partial}u - \rho\|^2$, where u is a solution of $\frac{1}{2}\Delta u = \Sigma_i\,\rho_{i\bar{i}}$. When the complete Kähler manifold M has nonnegative holomorphic bisectional curvature $\Delta\|i\partial\bar{\partial}u - \rho\|^2 \geq 0$. When ρ grows suitably, we can choose u so that $\|i\partial\bar{\partial}u - \rho\|^2$ decays to zero at infinity. The sub-mean value inequality implies that $i\partial\bar{\partial}u = \rho$. We use this technique to prove an isometry theorem (isometric to $\mathbb{C}^n$ with the Euclidean metric) on complete Kähler manifolds of nonnegative holomorphic bisectional curvature when the scalar curvature grows like $1/r^{2+\epsilon}$ in terms of the geodesic distance r, under certain auxiliary conditions. Under similar conditions but assuming the scalar curvature grows like $1/r^2$, we can show that M is a Stein manifold.

The second technique in solving $i\partial\bar{\partial}u = \rho$ is based on the L^2-estimate of $\bar{\partial}$ of Andreotti-Vesentini [1] on complete Kähler manifolds. Suppose M is a complete Kähler manifold with a pole (i.e., the exponential map at this point is a diffeomorphism) such that the

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sectional curvature is bounded by $\pm A_\epsilon/(1+\gamma^2)^{1+\epsilon}$, where A_ϵ is a suitable constant depending on ϵ; then by suitably choosing weight functions on subdomains and using the Harnack inequality of Moser [11], the equation $i\partial\bar{\partial}u = \rho$ can be solved with u bounded whenever ρ grows like $1/r^{2+\epsilon}$. We make two applications of this method. When M has sectional curvature bounded by $\pm A_\epsilon/(1+r^2)^{1+\epsilon}$ and possesses a pole, we prove that M is biholomorphic to $\mathbb{C}^n$. When M has nonpositive sectional curvature bounded by $-C/1+r^{2+\epsilon}$ from below, $n \geq 2$, we show that M is actually isomertrically biholomorphic to $\mathbb{C}^n$. For this we combine the above method with some intermediate results of Siu and Yau [13].

After completion of our research, we conjectured that the analogue of parts of our results should also hold for Riemannian manifolds. Namely, (1) of Theorem 2.1 should be true if we replace bisectional curvature by sectional curvature; and the last statement of Theorem 2.2 should be true in general. Recently, Gromov told us that he could prove our conjecture.*

We remark that Bishop and Goldberg [17], Goldberg and Kobayashi [18] obtained long ago Bochner identities basically equivalent to the one we need. For the sake of completeness, however, we have included here a proof of the identity. For the convenience of the non-specialist we have also included standard iteration techniques of Moser [12] necessary for our estimates in §1.

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* Recently R. Greene and H.-H. Wu [19] have also announced similar theorems in Riemannian geometry.

§1. The Poincaré-Lelong equation on complete Kähler manifolds with nonnegative holomorphic bisectional curvature

(1.1) A BOCHNER IDENTITY FOR THE POINCARE-LELONG EQUATION

The crux of the argument in this section is the following Bochner identity on complete Kähler manifolds of nonnegative holomorphic bisectional curvature.

PROPOSITION: *Let M be a complete Kähler manifold of nonnegative holomorphic bisectional curvature. Suppose ρ is a d-closed $(1,1)$ form on M and f is the trace of ρ with respect to the Kähler metric. Let u be a solution of $\frac{1}{2}\Delta u = f$. Then $\|i\partial\bar{\partial}u-\rho\|^2$* is subharmonic, where $\|\cdot\|$ denotes norms measured in terms of the Kähler metric.

PROOF: We assume without loss of generality that ρ is a real 2-form. In local coordinates $\rho = i\sum \rho_{i\bar{j}}\, dz_i \wedge d\bar{z}_j$, $(\rho_{i\bar{j}})$ being hermitian symmetric. The Laplace-Beltrami operator is given by $\Delta u = 2\sum_{i,j} g^{i\bar{j}}(\partial^2 u/\partial z, \partial \bar{z}_j)$ where $2\, Re \sum_{i,j} g_{i\bar{j}}\, dz^i \otimes d\bar{z}^j$ is the Kähler metric on M and $(g^{i\bar{j}})$ is the inverse matrix of $(g_{i\bar{j}})$. The trace of ρ is defined by $f = \text{trace}(\rho) = \sum_{i,j} g^{i\bar{j}}\rho_{i\bar{j}}$. Fix a point x on X. We can choose a complex geodesic coordinate system (z_i) at x, so that $g_{i\bar{j}}(x) = \delta_{i\bar{j}}$ and $(\partial/\partial z_k)g_{i\bar{j}}(x) = (\partial/\partial \bar{z}_k)g_{i\bar{j}}(x) = 0$. Write $v = i\partial\bar{\partial}u - \rho = i\sum_{i,j} v_{i\bar{j}}\, dz^i \wedge d\bar{z}^j$. Then, in a neighborhood of x,

(i)
$$\|i\partial\bar{\partial}u-\rho\|^2 = \sum_{i,j,k,l} g^{i\bar{l}} g^{k\bar{j}} v_{i\bar{j}} \bar{v}_{l\bar{k}}.$$

From the equality

(ii)
$$\sum g^{i\bar{l}} g_{n\bar{l}} = \delta^i_n$$

and

$$\frac{\partial}{\partial z_k} g_{i\bar{j}}(x) = \frac{\partial}{\partial \bar{z}_k} g_{i\bar{j}}(x) = 0, \quad g_{i\bar{j}}(x) = \delta_{i\bar{j}},$$

it follows that at x, by differentiating (ii),

(iii)
$$\frac{\partial^2 g^{i\bar{l}}}{\partial z_p \partial \bar{z}_q}(x) = -\frac{\partial^2 g_{l\bar{i}}}{\partial z_p\, \partial \bar{z}_q}(x).$$

The curvature tensor at x in terms of the coordinates (z_i) is given by

$$R_{i\bar{j}k\bar{l}}(x) = \frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l}(x).$$

It follows from (i) and (iii) that, at x,

$$\text{(iv)} \qquad \tfrac{1}{2}\Delta\|i\partial\bar{\partial}u - \rho\|^2(x) = \Big(\sum_{i,j,l,p} -R_{l\bar{i}p\bar{p}}v_{i\bar{j}}\bar{v}_{l\bar{j}} + \sum_{i,j,k,p} -R_{j\bar{k}p\bar{p}}v_{i\bar{j}}\bar{v}_{i\bar{k}}\Big)(x)$$
$$+\sum_{i,j,p}\left(\left|\frac{\partial v_{i\bar{j}}}{\partial z_p}\right|^2 + \left|\frac{\partial v_{i\bar{j}}}{\partial \bar{z}_p}\right|^2\right)(x)$$
$$+\sum_{i,j,p}\left(\frac{\partial^2 v_{i\bar{j}}}{\partial z_p\partial\bar{z}_p}\bar{v}_{i\bar{j}}\right)(x) + \sum_{i,j,p}\left(v_{i\bar{j}}\frac{\partial^2 \bar{v}_{i\bar{j}}}{\partial z_p\partial\bar{z}_p}\right)(x).$$

Recall that

$$\tfrac{1}{2}\Delta u = \sum_{k,l} g^{k\bar{l}}u_{k\bar{l}} = \sum_{k,l} g^{k\bar{l}}\rho_{k\bar{l}},$$

so that

$$\text{(v)} \qquad \sum_{k,l} g^{k\bar{l}}v_{k\bar{l}} = \sum_{k,l} g^{k\bar{l}}(u_{k\bar{l}} - \rho_{k\bar{l}}) = 0.$$

Differentiating (v), we obtain, at x,

$$\sum_p \frac{\partial^2 v_{p\bar{p}}}{\partial z_i\partial\bar{z}_j}(x) - \sum_{k,l} R_{l\bar{k}i\bar{j}}v_{k\bar{l}} = 0.$$

Since $d\rho = 0$ and ρ is of type $(1,1)$, locally $\rho = i\partial\bar{\partial}w$, so that $v_{i\bar{j}} = (\partial^2 v/\partial z_i\partial\bar{z}_j)$ for some smooth v, and

$$\frac{\partial^2 v_{p\bar{p}}}{\partial z_i\partial\bar{z}_j} = \frac{\partial^2 v_{i\bar{j}}}{\partial z_p\partial\bar{z}_p}.$$

Hence, at x,

$$\text{(vi)} \qquad \sum_{i,j,p}\frac{\partial^2 v_{i\bar{j}}}{\partial z_p\partial\bar{z}_p}\bar{v}_{i\bar{j}} = \sum_{i,j,k,l} R_{l\bar{k}i\bar{j}}v_{k\bar{l}}\bar{v}_{i\bar{j}}.$$

Since $v_{i\bar{j}}$ is hermitian symmetric, after a unitary change of coordinates, $v_{i\bar{j}} = a_i\delta_{ij}$, a_i real. Substituting (vi) into (iv),

$$\begin{aligned}\tfrac{1}{2}\Delta\|i\partial\bar{\partial}u-\rho\|^2 &\geq \sum_{i,p} -2R_{i\bar{i}p\bar{p}}a_i^2 + 2\sum_{i,p} R_{i\bar{i}p\bar{p}}a_ia_p \\ &= \sum_{i,p} -2R_{i\bar{i}p\bar{p}}(a_i^2 - 2a_ia_p + a_p^2) \\ &= \sum_{i,p} -2R_{i\bar{i}p\bar{p}}(a_i - a_p)^2.\end{aligned}$$

Since M has nonnegative holomorphic bisectional curvature,

$$\sum_{i,j,k,l} R_{i\bar{j}k\bar{l}}a_i\bar{a}_jb_k\bar{b}_l \leq 0 \quad \text{for all} \quad (a_i), (b_k).$$

In particular, $R_{i\bar{i}p\bar{p}} \leq 0$, so that

$$\tfrac{1}{2}\Delta\|i\partial\bar{\partial}u-\rho\|^2 \geq 0,$$

proving the proposition.

(1.2) Conditions for reducing the Poincaré-Lelong equation by taking traces

To make use of the Bochner identity, we show that $\|i\partial\bar{\partial}u-\rho\|^2$ decays to zero at infinity under suitable growth conditions. More precisely, we have

THEOREM 1.1: *Let M be a complete Kähler manifold of complex dimension $n \geq 2$. Suppose M has nonnegative holomorphic bisectional curvature bounded by c/r^2 and volume $(B(x,r)) \geq cr^{2n}$, where $B(x,r)$ denotes geodesic balls and $c > 0$. Suppose furthermore that $\|\rho\| \leq (c_1/r^2)$, and $f = \mathrm{trace}(\rho)$ where $\|\cdot\|$ denotes norms measured in terms of the Kähler metric. Then, there exists a solution u of $\frac{1}{2}\Delta u = f$, such that u is of order* $O(\log r)$ *and satisfies automatically $i\partial\bar{\partial}u = \rho$.*

For the proof of the theorem we need the following gradient estimate on Riemannian manifolds due to Cheng and Yau [4]. *Gradient estimate on geodesic balls* (Cheng and Yau [4]): Let u be a C^2 function on a Riemannian manifold M. Suppose on the geodesic ball $B(x_0, a)$ u satisfies

$$|\Delta u| \leq C_1(u+c), \quad \|\nabla(\Delta u)\| \leq C_2(u+c).$$

Then, $\|\nabla u\| \le C_3(u+c)(a^2/a^2-r^2)[\max(C_1, C_2^{1/3}, a|K|+1/a)]$, where $r(x)=d(x_0,x)$ is the geodesic distance, and the Ricci curvature dominates $-K$ on $B(x_0,a)$.

PROOF OF THEOREM: First we solve $\frac{1}{2}\Delta u = f$ with $|u| = 0(\log r)$. By Croke [6] the isoperimetric inequality is valid for all compact subdomains of M. (For this we only need nonnegative Ricci curvature and the condition on volume growth.) It follows that the Sobolev inequality with compact support is valid. Let v_R be the solution on $B(R)$ of $\Delta v_R = \chi_{B(r)}$, $v_R \equiv 0$ on $\partial B(r)$, where $B(r) = B(x_0, r)$, with x_0 fixed, $r \le R$, and χ stands for characteristic functions. By the Sobolev inequality

$$\left(\int_{B(R)} |v_R|^{2n/n-1}\right)^{n-1/2n} \le c\left(\int_{B(R)} |\nabla v_R|^2\right)^{1/2},$$

where c is an absolute constant. Write $k = (n/n-1)$. By the Hölder inequality

$$\begin{aligned}
\|v_R\|_{2k} &\le C_1\left(\int_{B(R)} v_R \Delta v_R\right)^{1/2}\\
&= C_1\left(\int_{B(R)} v_R \chi_{B(r)}\right)^{1/2}\\
&\le C_1\left[\left(\int_{B(R)} |v_R|^{2k}\right)^{1/2k}\left(\int_{B(R)} |\chi_{B(r)}|^{2/n+1}\right)^{n+1/2n}\right]^{1/2}\\
&= C_1\|v_R\|_{2k}^{1/2}\|\chi_{B(r)}\|_{2n/n+1}^{1/2}, \quad \text{so that}\\
\|v_R\|_{2k} &\le C_1^2\|\chi_{B(r)}\|_{2n/n+1}.
\end{aligned}$$

To iterate the argument, for any integer $p \ge 0$

$$\begin{aligned}
&\left(\int_{B(R)} |v_R|^{2k^{p+1}}\right)^{1/2k}\\
&= \left(\int_{B(R)} |v_R^{k^p}|^{2k}\right)^{1/2k}\\
&\le C_1\left(\int_{B(R)} |\nabla v_R^{k^p}|^2\right)^{1/2}\\
&= C_1\left(\int_{B(R)} k^{2p} v_R^{2k^p-2}|\nabla v_R|^2\right)^{1/2}\\
&= C_1\left(\int_{B(R)} \frac{k^{2p}}{2k^p-1}\, v_R^{2k^p-1}\chi_{B(R)}\right)^{1/2}\\
&\le C_2 k^{p/2}\|v_R\|_{2k^p}^{k^p-(1/2)}\|\chi_{B(r)}\|_{2k^p}^{1/2}.
\end{aligned}$$

So,

$$\|v_R\|_{2k^{p+1}} \le (C_2 k^{p/2})^{1/kp} \|v_R\|_{2k^p}^{1-(1/2k^p)} \|\chi_{B(R)}\|_{2k^p}^{1/2k^p}.$$

By iteration

$$\|v_R\|_{2k^{p+1}} \le \left(\prod_{q=1}^{p} C_2^{1/k^q}\right)\left(\prod_{q=1}^{p} k^{q/2k^q}\right)$$
$$\times \left(\prod_{q=1}^{p} \|\chi_{B(R)}\|_{2k^q}^{1/2k^q(1-(1/2k^{q+1}))\cdots(1-(1/2k^p))}\right)\|v_R\|_{2k}^{1-(1/2k)}.$$

To get uniform estimates of v_R we show that the infinite products obtained by letting $p \to \infty$ converge. In fact

$$\log\left(\prod_{q=1}^{p} C_2^{1/k^q}\right) = \left(\sum_{q=1}^{p} \frac{1}{k^q}\right) \log C_2 \le C_3 \quad \text{for all} \quad p$$

$$\log\left(\prod_{q=1}^{p} k^{q/2k^q}\right) \le \sum_{q=1}^{p} \frac{q}{2k^q} \log k \le C_4 \quad \text{for all} \quad p.$$

Since $\|\chi_{B(r)}\|_\alpha = [\mathrm{vol}\, B(r)]^{1/\alpha}$, and vol $(B(r)) \le r^{2n}$ because M has non-negative Ricci curvature,

$$\log\|v_R\|_{2k^{p+1}} \le C_5 + \log r\left[\sum_{q=1}^{p} \frac{n}{k^q}\left(\frac{1}{2k^q}\right)\left(1-\frac{1}{2k^{q+1}}\right)\cdots\left(1-\frac{1}{2k^p}\right)\right.$$
$$\left. + (n+1)\left(1-\frac{1}{2k}\right)\left(1-\frac{1}{2k^2}\right)\cdots\left(1-\frac{1}{2k^p}\right)\right]$$
$$+ \left(1-\frac{1}{2k}\right)\|v_R\|_{2k}.$$

Clearly the term inside the bracket $\le (n/2)\sum_{q=1}^{\infty}(1/k^{2q}) + (n+1) < \infty$. Since $\|v_R\|_{2k} \le C_1^2 \|\chi_{B(r)}\|_{2n/n+1}$, we obtain by taking limits that

$$\sup_{B(R)} |v_R| \le C r^\beta.$$

Notice that $\sup_{B(R)} |v_R|$ is now independent of R. Let $v = \lim_{R\to\infty} v_R$. By scaling the metric g so that $g = rg'$, $\Delta' v$ (with respect to g'), is then $r^2 \Delta v$, which therefore gives

$$\sup_{B(R)} |v| \le C r^2 \quad \text{for all} \quad R > r.$$

(Recall that $\Delta v = \chi_{B(R)}$, obtained by solving Dirichlet boundary problems on subdomains.)

Fix a number $R > 1$. The trace of ρ can be decomposed as $\Sigma_{p=1}^{\infty} f\chi_{B(R^p)-B(R^{p-1})}$. Let u_p solve $\Delta u_p = f\chi_{B(R^p)-B(R^{p-1})}$, obtained by solving on subdomains with Dirichlet boundary conditions and taking limits. Since $|f| = 0(1/r^2)$, by the above $\sup_M |u_p| \leq C$. Let $u = \Sigma_{p=1}^{\infty}(u_p - u_p(x_0))$. u_p is harmonic on $B(R^{p-1})$. Using the Sobolev inequality with compact support as above and the abstract John-Nirenberg inequality in Bombieri and Giusti [2], the iteration technique of Moser [12] can be adapted to prove the Harnack inequality for positive harmonic functions on geodesic balls in M and hence the Hölder estimate

$$|u_p(x) - u_p(x_0)| \leq C\left(\frac{d(x_0, x)}{R^{p-1}}\right)^{\gamma}$$

for some $\gamma > 0$, for $d(x_0, x) < R^{p-1}$.
From this clearly $u = \Sigma_{p=1}^{\infty}(u_p - u_p(x_0))$ converges and $u = 0(\log r)$.

Estimate for the Green function

The Green function with pole x_0, $G(x_0, y)$, exists on the manifold M. Moreover, there exist positive constants A and B such that $(A/d(x_0, y)^{2n-2}) \leq |G(x_0, y)| \leq (B/d(x_0, y)^{2n-2})$. To prove this, observe first that $G_R(x_0, y)$, the Green function on the geodesic ball $B(x_0, R)$ with pole x_0, is always well defined, harmonic on $B(x_0, R) - \{x_0\}$ and positive. Let $p = d(x_0, y)$ and $R > p$. On $B(y, \frac{1}{2}p)$, by the Harnack inequality

$$\inf_{z \in B(y, 1/2p)} |G_R(x_0, z| > C_M \sup_{z \in B(y, 1/2p)} |G_R(x_0, z)|,$$

where C_M is a constant depending only on M. Let v_R be the solution of $\Delta\, v_R = \chi_{B(y, 1/2p)}$ on $B(x_0, R)$, $v_R\,|_{\partial B(x_0, R)} \equiv 0$. By the estimates in Theorem 1,

$$\sup_{B(x_0, R)} |v_R| \leq C d(x_0, y)^2.$$

From

$$|v_R(x_0)| \geq \inf_{z \in B(y, 1/2p)} |G_R(x_0, z)| \cdot \text{volume}(B(y, \tfrac{1}{2}p))$$

and

$$\text{volume}(B(y, \tfrac{1}{2}p)) \geq Cd(x_0, y)^{2n},$$

and the Harnack inequality, we obtain

$$\sup_{z \in B(y, 1/2p)} |G_R(x_0, z)| \leq \frac{A}{d(x_0, y)^{2n-2}}.$$

From a standard comparison theorem

$$|G_R(x_0, y)| \geq B\left(\frac{1}{d(x_0, y)^{2n-2}} - \frac{1}{R^{2n-2}}\right).$$

Taking limits as $R \to \infty$ we obtain the estimate for the Green function.

Gradient estimates of u

The Green function $G(x; y)$ is harmonic in x on $M - \{y\}$ and positive. From the estimate of the Green function $(A/d(x, y)^{2n-2}) \leq |G(x, y)| \leq (B/d(x, y)^{2n-2})$ (A and B being independent of x) and the gradient estimate of Cheng and Yau [4] it follows that $\|\nabla_y G(x, y)\| \leq (C/d(x, y)^{2n-1})$. From Riesz representation the solution of $\frac{1}{2}\Delta u = f$ we obtained can be given by

$$u(x) = \lim_{R \to \infty} \int_{B(x_0, R)} (G_R(x; y) - G_R(x_0; y))f(y) + C$$

where G_R stands for the Green kernel on $B(x_0, R)$, and $f(y) = 0(1/r^2)$. It follows by differentiating the Green kernel that $\|\nabla u\| = 0(1/r)$.

Second order estimates of u

We are going to estimate the average of $\|\partial\bar{\partial}u\|^2$ over geodesic balls. Fix a base point x_0. Let ϕ be a Lipschitz cut off function on $B(x_0; R)$, $\phi \equiv 1$ on $B(x_0; R/2)$, $\phi \equiv 0$ outside $B(x_0; 3R/4)$ and $\|\nabla\phi\| \leq (C/R)$ on $B(x_0; R)$. We first assume that $B(x_0, R)$ lies in a coordinate neighborhood with holomorphic coordinates (z_j). With respect to this coordinate system, the covariant and contravariant metric tensor will be denoted by $(g_{i\bar{j}})$ and $(g^{i\bar{j}})$. u_i, $u_{i\bar{j}}$ will mean $\partial u/\partial z_i \cdot \partial^2 u/\partial z_i \partial \bar{z}_j$, etc; and $\partial g^{i\bar{j}}/\partial z_k$ will be written $g^{i\bar{j}}_{,k}$ etc. Then, integrating by parts and assuming u real

$$\int_{B(x_0; R)} \phi^2 \|\partial\bar{\partial}u\|^2$$

$$= \int_{B(x_0;\,R)} \sum_{i,j,k,l} \phi^2 g^{i\bar{k}} g^{l\bar{j}} u_{i\bar{j}} \bar{u}_{k\bar{l}} \det(g_{p\bar{q}}) \left(\frac{i}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$$

$$= \int_{B(x_0;\,R)} - \sum_{i,j,k,l} [2\phi\phi_{\bar{j}} g^{i\bar{k}} g^{l\bar{j}} u_i u_{l\bar{k}} \det(g_p \bar{q})$$
$$+ \phi^2 g^{i\bar{k}}_{,\bar{k}} g^{l\bar{j}} u_i u_{l\bar{j}} \det(g_{p\bar{q}})$$
$$+ \phi^2 g^{i\bar{k}} g^{l\bar{j}}_{,k} u_i u_{l\bar{j}} \det(g_{p\bar{q}})$$
$$+ \phi^2 g^{i\bar{k}} g^{l\bar{j}} u_i u_{l\bar{k}} \det(g_{p\bar{q}})_{,j}$$
$$+ \phi^2 g^{i\bar{k}} g^{l\bar{j}} u_i u_{l\bar{k}\bar{j}} \det(g_{p\bar{q}})] \left(\frac{i}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$$

We then perform integration by parts on the last term to get

$$\int_{B(x_0;\,R)} \sum_{i,j,k,l} [\phi^2 g^{i\bar{k}} g^{l\bar{j}} u_i u_{l\bar{k}\bar{j}} \det(g_{p\bar{q}})] \left(\frac{i}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$$

$$= \int_{B(x_0;\,R)} - \sum_{i,j,k,l} [2\phi\phi_{\bar{k}} g^{i\bar{k}} g^{l\bar{j}} u_i u_{l\bar{j}} \det(g_{p\bar{q}})$$
$$+ \phi^2 g^{i\bar{k}}_{,\bar{k}} g^{l\bar{j}} u_i u_{l\bar{j}} \det(g_{p\bar{q}})$$
$$+ \phi^2 g^{i\bar{k}} g^{l\bar{j}}_{,\bar{k}} u_i u_{l\bar{j}} \det(g_{p\bar{q}})$$
$$+ \phi^2 g^{i\bar{k}} g^{l\bar{j}} u_i u_{l\bar{j}} \det(g_{p\bar{q}})_{,\bar{k}}$$
$$+ \phi^2 g^{i\bar{k}} g^{l\bar{j}} u_{i\bar{k}} u_{l\bar{j}} \det(g_{p\bar{q}})] \left(\frac{i}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n.$$

The two integration by parts are obtained by applying Stokes' Theorem to the compactly supported $(2n-1)$ forms

$$\Phi_1 = \phi^2 \sum_{i,j,k,l} g^{i\bar{k}} g^{l\bar{j}} u_i u_{l\bar{k}} \det(g_{p\bar{q}})$$
$$\left(\frac{i}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_j \wedge \widehat{d\bar{z}_j} \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$$

$$\phi_2 = \phi^2 \sum_{i,j,k,l} g^{i\bar{k}} g^{l\bar{j}} u_i u_{l\bar{j}} \det(g_{p\bar{q}})$$
$$\left(\frac{i}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots dz_k \wedge \widehat{d\bar{z}_k} \wedge \cdots dz_n \wedge d\bar{z}_n$$

where $\wedge$ denotes removal of the differential form. Since both are obtained by suitably contracting (with the contravariant metric tensor) and taking wedge products of the forms $\partial\bar{\partial}u$, ∂u and the volume form, they are in fact independent of the choice of local coordinates. Integration by parts is therefore valid on any $B(x_0; R)$ and the integrand can be computed pointwise by using complex geodesic coordinates, $(g_{i\bar{j}}(x) = \delta_{ij}$ and $(\partial g_{i\bar{j}}/\partial z_k)(x) = (\partial g_{i\bar{j}}/\partial \bar{z}_k)(x) = 0)$, which simplies to the form

$$\int_{B(x_0;\, R)} \phi^2 \|\partial\bar{\partial}u\|^2 = \int_{B(x_0;\, R)} \phi^2(\tfrac{1}{2}\Delta u)^2 - 2\phi\langle \partial u \wedge \bar{\partial}\phi, \partial\bar{\partial}u\rangle$$
$$+ 2\phi\langle \partial u, \partial\phi\rangle(\tfrac{1}{2}\Delta u).$$

Recall that $\|\nabla u\| = 0(1/r)\Delta u = 0(1/r^2)$ and $\|\nabla\phi\| \leq (C/R)$ on $B(x_0; R)$. From the Schwarz inequality it follows that for R large

$$\int_{B(x_0;\, R)} \phi^2 \|\partial\bar{\partial}u\|^2 \leq \frac{C}{R}\int_{B(x_0;\, R)} \frac{C_1^2}{1+r^2} + \sqrt{\int_{B(x_0;\, R)} \phi^2\|\partial\bar{\partial}u\|^2}$$
$$+ \int_{B(x_0;\, R)} \frac{C_2}{1+r^4}.$$

Therefore,

$$\int_{B(x_0;\, R)} \phi^2 \|\partial\bar{\partial}u\|^2 \leq C_3 R^{2n-3} + C_4.$$

By Proposition (1.1) $\|i\partial\bar{\partial}u - \rho\|^2$ is subharmonic. Since $\rho = 0(1/r^2)$, at any x with $d(x_0; x) < (R/2)$

$$\|i\partial\bar{\partial}u - \rho\|^2(x)$$
$$\leq \frac{C_5}{R^{2n}}\int_{B(x_0;\, R)} 2\|\partial\bar{\partial}u\|^2 + 2\|\rho\|^2 \text{ (sub-mean value inequality)}$$
$$\leq \frac{C_6}{R^3},$$

where the sub-mean value inequality is obtained by using Moser's iteration technique as above. Fixing x and letting $R \to \infty$ we conclude that $\|i\partial\bar{\partial}u - \rho\|^2(x) = 0$ proving Theorem 1.1.

(1.3) Application to Kähler geometry

Finally, we make two applications of the Bochner formula to study noncompact complete Kähler manifolds of nonnegative holomorphic bisectional curvature. The results are summarised in the following theorem.

THEOREM 1.2: *Let M be a complete Kähler manifold of nonnegative holomorphic bisectional curvature of dimension $n \geq 2$. Let R denote the scalar curvature and $B(x, r)$ denote geodesic balls. Then*

(1) *If $R = 0(1/r^{2+\epsilon})$ volume $(B(x, r)) \geq Cr^{2n}$ $(C > 0)$, then M is isometrically biholomorphic to $\mathbb{C}^n$ with the Euclidean metric under either of the following additional assumptions:*
   (i) *the Riemannian sectional curvatures are nonnegative, or*
   (ii) *the complex manifold M is Stein.*

(2) *If the Ricci curvatures are positive and there exist positive constants C_1, C_2 such that $(C_1/1 + r^2) \leq R \leq (C_2/1 + r^2)$, and volume $(B(x, r)) \geq Cr^{2n}$ $(C > 0)$, then M is a Stein manifold.*

PROOF: (1) By Theorem 1.1 one can solve for $\partial\bar{\partial}u$ = Ricci form by reducing it to $\frac{1}{2}\Delta u = R$. When $R = 0(1/r^{2+\epsilon})$, the solution obtained in Theorem 1 is actually bounded. Furthermore, by a simple estimate of the infinite behavior of the solution u in terms of harmonic measures, which can be estimated by the Poincaré inequality, the solution u thus obtained is actually $0(1/r^{\alpha})$ for some $\alpha > 0$. We have thus obtained a smooth bounded plurisubharmonic function u (since the Ricci form is positive semidefinite) decaying to zero at infinity. To prove Theorem 1.2(1), it suffices to show that $\partial\bar{\partial}u = 0$, so that the Ricci form and hence the holomorphic bisectional curvatures vanish identically. First we show that $(\partial\bar{\partial}u)^n = 0$. In fact, if $\partial B(x_0, r)$ is smooth,

$$\int_{B(x_0, r)} (\partial\bar{\partial}u)^n = \int_{\partial B(x_0, r)} \bar{\partial}u \wedge (\partial\bar{\partial}u)^{n-1}.$$

Since $\partial\bar{\partial}u$ = Ricci form, $\|\partial\bar{\partial}u\| = 0(1/r^{2+\epsilon})$. By the estimates of ∇u in Theorem 2, $\|\bar{\partial}u\| = 0(1/r)$. Furthermore, since M carries nonnegative Ricci curvature, the exponential map at x_0 is volume decreasing, implying volume $(\partial B(x_0, r)) = 0(r^{2n-1})$. Hence,

$$\int_{B(x_0, r)} (\partial\bar{\partial}u)^n = 0\left(\frac{1}{r^{(n-1)\epsilon}}\right).$$

But $n \geq 2$ and $(\partial\bar{\partial}u)^n \geq 0$. Taking the limit as $r \to \infty$, $(\partial\bar{\partial}u)^n \equiv 0$ on M identically. To show that $u \equiv 0$ we make use of one of the additional assumptions. If we assume (ii) the complex manifold M is Stein, then M can be embedded as a closed complex submanifold of some $\mathbb{C}^N$, with coordinates $(z_1, \ldots, z_N)$. Let φ denote the restriction of $\Sigma_{i=1}^N |z_i|^2$ to M. Suppose u is not identically zero. Then $M_c = \{u < c\}$ is relatively compact for $c < 0$. Let x be a point on ∂M_c such that φ is maximum. We can choose c such that ∂M_c is smooth. Then x is a strictly pseudoconvex boundary point of M_c. Since u is a defining function for M_c, $\partial\bar{\partial}u$ is positive definite on the complex tangent space of ∂M_c at x. (Strict pseudoconvexity is independent of the choice of a C^2 defining function.) $\partial\bar{\partial}e^u = e^u\partial\bar{\partial}u + e^u\partial u \wedge \bar{\partial}u$ is then positive definite at x. But the argument in the previous paragraph shows $(\partial\bar{\partial}e^u)^n \equiv 0$ too, giving a contradiction. Hence $u \equiv 0$.

If we assume (i) the Riemannian sectional curvatures are nonnegative, we can make use of the geometry of such manifolds in Cheeger and Gromoll [3]. Under the growth condition, volume $(B(x, r) \geq Cr^{2n}$, we are going to show that there exists a geodesically convex function v whose minimum is attained at a single point, obtained from the Busemann functions. Then by piecing v with the function u we obtain a bounded plurisubharmonic function u' which is strictly plurisubharmonic at one point in a weak sense. Finally, we justify by smoothing arguments that $\int_M (\partial\bar{\partial}u')^n > 0$. This contradicts the fact that $(\partial\bar{\partial}u')^n \equiv 0$, similarly obtained as $(\partial\bar{\partial}u)^n \equiv 0$. The proof is pretty long and will involve a number of lemmas.

From now on M is assumed to have nonnegative sectional curvature. Let γ be a geodesic ray on M. Then $g_\gamma(x) = \lim_{t\to\infty}(d(\gamma(t), x) - t)$ is called the Busemann function associated to γ; $-g_\gamma$ is geodesically convex by Cheeger and Gromoll [3]. For any set $E \subset M$ we define the totally convex set $C_E = \{x \in M : -g_\gamma(x) \leq 0$ for all geodesic rays issuing from $q \in E\}$. Then sets C_E are compact and totally convex. The set of nonempty C_E can be partially ordered by $C_{E_1} \leq C_{E_2}$ if $E_1 \supset E_2$. We can choose a minimal C_E which has to be nonempty by compactness. We assert that C_E must reduce to a single point. Otherwise pick $a, b \in C_E$ and join them by a geodesic ω, which must lie on C_E by geodesic convexity. Pick a point c in the interior of the curve ω, and consider all geodesics issuing from c. We have the following lemma.

LEMMA 1: *Suppose M is the complete Kähler manifold in Theorem 1.2(1) satisfying the condition (ii), the Riemannian sectional curvatures are nonnegative. Let ω, c be as in the last paragraph. Then, all geodesic rays γ from c must be perpendicular to ω. Moreover, this implies that, for any $\epsilon > 0$, volume $(B(c, r)) \leq \epsilon r^{2n}$ for r large enough. Hence by contradiction C_E must reduce to a single point.*

Proof of Lemma: Suppose γ is a geodesic ray from c which is not perpendicular to ω. Then, for a fixed t, $t - d(\gamma(t), x) > 0$ for some $x \in \omega$. In particular, $-g_\gamma(x) > 0$ so that x lies outside the half space defined by $\{-g_\gamma(x) \le 0\}$, contradicting with the minimality of C_E. Define now a cone K_α in the tangent space T_c of c as follows. $v \in K_\alpha$ if it makes an angle $\ge \alpha$, $\alpha < (\pi/2)$ fixed but arbitrary, with the tangent space of ω at c. We claim that $\exp_c(K_\alpha)$ almost covers M in the sense that the complement is relatively compact. Otherwise pick a sequence $\{x_n\}$ outside $\exp_c(K_\alpha)$, and join them to c by minimal geodesics γ_n. The γ_n converges to a geodesic ray intersecting ω at an angle $\le \alpha < (\pi/2)$, contradicting the first assertion. Since the exponential map is volume decreasing, volume $(B(c, r)) \le \epsilon r^{2n}$ for r large enough.

Now we will piece the plurisubharmonic functions u from $\frac{1}{2}\Delta u =$ scalar curvature and v coming from the Busemann functions. Suppose $b < a < 0$. Let ρ be a smooth function on the negative real axis such that $\rho(t) = 0$ for $t \ge a$, $\rho(t) = 1$ for $t \le b$, $0 \le \rho \le 1$ in between. Consider the function $Ce^u + \rho(u)\exp(\alpha \exp v)$

$$
\begin{aligned}
\partial\bar\partial(\rho(u)\exp(\alpha \exp v) &= \rho(u)\partial\bar\partial \exp(\alpha \exp v) + 2Re\, \partial\rho(u) \\
&\quad \wedge \bar\partial \exp(\alpha \exp v) + \partial\bar\partial\rho(u)\cdot \exp(\alpha \exp v) \\
&= \exp(\alpha \exp v)[\rho(u)(\alpha \exp v + \alpha^2 \exp 2v)\partial v \wedge \bar\partial v \\
&\quad + \rho(u)\alpha \exp v\, \partial\bar\partial v + 2Re(\alpha \exp v)\rho'(u)\partial u \wedge \bar\partial v \\
&\quad + \rho''(u)\partial u \wedge \bar\partial u + \rho'(u)\partial\bar\partial u] \\
\partial\bar\partial C e^u &= Ce^u \partial\bar\partial u + Ce^u \partial u \wedge \bar\partial u.
\end{aligned}
$$

There exists a C_0 (depending on ρ) such that

$$
\begin{aligned}
C_0 e^u \partial\bar\partial u + C_0 e^u \partial u \wedge \bar\partial u + \exp(\alpha \exp v)\rho''(u)\partial u \wedge \bar\partial u \\
+ \exp(\alpha \exp v)\rho'(u)\partial\bar\partial u \ge 0.
\end{aligned}
$$

We use the inequality

$$
\partial(Au + v) \wedge \bar\partial(Au + v) = A^2 \partial u \wedge \bar\partial u + 2ARe\partial u \wedge \bar\partial v + \partial v \wedge \bar\partial v \ge 0
$$

in the sense of distribution. We can choose ρ so that $\rho'' > 0$ on $[a - \epsilon, a]$ and

$$
\sqrt{\rho''\rho} \ge \tfrac{1}{2}\rho' \quad \text{on} \quad [a - \epsilon, a].
$$

Then, for suitable $\alpha > 0$,

$$\rho(u)(\alpha \exp v + \alpha^2 \exp 2v)\partial v \wedge \bar{\partial} v + 2Re(\alpha \exp v)\rho'(u)\partial u \wedge \bar{\partial} v$$
$$+ \rho''(u)\partial u \wedge \bar{\partial} u \geq 0 \quad \text{on} \quad [a - \epsilon, a],$$

and

$$\rho(u)(\alpha \exp v + \alpha^2 \exp 2v)\partial v \wedge \bar{\partial} v + 2Re(\alpha \exp v)\rho'(u)\partial u \wedge \bar{\partial} v$$
$$+ (\rho''(u) + C_1 e^u)\partial u \wedge \bar{\partial} u \geq 0 \quad \text{on} \quad [b, a - \epsilon],$$

so that if $u' = (C_0 + C_1)e^u + \rho(u)\exp(\alpha \exp v)$,

$$\partial\bar{\partial} u' \geq 0 \text{ on } M \text{ in the sense of distribution.}$$

Recall that the geodesically convex function v attains a unique minimum at some point c, $v(c) = 0$. Fix a small holomorphic coordinate neighborhood U of c with coordinates $z_1 \dots z_n$ such that $z_i(c) = 0$. We can assume $B(1) \subset U$, and $v \geq \delta' > 0$ on $\partial B(1)$. We claim the following is true for plurisubharmonic functions.

LEMMA 2: *Let φ be a smooth weakly plurisubharmonic function on a neighborhood of $\overline{B(1)}$ such that $\varphi(0) = 0$, $\varphi \geq 0$, $\varphi(z) \geq 1$ on $\partial B(1)$, and $\partial(z) \leq \frac{1}{4}$ on $B(\alpha)$, $0 < \alpha < 1$. Then*

$$\int_{B(1)} (\partial\bar{\partial}\varphi)^n \geq C > 0, \ c \text{ independent of } \varphi.$$

PROOF: Write $r^2 = \Sigma |z_i|^2$, $\varphi - (r^2/2) \geq \frac{1}{2}$ on $\partial B(1)$, $(\varphi - (r^2/2))(0) = 0$. Write $U(a) = \{z \in \overline{B(1)} = \varphi(z) - (|z|^2/2) \leq a\}$. When $n = 1$,

$$\int_{U(1/2)} \left(\varphi - \frac{r^2}{2} - \frac{1}{2}\right)\Delta\left(\varphi - \frac{r^2}{2}\right) = -\int_{U(1/2)} \left|\nabla\left(\varphi - \frac{r^2}{2}\right)\right|^2 \leq 0.$$

Since $\varphi - (r^2/2) - \frac{1}{2} = 0$ on $\partial U(\frac{1}{2})$

$$\int_{U(1/2)} \left(\frac{1}{2} + \frac{r^2}{2} - \varphi\right)\Delta\varphi \geq \int_{U(1/2)} \left(\frac{1}{2} + \frac{r^2}{2} - \varphi\right)\Delta\left(\frac{r^2}{2}\right)$$
$$\leq \frac{1}{4}\int_{B(\delta)} \Delta\left(\frac{r^2}{2}\right) \quad \text{since} \quad \varphi(z) \leq \frac{1}{4} \quad \text{on} \quad B(\delta)$$
$$= \frac{1}{2}\,\text{vol}\,(B(\delta)) > 0.$$

Hence $\int_{B(1)} \Delta\varphi \geq \int_{U(1/2)} \Delta\varphi$ (since $0 \leq \frac{1}{2} + (r^2/2) - \varphi \leq 1$ on $U(\frac{1}{2})$)

$$\geq \tfrac{1}{2}\operatorname{vol}(B(\delta)) = C > 0.$$

In higher dimensions we proceed by induction

$$\begin{aligned}&\int_{U(1/2)} \left(\varphi - \frac{r^2}{2} - \frac{1}{2}\right)\partial\bar\partial\left(\varphi - \frac{r^2}{2}\right) \wedge (\partial\bar\partial\varphi)^{n-1}\\ &= -\int_{U(1/2)} \partial\left(\varphi - \frac{r^2}{2}\right) \wedge \bar\partial\left(\varphi - \frac{r^2}{2}\right) \wedge (\partial\bar\partial\varphi)^{n-1} \leq 0\end{aligned}$$

since $(\partial\bar\partial\varphi)^{n-1} \geq 0$. Hence,

$$\int_{U(1/2)} \left(\frac{1}{2} + \frac{r^2}{2} - \varphi\right)(\partial\bar\partial\Phi)^n \geq \int_{U(1/2)} \left(\frac{1}{2} + \frac{r^2}{2} - \varphi\right)\partial\bar\partial\left(\frac{r^2}{2}\right) \wedge (\partial\bar\partial\varphi)^{n-1}.$$

By the same method and induction,

$$\begin{aligned}&\int_{U(1/2)} \left(\frac{1}{2} + \frac{r^2}{2} - \varphi\right)\left(\partial\bar\partial\left(\frac{r^2}{2}\right)\right)^p \wedge (\partial\bar\partial\varphi)^{n-p}\\ &\geq \int_{U(1/2)} \left(\frac{1}{2} + \frac{r^2}{2} - \varphi\right)\left(\partial\bar\partial\left(\frac{r^2}{2}\right)\right)^{p+1} \wedge (\partial\bar\partial\varphi)^{n-p-1};\end{aligned}$$

hence,

$$\begin{aligned}\int_{B(1)} (\partial\bar\partial\varphi)^n &\geq \int_{U(1/2)} (\partial\bar\partial\varphi)^n \geq \frac{1}{4}\int_{B(\delta)} \left(\partial\bar\partial\left(\frac{r^2}{2}\right)\right)^n\\ &= \frac{1}{4} 2^n \text{ volume } (B(\delta)) = C > 0.\end{aligned}$$

Recall now $u' = Ce^u + \rho(u)\exp(\alpha \exp v)$ is such that $\partial\bar\partial u \geq 0$ (in the sense of distribution), $v(c) = 0$, and c is the unique minimal point of v. u' is smooth outside a compact set K. Cover M by a locally finite family of open coordinate balls (U_i) such that U_1 is an open neighborhood of c. We assume further that $U_i \Subset U_i'$, where U_i' are coordinate balls. For each U_i intersecting K we define $u'_{\epsilon,i}$ by smoothing u' with the standard symmetric kernels. On each U_i disjoint from K, define $u'_{\epsilon,i} = u'$. Each $u'_{\epsilon,i}$ is then smooth and plurisubharmonic, and converges pointwise to u'. Recall that v is a supremum of Busemann functions so that $|v(x) - v(y)| \leq d(x, y)$. Hence, $u'_{\epsilon,i}$ actually converges to u' uniformly on U_i. By taking $\rho_1 \equiv 1$ in a neighborhood $U \Subset U_1$ of c and applying Lemma 2 to U, we have $\int_U (\partial\bar\partial u')^n \geq C > 0$, for some C

independent of ϵ. From the definition of $u'_{\epsilon,i}$ and $\Sigma_i \rho_i = 1$,

$$\partial\bar{\partial}u'_\epsilon = \sum_i \rho_i \partial\bar{\partial}u'_{\epsilon,i} + \sum_i \partial\rho_i \wedge \bar{\partial}u'_{\epsilon,i} + \sum_i (\partial\bar{\partial}\rho_i)u'_{\epsilon,i}$$

$$= \sum_i \rho_i \partial\bar{\partial}u'_{\epsilon,i} + \sum_i \partial\rho_i \wedge \bar{\partial}(u'_{\epsilon,i} - u') + \sum_i \partial\bar{\partial}\rho_i(u'_{\epsilon,i} - u').$$

Since $u'_{\epsilon,i} \neq u'$ only possibly for $U_i \cap K \neq \phi$, by integrating by parts

$$\int_{M-U} (\partial\bar{\partial}u'_\epsilon)^n \geq -C(\epsilon) \sup_{\substack{0\leq k\leq n-1 \\ i}} \int \|(\partial\bar{\partial}u'_{\epsilon,i})^k\|,$$

where $C(\epsilon) \to 0$ as $\epsilon \to 0$.
The last integral is nevertheless bounded independent of ϵ and i, by the following lemma of Chern-Levine-Nirenberg [5].

LEMMA 3 (Chern-Levine-Nirenberg [5]): *Suppose $\Delta_1 \Subset \Delta$ are polydiscs. $u_1, \ldots, u_r$ are C^2 plurisubharmonic functions in Δ with $0 < u_i < 1$. Let $J = (j_1, \ldots, j_r)$, $K = (k_1, \ldots, k_r)$, $1 \leq j_1 < \cdots < j_r \leq n$ and $1 \leq k_1 < \cdots < k_r \leq n$ be multi-indices. $U_{J\bar{K}}$ will denote the coefficient of $dz_{j_1} \wedge d\bar{z}_{k_1} \wedge \cdots \wedge dz_{j_r} \wedge d\bar{z}_{k_r}$ in $\partial\bar{\partial}u_1 \wedge \cdots \wedge \partial\bar{\partial}u_r$. dV will denote the Euclidean volume. Then*

$$\int_{\Delta_1} |U_{J\bar{K}}|\, dV \leq C', \textit{ independent of } u_i.$$

From this lemma it follows that

$$(*) \qquad \int_M (\partial\bar{\partial}u'_\epsilon)^n \geq C - C(\epsilon)C'.$$

Since u'_ϵ is smooth and equal to u outside a compact set, the argument at the beginning of the proof implies that

$$\int_M (\partial\bar{\partial}u'_\epsilon)^n = 0,$$

which is a plain contradiction with (*) when ϵ is sufficiently small. Hence in the case of (i) the Riemannian sectional curvatures are nonnegative $u \equiv 0$, so that M is isometrically biholomorphic to $\mathbb{C}^n$ too.

PROOF OF THEOREM 1.2(2): We assume the Ricci form is positive definite and there exists $C_1, C_2 > 0$, $(C_1/1+r^2) \leq R \leq (C_2/1+r^2)$, and volume $(B(x, r)) \geq Cr^{2n}$. By Theorem 1 we can solve $\partial\bar{\partial}u =$ Ricci form by reducing it to $\frac{1}{2}\Delta u = R$ (scalar curvature). To prove Theorem 2(2), by Grauert's solution of the Levi problem, it suffices to show that u is an exhaustion function. u is strictly plurisubharmonic because the Ricci form is positive definite. Fix some $r > 1$. We can write $R = \Sigma_{p=1}^{\infty} R_p$, where R_p is zero outside $B(x_0, r^p)$ and $(C_3/r^{2p}) \leq R_p \leq (C_4/r^{2p})$, $C_3, C_4 > 0$ fixed. Let u_p be the solution of $\frac{1}{2}\Delta u_p = R_p$ on M, $u_p(x_0) = 0$. By the supremum estimate in Theorem 1 for the Laplacian, $\sup|u_p| \leq C$ independent of p. Consider the solution u of $\frac{1}{2}\Delta u = R$ obtained by taking $u = \Sigma_{p=1}^{\infty} u_p$. Fix a base point x_0, and fix x such that $r^q \leq d(x_0; x) \leq r^{q+1}$. We are interested in estimating a lower bound $u(x)$. Let $u_{p,k}$ be the solution of $\frac{1}{2}\Delta u_{p,k} = R_p$ on $B(x_0, k)$, $u_{p,k} \equiv$ constant on $\partial B(x_0, k)$ and $u_{p,k}(0) = 0$. Then u_p is the limit of $u_{p,k}$ uniformly on compact sets. From a standard comparison theorem $\Delta d(x_0, x)^2 \leq 4n$. Therefore, there exists a constant $C' > 0$ such that $u_{p,k} \geq C'$ on $\partial B(x_0, k)$ whenever $k \geq r^p$. Fix now any x such that $r^q \leq d(x_0, x) \leq r^{q+1}$,

$$u(x) = \sum_{p \leq q} u_p(x) + \sum_{p \geq q+1} u_p(x).$$

We estimate the terms $w_1(x) = \Sigma_{p\leq q} u_p(x)$ and $w_2(x) = \Sigma_{p\geq q+1} u_p(x)$ separately. For $p \leq q$, u_p is harmonic at $x(\frac{1}{2}\Delta u_p = 0$ outside $B(x_0, r^p))$. By the estimates in Theorem 1, $u_p \geq -C$ on $\partial B(x_0, r^p)$. By comparing to harmonic measures using estimates for the Green function at x_0, we obtain

$$u_p(x) \geq -C + (C + C')\left(1 - \frac{C'' r^{p(2n-2)}}{d(x_0, x)^{2n-2}}\right)(C'' \geq 1).$$

There exists an m such that for $q \geq p + m$, we have

$$u_p(x) \geq \epsilon > 0.$$

Let α be the absolute value of $1 - C''$. Then,

$$w_1(x) \geq \epsilon(q - m) - m(C(1 + \alpha) + C'\alpha).$$

To estimate w_2, by the gradient estimate on Riemannian manifolds (Cheng and Yau [4]), $\|\nabla_y G(x, y)\| = 0(1/d(x, y)^{2n-1})$, which gives by integration $\|\nabla u\| \leq (C_1/r^p)$ on all of $B(x_0, r^p)$. Hence,

$$w_2(x) \geq -\sum_{p \geq q+1} \frac{C_1}{r^p} d(x_0; x) \geq -\sum_{i=0}^{\infty} \frac{C_1}{r^i} \geq -C_2.$$

Finally,

$$u(x) = w_1(x) + w_2(x) \geq \epsilon(q-m) - m(C(1+\alpha) + C'_\alpha) - C_2$$

diverges to infinity as $d(x_0, x) \to \infty$. (Actually, $u(x) \geq \epsilon \log d(x_0, x) - C_3$.) This finishes the proof of Theorem 1.2.

§2. The Poincaré-Lelong equation on manifolds with a pole

(2.1) Conditions for reducing the equation by taking traces

In this section we study the Poincaré-Lelong equation $\partial\bar{\partial}u = \rho$ on a complete Kähler manifold with a pole p (i.e., the exponential map at p is a diffeomorphism) under suitable curvature assumptions. The basic tool comes from the L^2-estimate of $\bar{\partial}$ on complete Kähler manifolds as developed by Andreotti and Vesentini [1]. Since we shall allow M to have curvature of mixed signs, the weight functions in solving $\bar{\partial}$ (Hörmander [9]) will come from geometric comparison with a certain model with nonpositive (radial) curvature.

In Theorem 2.1 we study sufficient growth conditions on the curvature tensor of M and on the (1, 1) form ρ in order to solve $\partial\bar{\partial}u = \rho$ by reducing it to $\frac{1}{2}\Delta u = \text{trace}(\rho)$. In Theorem 2.2 we shall apply this to find conditions under which M is isometric or biholomorphic to $\mathbb{C}^n$.

THEOREM 2.1: *Let M be a complete Kähler manifold of dimension $n \geq 2$ such that the exponential map at p is a diffeomorphism. Let $r(x)$ denote geodesic distances from p. Suppose the Riemannian sectional curvature is bounded by*

$$-\frac{A_\epsilon}{(1+r^2)^{1+\epsilon}} \leq \text{sectional curvature} \leq \frac{A_\epsilon}{(1+r^2)^{1+\epsilon}}$$

where A_ϵ is a suitably small constant depending on ϵ, and suppose ρ is a closed (1, 1) form such that $\|\rho\| = 0(1/r^2)$ ($\|\ \|$ denotes norms in the given Kähler metric). Then, there exists a unique solution of $\frac{1}{2}\Delta u = \text{trace}(\rho)$ such that $u = 0(\log r)$. Moreover, u automatically satisfies the Poincaré-Lelong equation $i\partial\bar{\partial}u = \rho$.

Before the main part of the proof, we shall need a few lemmas. First, we shall need the following criterion which guarantees that the exponential map at p is a quasi-isometry.

LEMMA 1 (on quasi-isometry) (Greene and Wu [7]): *Let M be a complete Riemannian manifold with a pole p. Let $r(x)$ denote the geodesic distance from the pole p and $\partial/\partial r_p$ denote the radial tangent vector of unit length, at any $x \neq p$. Let $K(X, Y)$ denote the Riemannian sectional curvature of the 2-plane generated by the tangent vectors X and Y at a point x.*

Suppose now for $x \neq p$,

$$-C_1(r) \leq K\left(\frac{\partial}{\partial r_p}, X\right)(x) \leq C_2(r), \quad C_1(r), \quad C_2(r) \geq 0$$

and furthermore

$$\int_0^\infty rC_1(r) < \infty, \quad \int_0^\infty rC_2(r) \leq 1.$$

Then, the exponential map at p is a quasi-isometry.

To prove Theorem 2.1, we observe first that by making A_ϵ sufficiently small, the complete Kähler manifold M in Theorem 2.1 satisfies the Quasi-isometry Theorem. The Laplace-Beltrami operator is therefore uniformly elliptic in terms of normal geodesic coordinates at p. By classical estimates of the Green kernel $G(x, y)$ (cf. Stampacchia [15]) there exists constants C_1, C_2 such that

$$\frac{C_1}{d(x, y)^{2n-2}} \leq |G(x, y)| \leq \frac{C_2}{d(x, y)^{2n-2}}$$

where $d(x, y)$ denotes geodesic distances on M. Moreover, by the Hölder estimates of harmonic functions of Di Giorgi-Nash-Moser (cf. Moser [12]) and the same estimate as in § 1, Theorem 1.1, it follows that we can prove $\frac{1}{2}\Delta u_0 = \text{trace}(\rho)$ with $u_0 = 0(\log r)$. To prove Theorem 2.1 of this section, we shall produce a solution u of $i\partial\bar{\partial}u = \rho$ such that $u = O(r^\delta)$ with $\delta <$ Hölder exponent γ in the Harnack inequality on M. It follows that $u - u_0$, being harmonic with growth order $0(r^\delta)$ must be a constant, so that in fact $i\partial\bar{\partial}u_0 = \rho$. To find u we do this by solving $\bar{\partial}$ with L^2-estimates, using results of Andreotti-Vesentini [1] and

Hörmander [9]. For the construction of weight functions we shall need geometric comparison with $\mathbb{C}^n$ equipped with a certain Hermitian metric. We collect information on this model in the following lemma.

LEMMA 2: *Let $\epsilon > 0$ be given and $A_\epsilon \geq 0$ be such that $\int_0^\infty (rA_\epsilon/(1+r^2)^{1+\epsilon})\,dr \leq 1$. Then there exists a smooth Hermitian metric on $\mathbb{C}^n$ invariant under the orthogonal group such that the curvature $K(X, (\partial/\partial r_0))$ of any 2-plane containing the radial tangent vector with respect to the pole 0 is equal to $(A_\epsilon/(1+r^2)^{1-\epsilon})$. Suppose μ is the largest positive number such that for the exponential map $\exp_0$ at the origin,*

$$\mu|v| \leq \|\exp_0 v\| \leq |v|,$$

where v is a tangent vector on $\mathbb{R}^{2n} = T_0(\mathbb{C}^n)$ and $|\cdot|$, $\|\cdot\|$ denote norms on $\mathbb{R}^{2n}$ and on $\mathbb{C}^n$ equipped with the above metric respectively. Write $s = \sum_{i=1}^n |z_i|^2$, $(z_1, \ldots, z_n)$ being the complex coordinates on $\mathbb{C}^n$. Then, for any small $\delta > 0$, there exists constants $C_1(\delta, \epsilon, A_\epsilon)$ and $C_2(\epsilon, A_\epsilon)$ such that

$$C_1(\delta, \epsilon, A_\epsilon) r^{(1/\mu)-\delta} \leq s \leq C_2(\epsilon, A_\epsilon) r^{1/\mu}.$$

PROOF OF LEMMA 2: According to Greene and Wu [7, p. 58], given any smooth function $K(r) \geq 0$ such that $\int_0^\infty rK(r) \leq 1$ there exists a one-dimensional rotationally symmetric model N with a smooth metric such that the Gaussian curvature is given by $K(r)$, where r denotes the geodesic distance from the pole. By Blanc-Fiala-Huber (cf. [8]), with the canonical conformal structure, N is biholomorphic to the complex plane, giving a metric $\eta(s)\,dz \otimes d\bar{z}$ on $\mathbb{C}$, $s = |z|$, written $dr^2 + f^2(r)\,d\theta^2$ in normal geodesic polar coordinates (r, θ). On $\mathbb{C}^n$, consider the metric $\eta(s)(dz_1 \otimes d\bar{z}_1 + \cdots + dz_n \otimes d\bar{z}_n)$. By symmetry the metric is $dr^2 + f^2(r)\,d\Theta^2$, in geodesic polar coordinates, where $d\Theta^2$ is the canonical metric on the Euclidean unit sphere. Letting $K(r) = (A_\epsilon/(1+r^2)^{1+\epsilon})$ we get the metric in Lemma 2.

We denote this metric by N_ϵ. If (M_1, p_1), (M_2, p_2) are two Riemmanian manifolds of dimension m with poles p_1, p_2 respectively, we say that M_1 is more positive than M_2, $(M_1, p_1) \geq (M_2, p_2)$ to mean that if $x_1 = \exp_{p_1}(v)$, $x_2 = \exp_{p_2}(v) v \in \mathbb{R}^m$, then for any tangent vectors X_1, X_2, $X_1 \perp (\partial/\partial r_{p_1}) X_2 \perp (\partial/\partial r_{p_2})$ at x_1, x_2 respectively, $K(\partial/\partial r_{p_1}, X_1)(x_1) \geq K(\partial/\partial r_{p_2}, X_2)(x_2)$. In this sense $(N_\epsilon, 0)$ is more positive than (M, p) in the hypothesis of Theorem 2.1.

To estimate $r(z) = d_{N_\epsilon}(0, z)$ in terms of $s = (\Sigma |z_i|^2)^{1/2}$, it suffices to describe an explicit biholomorphism $\Phi : N_\epsilon \cong \mathbb{C}$ in the case of dimension 1. Let (r, θ) be geodesic polar coordinates on $(N_\epsilon, 0)$ and $r = r(s)$, where $s = |\Phi(r, \theta)|$ is independent of θ. The metric on N_ϵ can be written as

$$dr^2 + f^2(r)\, d\theta^2 = r'(s)\, ds^2 + f^2(r)\, d\theta^2.$$

Since the metric is Hermitian on $\mathbb{C}$, we have

$$\frac{f^2(r)}{(r'(s))^2} = s^2.$$

Since $f(r) > 0$, $r'(s) > 0$, we have

$$\frac{f(r)}{r'(s)} = s, \quad \frac{dr}{f(r)} = \frac{ds}{s}.$$

Integration then gives

(1) $$\log s = C + \int_1^r \frac{dr}{f(r)}$$

By Lemma 1 on quasi-isometry (Greene and Wu [7]) there exists $\mu > 0$ such that, for any tangent vector v on $T_0(N_\epsilon) = \mathbb{R}^2$,

$$\mu |x| \leq \|\exp_0 x\| \leq |x|, \quad \text{so that} \quad \mu r \leq f(r) \leq r.$$

More precisely, $f(r)$ satisfies the Jacobi equation

$$f''(r) = -\frac{A_\epsilon}{(1 + r^2)^{1+\epsilon}} f(r) = -K(r) f(r),$$

and hence

$$f'(r) = 1 + \int_0^r f''(r)\, dr = 1 - \int_0^r K(r) f(r)\, dr$$

where $f(r) < r$ at some point unless $K(r) \equiv 0$.

If we set $\mu = 1 - \int_0^\infty K(r) f(r)$, then $f'(r)$ decreases from 1 to μ, $\lim_{r\to\infty} f'(r) = \mu$, and $f(r)/r$ decreases with $\lim_{r\to\infty}(f(r)/r) = \mu$. From (1) it follows that given any $\delta > 0$ small enough, there exists constants

$C_1(\delta, \epsilon, A_\epsilon)$ and $C_2(\epsilon, A_\epsilon)$ such that

$$C_1(\delta, \epsilon, A_\epsilon) r^{(1/\mu)-\delta} \leq s \leq C_2(\epsilon, A_\epsilon) r^{1/\mu} + 1,$$

proving Lemma 2. We shall henceforth call μ the modulus of quasi-isometry of the manifold $(N_\epsilon, 0)$.

The next lemma collects information on weight functions used in solving $\bar\partial$. Let $\tau_{N_\epsilon}: \mathbb{R}^{2n} \to N_\epsilon$ be the exponential map at the origin 0 of N_ϵ and $\tau_M: \mathbb{R}^{2n} \to M$ be the one at the pole p of M. $\tau_M \tau_{N_\epsilon}^{-1}$ defines a diffeomorphism $N_\epsilon \to M$ preserving distances to the poles. $(\tau_M \tau_{N_\epsilon}^{-1})_* f(r)$ will also be written $f(r)$ on the manifold M for any radial function $f(r)$ on N_ϵ. In case of ambiguity we write f_{N_ϵ} and f_M for the two functions on N_ϵ and M respectively.

LEMMA 3: *Let M be the complete Kähler manifold as in the hypothesis of Theorem 2.1, and let $s = (\Sigma\, |z_i|^2)^{1/2}$ be first defined on N_ϵ in terms of the underlying complex coordinates. $s = s(r)$ is a radial function. If $s_M = (\tau_M \tau_{N_\epsilon}^{-1})_* s(r)$ on M, then for any $x \in M$, $x \neq p$, and any tangent vector v of type $(1,0)$ at x, we have the inequalities*

(*i*) $\langle \partial\bar\partial \log(1 + s_M^p),\ v \wedge \bar v\rangle \geq (C_1(p, \delta, \epsilon, A_\epsilon)/1 + r^{2+2\delta+(p/\mu)})\|v\|^2, \quad p > 0,\ \delta > 0$

(*ii*) $\langle \partial\bar\partial s_M^2, v \wedge \bar v\rangle \geq C_2(\delta, \epsilon, A_\epsilon)(1 + r^{(2/\mu)-2-2\delta} + 1)\|v\|^2,\ \delta > 0$

(*iii*) $\langle \partial\bar\partial \log s_M, v \wedge \bar v\rangle \geq 0$

where μ stands for the modulus of quasi-isometry of N_ϵ.

SKETCH OF PROOF: In (1) for example we need to compute

$$\begin{aligned}\langle \partial\bar\partial \log(1 + s_M^p), v \wedge \bar v\rangle &= \tfrac14 H(\log(1 + s_M^p))(Re(v), Re(v)) \\ &\quad + \tfrac14 H(\log(1 + s_M^p))(Im(v), Im(v)),\end{aligned}$$

where $H(\log(1 + s_M^p))$ denotes the real Hessian. By standard comparison theorems (cf. Greene and Wu [7] and Siu and Yau [13])

$$\begin{aligned}&\tfrac14 H(\log(1 + s_M^p))((Re(v), Re(v)) + \tfrac14 H(\log(1 + s_M^p))(Im(v), Im(v)),\\ &\geq \tfrac14 H(\log(1 + s_{N_\epsilon}^p))(X, X) + \tfrac14 H(\log(1 + s_{N_\epsilon}^p))(Y, Y)\end{aligned}$$

where X and Y are orthonormal tangent vectors at $x_0 = \tau_{N_\epsilon}\tau_M^{-1}(x)$ such that the angle which X makes with $\partial/\partial r_{N_\epsilon}$ equals to the angle between $\partial/\partial r_M$ and $Re(v)$ (same for Y and $Im(v)$).

If $JX = Y$, where J is the J-operator on N_ϵ, then the last two quantities simply give $\langle \partial\bar\partial(\log(1 + s_{N_\epsilon}^p), (X + \sqrt{-1}JX) \wedge$

$(X+\sqrt{-1}JX))$. In general, since the metric on N_ϵ is of the form $\eta(s)(dz_1\otimes d\bar{z}_1+\cdots+dz_n\otimes d\bar{z}_n)$. X and Y are orthogonal in the Euclidean metric. By means of a real orthogonal change of coordinates, we can change the complex structure of the underlying linear space $\mathbb{R}^{2n}$ without affecting the metric. With such a change X and Y would generate a complex plane $Y=J'X$ with the new J-operator. Given these considerations, it remains to estimate the complex Hessians of the functions on N_ϵ by direct computation and making use of the inequality $C_1(\delta,\epsilon,A_\epsilon)r^{(1/\mu)-\delta}\le s\le C_2(\epsilon,A_\epsilon)r^{1/\mu}$ in Lemma 2.

Proof of Theorem 2.1: In using the weight function $\log(1+s^p)$ on M, to take care of the non-differentiability at the pole for $0<p<1$, we introduce once and for all a smooth convex function χ on the real line such that $\chi(t)\equiv 1$ for $t\le 1$ and $\chi(t)=t$ for $t\ge 2$. Then the function $\log\chi(1+s^p)$ is smooth and plurisubharmonic at p. The function $\log(1+s^2)$, which is smooth everywhere, will be used for the part near the pole in solving $\bar\partial$ with L^2-estimates.

Let $\rho=\Sigma_{i,j=1}^{2n}\rho_{ij}\,dx^i\wedge dx^j$ in terms of normal geodesic coordinates $(x_1,\ldots,x_{2n})$ at the pole p of M. We first solve $dv=\rho$ on M by the Poincaré lemma. Then, decomposing v into the $(1,0)$ component $v_{1,0}$ and $(0,1)$ component $v_{0,1}$ in the complex structure of M, $dv=0$ means

$$\begin{cases}\partial v_{1,0}=0\\ \bar\partial v_{0,1}=0\\ \bar\partial v_{1,0}+\partial v_{0,1}=\rho.\end{cases}$$

On the geodesic ball $B(p,R)$ we solve $\bar\partial u_{R,0}=v_{0,1}$. Without loss of generality we can assume that ρ is a real 2-form. Then v is real (from the Poincaré lemma) and $\bar v_{0,1}=v_{1,0}$. Putting $u_R=u_{R,0}-\bar u_{R,0}$, u_R is real and $\partial\bar\partial u_R=\partial v_{0,1}+\bar\partial v_{1,0}=\rho$. This approach was used in Henkin [6] and Skoda [12] in studying holomorphic functions in the Nevanlinna class on strictly pseudoconvex domains.

The Poincaré Lemma: *Let* $F=[0,1]\times X\to X$ be defined by $F(t,x)=tx$.

Define v by $v(x)=\int_{t=0}^{t=1}F^*(\rho(tx))$. By the Poincaré lemma $dv=\rho$.

$$F^*(\rho(tx))=\sum_{i,j=1}^{2n}\rho_{ij}(tx)(t\,dx_i+x_i\,dt)(t\,dx_j+x_j\,dt)$$

$$v_j(x)=\int_0^1\Big(\sum_i\rho_{ij}(tx)tx_i\,dt)\,dx_j,\quad v=\sum_j v_j(x)\,dx_j.$$

If $x = rw$, $|w| = 1$. Then,

$$v_j(x) = \int_0^r \sum_i \rho_{ij}\left(\frac{sx}{r}\right)\frac{sx_i}{r}\frac{ds}{r}$$

$$= \frac{1}{r}\int_0^r \sum_i \rho_{ij}(sw)sw_i\, ds$$

$$v_j(x) \le \frac{C_1}{r}\int_0^r \frac{1}{(1+s)^2}\, s\, ds = 0\left(\frac{\log r}{r}\right).$$

Let $\Sigma_{i,j}\, g_{ij}\, dx^i \otimes dx^j$ be the metric on M in terms of normal geodesic coordinates at p. The volume form is given by $\sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^{2n}$. By Lemma 1 (on quasi-isometry, Greene and Wu [7]) the metric is uniformly equivalent to the Euclidean metric. In particular, $\det(g_{ij})$ is bounded between positive constants. $v = v_{0,1} + v_{1,0}$ is the decomposition of v into $(0, 1)$ and $(1, 0)$ components. Since $v_{0,1}$ and $v_{1,0}$ are perpendicular, $\|v_{0,1}\| = 0(\log r/r)$. Hence

$$\int_{B(R)} \|v_{0,1}\|^2\sqrt{G}\, dx_1 \wedge \cdots \wedge dx_{2n}, \quad G = \det(g_{ij})$$

$$= \int_1^B \int_{\partial B(r)} \|v_{0,1}(x)\|^2\sqrt{G}\, d\sigma_r(x)\, dr + C_1$$

(where $d\sigma_r$ = Euclidean volume element on $\partial B(r)$)

$$\le C_2\int_1^R r^{2n-1}\frac{(\log r)^2}{r^2}\, dr + C_3$$

$$= 0(R^{2n-2}(\log R)^2)$$

To solve $\bar\partial u_{R,0} = v_{0,1}$ we need the following version of L^2-estimates of $\bar\partial$ on M (cf. Siu and Yau [13, p. 244]).

LEMMA 4 (L^2-estimates of $\bar\partial$): *Suppose M is a complete Kähler manifold of complex dimension n. Let φ be a smooth plurisubharmonic function on M and c be a positive continuous function on M. If, for any tangent vector v of type $(1, 0)$ at any point x of M,*

$$\langle \partial\bar\partial\varphi + \mathrm{Ric}, v \wedge \bar v\rangle \ge c(x)\|v\|^2,$$

where Ric stands for the Ricci form on M. Then, for every $C^\infty(0, 1)$ *form g on M with $\bar\partial g = 0$, there exists a C^∞ function u on M such that*

$\bar\partial u = g$ and

$$\int_M \|u\|^2 e^{-\varphi} \le \int_M \|g\|^2 \frac{e^{-\varphi}}{c}.$$

For $(n, 1)$ forms a similar statement holds if

$$\langle \partial\bar\partial\varphi, v \wedge \bar v\rangle \ge c(x)\|v\|^2.$$

Furthermore, similar existence theorems and estimates hold on Stein subdomains of M.

REMARK: For the last statement, we can, in a standard manner, reduce to the case of a strictly pseudoconvex subdomain Ω which is defined by $\{\Psi < 0\}$ where Ψ is a smooth strictly plurisubharmonic function on $\bar\Omega$. In this case one can follow Hörmander [9] to dispose of the boundary term in the estimate of the square Laplacian for forms, using Morrey's trick.

By the Poincaré lemma we solved $dv = \rho$, $\|v\| = 0(\log r/r)$. On the geodesic ball $B(\rho, R)$, which is Stein and defined by $\{s^2 - s^2(R) < 0\}$ (s^2 is strictly plurisubharmonic on M by Lemma 2), we are going to solve $\bar\partial u_{R,0} = v_{0,1}$. By Lemma 2, when A_ϵ is sufficiently small, there exist constants α, $\beta > 0$ such that

$$\partial\bar\partial(\alpha \log(1 + s^2) + \beta \log \chi(1 + s^p)) + \mathrm{Ric} \ge 0 \quad \text{on} \quad M,$$

where p is chosen such that $(p/\mu) < \epsilon$, μ = modulus of quasi-isometry of the model N_ϵ. Moreover, if we fix p, then α and β can be chosen to decrease to zero as A_ϵ decreases to zero. We now choose the weight function $\varphi = \alpha \log(1 + s^2) + \beta \log \chi(1 + s^p) + (s^2/s(R)^2)$, then

$$\langle \partial\bar\partial\varphi + \mathrm{Ric}, v \wedge \bar v\rangle \ge \frac{C}{s(R)^2}\|v\|^2,$$

for any tangent vector of type $(1, 0)$ at $x \in B(p, R)$, again by Lemma 2, where we have used the weaker fact that $\partial\bar\partial s_M^2$ dominates a positive multiple of the Kähler form on M. By the L^2-estimate of $\bar\partial$ (Lemma 3), there exists $u_{R,0}$ satisfying $\bar\partial u_{R,0} = v_{0,1}$ such that

$$\begin{aligned}\int_{B(R)} |u_{R,0}|^2 e^{-\varphi}\sqrt{G}\, dx_1 \wedge \cdots \wedge dx_{2n} &\le s(R)^2 \int_{B(R)} \|v_{0,1}\|^2 e^{-\varphi}\sqrt{G}\, dx_1 \wedge \cdots \\ &\quad \wedge dx_{2n} \\ &= 0(R^{2n-2+(2/\mu)}(\log R)^2).\end{aligned}$$

Recall that $\varphi = \alpha \log(1+s^2) + \beta \log \chi(1+s^p) + (s^2/s(R)^2)$. Putting $q = 2\alpha + p\beta$, then

$$\begin{aligned}\int_{B(R)} |u_{R,0}|\, dx_1 \wedge \cdots \wedge dx_{2n} &\leq \left(\int_{B(R)} |u_{R,0}|^2(1+s)^{-q}\, dx_1 \wedge \cdots \wedge dx_{2n}\right)^{1/2} \\ &\quad \times \left(\int_{B(R)} (1+s)^q\, dx_1 \wedge \cdots \wedge dx_{2n}\right)^{1/2} \\ &= 0(R^{n-1+(1/\mu)} \log R) 0(R^{n+(q/2\mu)}) \\ &= 0(R^{2n-1+(1/\mu)(1+(q/2))} \log R).\end{aligned}$$

Fix a number $a > 1$, and write $R_m = a^m m \geq 0$. For notational convenience we write $u_{(m)} = u_{R_m}$, where $u_R = u_{R,0} - \bar{u}_{R,0}$ solves $\partial\bar{\partial} u_R = \rho$ on $B(p, R)$. Now $u_{(m+1)} - u_{(m)}$ is a harmonic function on $B(p, R_m)$ and

$$\int_{B(R_m)} |u_{(m+1)} - u_{(m)}|\, dx_1 \wedge \cdots \wedge dx_{2n} = 0(R_m^{2n-1+(1/\mu)(1+(q/2))} \log R_m).$$

Since the exponential map at $p \in M$ is a quasi-isometry, the Laplace-Beltrami operator is uniformly elliptic in terms of normal geodesic coordinates at p. Hence the sub-mean value inequality for non-negative subharmonic functions holds on M. Applying the sub-mean value inequality to $\sup(u_{(m+1)} - u_{(m)}, 0)$ and $\sup(u_{(m)} - u_{(m+1)}, 0)$, we have

$$\sup_{B(R_{m-1})} |u_{(m+1)} - u_{(m)}| = 0(R_m^{(q+2/2\mu)-1} \log R_m).$$

For $m \geq 1$ we replace $u_{(m)}$ by $u_{(m)} - u_{(m)}(p) + u_{(m-1)}(p)$, so that $u_{(m)}(p) = u_{(0)}(p)$. We retain the notation $u_{(m)}$ for the new solutions after normalizing at the pole p. Fix $m \geq 1$ and let $x \in B(p, R_{m-1})$ and write $h_{m+k,m}(x) = u_{(m+k)}(x) - u_{(m)}(x)$. Now,

$$u_{(m+k)}(x) = u_{(m)}(x) + h_{m+1,m}(x) + h_{m+2,m+1}(x) + \cdots + h_{m+k,m+k-1}(x).$$

Since $\sup_{B(R_{m+l-1})} |h_{m+l+1,m+l}| = 0(R_{m+l}^{(q+2/2\mu)-1} \log R_{m+l})$, and by the Hölder estimate in the Harnack inequality, for any harmonic function h on $B(p, R)$, $h(p) = 0 |h(x)| \leq C \sup|h| d(p,x)^\gamma$, for some $\gamma > 0$ depending on the manifold M, we obtain the estimate for $u_{(m+k)}(x)$, $x \in B(p, R_{m-1})$

$$\begin{aligned}|u_{(m+k)}(x)| &= |u_{(m)}(x) + h_{m+1,m}(x) + \cdots + h_{m+k,m+k-1}(x)| \\ &\leq |u_{(m)}(x)| + \sum_{l=0}^{k-1} (C_1 R_{m+l}^{(q+2/2\mu)-1} \log R_{m+l} + C_2) C \left(\frac{d(p,x)}{R_{m+l}}\right)^\gamma.\end{aligned}$$

Recall that μ is the modulus of quasi-isometry of the model N_ϵ with (radial) curvature $(A_\epsilon/(1+r^2)^{1+\epsilon})$, $(p/\mu)<\epsilon$. α and β are such that

$$\partial\bar{\partial}(\alpha \log(1+s^2)+\beta \log x(1+s^p))+\mathrm{Ric}\geq 0.$$

The Ricci form is bounded by $(A_\epsilon/(1+r^2)^{1+\epsilon})$ times the Kähler form, and γ is the Hölder exponent in the Harnack inequality. Moreover, μ, p, α, β, γ depend on A_ϵ in such a way that μ approaches to 1 as A_ϵ approaches zero, and with p fixed for μ close to 1, α and β can be chosen to approach zero as A_ϵ shrinks to zero. The Hölder exponent γ can be taken $=\gamma_0$ whenever $A_\epsilon \leq A_\epsilon^0$ for some A_ϵ^0 fixed and sufficiently small. It follows, by fixing $\gamma=\gamma_0$ and taking A_ϵ small enough, so that $(q+2/2\mu)-1<\gamma_0$, we have for $x\in B(p,R_{m-1})$

$$\begin{aligned}|u_{(m+k)}(x)| &\leq |u_{(m)}(x)|+C_3\Big(\sum_{l=0}^{k-1} R_{m+l}^{(q+2/2\mu)-1-\gamma_0}\log R_{m+l}\Big)d(p,x)^{\gamma_0}+C_4\\ &\leq |u_{(m)}(x)|+C_5R_m^{(q+2/2\mu)-1}\log R_m+C_4.\end{aligned}$$

Finally we estimate $u_{(m)}(x)$. On the geodesic ball $B(p,R_m)$, $u_{(m)}$ satisfies the estimate

$$\int_{B(R_m)}|u_{(m)}(x)|\,dx_1\wedge\cdots\wedge dx_{2n}=0(R_m^{2n-1+(1/\mu)(1+(q/2))}\log R_m).$$

Recall that since the closed (1, 1) form ρ grows like $(1/r^2)$, there is a solution u' of $\frac{1}{2}\Delta u'=\mathrm{trace}(\rho)$ with $u'=0(\log r)$. Then u_m-u' is harmonic on R_m and

$$\int_{B(R_m)}|u_{(m)}-u'|\,dx_1\wedge\cdots\wedge dx_{2n}=0(R_m^{2n-1+(q+2/2\mu)}\log R_m).$$

Since $u_{(m)}-u'$ is harmonic, by the sub-mean value inequality, for $x\in B(p,R_{m-1})$,

$$|u_{(m)}(x)|\leq C_6R_m^{(q+2/2\mu)-1}\log R_m+C_7$$

$$|u_{(m+k)}(x)|\leq(C_5+C_6)R_m^{(q+2/2\mu)-1}\log R_m+(C_4+C_7).$$

independent of k.

It follows that the limit $\lim_{l\to\infty}u_{(l)}=u$ converges uniformly on compact sets, $\partial\bar{\partial}u=\rho$, and $|u|=0(r^{(q+2/2\mu)-1}\log r)$. Recall that $(q+2/2\mu)-1<\gamma_0$ (Hölder exponent), so that the harmonic function $u-u'$

is actually a constant by the Hölder estimate. Hence $\partial\bar{\partial}u' = \rho$, proving Theorem 2.1.

(2.2) Application to Kähler geometry

In this section we apply Theorem 1 to study complete Kähler manifolds with a pole. Results are summarized in the following theorem.

THEOREM 2: *Let M be a complete Kähler manifold of complex dimension n with a pole p. Let r denote geodesic distances from the pole p.*

Suppose $n \geq 2$ and the curvature of M is bounded by

$$-\frac{A_\epsilon}{(1+r^2)^{1+\epsilon}} \leq \text{sectional curvature} \leq \frac{A_\epsilon}{(1+r^2)^{1+\epsilon}}$$

where A_ϵ is a sufficiently small constant depending on ϵ. Then, M is biholomorphic to $\mathbb{C}^n$. In case of $n = 1$, M is biholomorphic to $\mathbb{C}$ if $(-C/1 + r^{2+\epsilon}) \leq$ Gaussian curvature. On the other hand, if $n \geq 2$ and there exists a constant $C \geq 0$, such that $(-C/1 + r^{2+\epsilon}) \leq$ sectional curvature ≤ 0, then M is isometrically bioholomorphic to $\mathbb{C}^n$.

PROOF: For the second part we shall make use of Theorem 2.1 and intermediate results of Siu and Yau [13]. In the latter article it was proved that M is biholomorphic to $\mathbb{C}^n$. In an intermediate step, they proved

LEMMA: *Let M be a complete Kähler manifold of nonpositive sectional curvature such that $(-C/1 + r^{2+\epsilon}) \leq$ sectional curvature ≤ 0. Then there exists a holomorphic n-form ξ of slow growth, such that for any $\delta > 0$, $\|\xi\| = 0(r^\delta)$. Moreover, ξ is invertible, unique up to a multiplicative constant, and $\|1/\xi\| = 0(r^\delta)$.*

We solve the equation $\partial\bar{\partial}u = \text{Ric}$. Since the Ricci grows like $1/r^{2+\epsilon}$ and the Green kernel is bounded by $C'/d(x, y)^{2n-2}$, there is a bounded solution u' of $\frac{1}{2}\Delta u' = -R$ (scalar curvature). On the other hand, by the technique of Theorem 2.1, we are going to solve $\partial\bar{\partial}u = -$Ricci form with $|u| = 0(r^\delta)$, $\delta > 0$ arbitrarily small. By the Hölder estimate of harmonic functions $u - u'$ is a constant, $\partial\bar{\partial}u' = -$Ricci form, so that there is a bounded plurisubharmonic function on M. Since M is biholomorphic to $\mathbb{C}^n$, $u' \equiv$ constant, Ricci form $\equiv 0$, and M is flat (sectional curvature $\equiv 0$).

In solving $\partial\bar{\partial}u$ = Ricci form, we follow Theorem 2.1, except that the equation $\bar{\partial}u_{R,0} = v_{0,1}$ on $B(p,R)$ is replaced by $\bar{\partial}(u_{R,0} \wedge \delta) = v_{0,1} \wedge \xi$ for $(n,0)$ forms. In this case we choose the weight function $\varphi = (r^2/R^2)$. Then $\langle \partial\bar{\partial}\varphi, v \wedge \bar{v}\rangle \geq (1/R^2)\|v\|^2$ for any tangent vector v of type $(1,0)$. We have no need to dominate the Ricci form in solving for $(n,0)$ forms, which creates trouble when C is too large. Finally $u_{R,0}$ is estimated from $1/\xi$ and $u_{R,0} \wedge \xi$.

To prove the first part we observe first that in case of dimension $n = 1$, since volume $(B(p,r)) = 0(r^2)$ by a standard comparison theorem, M is parabolic by a theorem of Cheng and Yau [4]. For $n \geq 2$, we are going to produce holomorphic functions of minimal degree $(f_1, \ldots, f_n)$ which will define a biholomorphism $F: M \to \mathbb{C}^n$. We shall solve $\bar{\partial}$ in the space of smooth $(n,0)$ forms. Let $z_1, \ldots, z_n$ be a system of local holomorphic coordinates at p defined on a neighborhood U. Let ρ be a smooth cut-off function such that $\rho \equiv 1$ in a neighborhood of p, and $\operatorname{Sup}\rho \Subset U$. Let $\varphi = \nu \log(1+s^2) + (2n+2)\log s$, $\nu > 0$. Then $\partial\bar{\partial}\varphi > 0$ and there exists a smooth $(n,0)$ form u_i such that

$$\bar{\partial}u_i = \bar{\partial}(\rho z_i\, dz_i \wedge \cdots \wedge dz_n) \quad \text{and} \quad \int_M \|u_i\|^2 e^{-\varphi} \leq \int_M \frac{\|\bar{\partial}(\rho z_i\, dz_1 \wedge \cdots \wedge dz_n)\|^2}{c}$$

where $\langle \partial\bar{\partial}\nu \log(1+s^2), v \wedge \bar{v}\rangle \geq c\|v\|^2$ on U for any tangent vector of type $(1,0)$.

Define $g_i = u_i - \rho z_i\, dz_1 \wedge \cdots \wedge dz_n$ (globally defined on M). g_i is holomorphic, and because of the singularity $(2n+2)\log s$, $u_i(p) = 0$. If $g_i = h_i\, dz_1 \wedge \cdots \wedge dz_n$ in a neighborhood of p, then $dh_i(p) = dz_i(p)$. Let g be the holomorphic n-form obtained from solving $\partial\bar{\partial}\log\|g\|^2$ = Ricci form, from Theorem 1. Since Ricci form $= 0(1/r^{2+\epsilon})$, both $\|g\|$ and $1/\|g\|$ are bounded. If $g = h\, dz_1 \wedge \cdots \wedge dz_n$ on U, then at p,

$$\frac{\partial}{\partial z_j}\left(\frac{g_i}{g}\right)(p) = \frac{1}{h(p)^2}\left(h(p)\frac{\partial h_i}{\partial z_j}(p) - h_i(p)\frac{\partial h}{\partial z_j}(p)\right) = -\frac{1}{h(p)}\delta_{ij}.$$

It follows that $F = (f_1, \ldots, f_n)$ defines a local biholomorphism at p. To get pointwise estimate for f_i, it follows from $1/\|g\| = 0(1)$ and $\int_M \|u_i\|^2 e^{-\varphi} < \infty$ that $\int_M (|f_i|^2/(1+s^2)^\nu s^{2n+2})\, dx_1 \wedge \cdots \wedge dx_{2n} < \infty$. By Lemma 2 to Theorem 1, for any $\delta > 0$, there exists constants

$C(\delta, \epsilon, A_\epsilon)$ and $C'(\epsilon, A_\epsilon)$ such that $C(\delta, \epsilon, A_\epsilon)r^{(1/\mu)-\delta} \le s \le C'(\epsilon, A_\epsilon)r^{1/\mu} + 1$, ($\mu \equiv$ modulus of quasi-isometry of the model N_ϵ).

By the sub-mean value inequality, for x such that $d(p, x) = R$,

$$|f_i(x)| \le \frac{C}{R^{2n}} \int_{B(x,(R/2))} |f_i|\, dx_1 \wedge \cdots \wedge dx_{2n}$$

$$\le \frac{C}{R^{2n}} \left(\int_{B(x,(R/2))} \frac{|f_i|^2}{(1+s^2)^\nu s^{2n}} \right)^{1/2} \left(\int_{B(x,(R/2))} (1+s^2)^\nu s^{2n+2} \right)^{1/2}$$

$$= 0(R^{1+n((1/\mu)-1)+\nu/\mu}).$$

Next we show that $F = (f_1, \ldots, f_n)$ is locally biholomorphic. We estimate $df_1 \wedge \cdots \wedge df_n$ by the following lemma.

LEMMA 2: (*Cauchy estimates for derivatives of holomorphic functions*): *Suppose M is a complete Kähler manifold with a pole p satisfying*

$$\frac{-A_\epsilon}{(1+r^2)^{1+\epsilon}} \le \text{sectional curvature} \le \frac{A_\epsilon}{(1+r^2)^{1+\epsilon}}\; r(x) = d(p, x)$$

where A_ϵ is sufficiently small, depending on ϵ. Let f be a holomorphic function on M. Then for t sufficiently small, $t > 0$ there exists a positive number ρ which depends only on t, n, A_ϵ, ϵ such that when

$0 < \eta \le 2(r(x) - \rho)$

(1) $$\|df(x)\|^2 \le \frac{C}{\eta^{2n+2}} (1 + r(x))^t \int_{B(\eta)} |f|^2$$

If η_0 is fixed, then for $\eta \le \eta_0$ and x arbitrary,

(2) $$\|df(x)\|^2 \le \frac{C_0}{\eta^{2n+2}} \int_{B(\eta)} |f|^2.$$

PROOF: The proof is exactly as in Siu and Yau [13] with obvious modifications regarding the sub-mean value inequality and the integral inequality

$$\int_{B(x,\lambda\eta)} \Delta u \le \frac{C}{\eta^2} \int_{B(x,\eta)} u \quad 0 < \lambda < 1, \quad u \ge 0 \text{ subharmonic}$$

obtained for example by using the fact that the Laplace-Beltrami operator is uniformly elliptic. The new metric on (1,0) forms with nonpositive curvature can now be defined by $\|u\|_0^2 = (\|u\|^2/(1+s^2)^q\chi(1+s^p))$ for appropriate p and q, which allows us to apply the sub-mean value inequality for vector bundles (pg. 239).

REMARK: The estimate for $\|df_i\|$ can also be obtained by using the gradient estimate for harmonic functions on Riemannian manifolds (Cheng and Yau [4]). Given the lemma, it follows that $(R = d(p, x))$

$$\|df_i(x)\| = 0(R^{n((1/\mu)-1)+(\nu/\mu)+(t/2)}),\ v > 0 \text{ arbitrary, } t > 0 \text{ sufficiently small}$$

$$\|df_1 \wedge \cdots \wedge df_n(x)\| = 0(R^{n^2((1/\mu)-1)+(n\nu/\mu)+(nt/2)})$$

Recall that when A_ϵ is sufficiently small, we can assume the Hölder exponent γ for harmonic functions to be fixed independent of A_ϵ. Then, $n^2((1/\mu)-1)+(n\nu/\mu)+(nt/2) < \gamma$ when μ is close to 1, with ν, t suitably chosen. If g is the holomorphic n-form obtained from Theorem 2.1 by solving $\partial\bar{\partial}\log\|g\|^2 =$ Ricci form. Then $df_1 \wedge \cdots \wedge df_n/g$ is a holomorphic function of order $0(R^{n^2((1/\mu)-1)+(n\nu/\mu)+(nt/2)})$, which is identically constant by the Hölder estimate. Finally, to prove Theorem 2.2 we show that F is biholomorphic. We shall present two different proofs. For both proofs we need the following lemma on holomorphic vector fields.

LEMMA 3: *Let X_i be holomorphic vector fields defined by inverting df_i, i.e., $\langle X_i, df_i\rangle = X_if_j = \delta_{ij}$ at each point x. Then X_i satisfies the growth condition for $d(p, x)$ sufficiently large*

$$C_1 d(p, x)^{-[n(1/\mu)-1)+(\nu/\mu)+(t/2)]} \leq \|X_i(x)\| \leq C_2 d(p, x)^{[n(1/\mu)-1)+(\nu/\mu)+(t/2)](n-1)}.$$

PROOF OF LEMMA: X_i is obtained by inverting the matrix $\partial f_i/\partial z_j$ in local coordinates. Since $\|df_1 \wedge \cdots \wedge df_n\|$ is bounded between positive constants,

$$\begin{aligned} \|X_i(x)\| &\leq C \sup_{i_1 < \cdots < i_{n-1}} \|df_{i_1} \wedge \cdots \wedge df_{i_{n-1}}\| \\ &= 0(R^{[n((1/\mu)-1)+(\nu/\mu)+(t/2)](n-1)}), \\ R &= d(p, x) \end{aligned}$$

(by using local coordinates such that $\dfrac{\partial}{\partial z_j}$ are orthonormal)

On the other hand,

$$\|X_i(x)\| = \sup_{\|\omega(x)\|=1} |\langle X_i(x), \omega(x)\rangle|, \ \omega(x) \text{ cotangent vector of type } (1,0) \text{ at } x.$$

Since $X_i(x)f_i(x) = 1$ and $\|df_i(x)\| = 0(R^{((1/\mu)-1)n+(\nu/\mu)+(t/2)})$, we have

$$\|X_i(x)\| \geq CR^{-[n((1/\mu)-1)+(\nu/\mu)+(t/2)]} \text{ for } R \text{ sufficiently large.}$$

We now prove the final part of Theorem 2.

PROPOSITION: *The map $F = (f_1, \ldots, f_n) = M \to \mathbb{C}^n$ is a biholomorphism.*

PROOF I: The first proof will be obtained by directly inverting the map F. Define an "exponential map" $\Phi : \mathbb{R}^{2n} \to M$ by integrating linear combinations of real imaginary parts of the holomorphic vector fields X_i as follows. Let $w_i = u_i + \sqrt{-1}v_i$ be local coordinates at some x with $w_i(x) = f_i(x)$. Then $X_i = \frac{1}{2}((\partial/\partial u_i) - \sqrt{-1}(\partial/\partial v_i))$, $(Re\, X_i)(Re\, f_j) = \frac{1}{2}\delta_{ij}$, $(Im\, X_i)(Im\, f_j) = \frac{1}{2}\delta_{ij}(Re\, X_i)(Im\, f_j) = (Im\, X_i)(Re f_j) = 0$, $Re\, X_i$ and $Im\, X_i$ being real and imaginary parts of X_i. For any unit vector $w = \Sigma_{i=1}^{2n} a_i e_i$, (e_i) the canonical basis of $\mathbb{R}^{2n}$, define $X_w = \Sigma_{i=1}^{n} 2a_i\, Re\, X_i + \Sigma_{i=1}^{n} 2a_{n+i}\, Im\, X_i$. Now define $\Phi(tw)$, $t \in \mathbb{R}$ to be the trajectory of X_w at time t, $\Phi(0) = p$. Φ is clearly smooth where defined, and $F(\Phi(b_1, \ldots, b_{2n})) = (b_1 + \sqrt{-1}b_{n+1}, \ldots, b_n + \sqrt{-1}b_{2n})$. We use normal geodesic coordinates $(x_1, \ldots, x_{2n})$ (with respect to p) on M. Then $\Phi(tw)$ is defined by the ordinary differential equation

$$\frac{d}{dt}\Phi(tw) = X_w(\Phi(tw)) = \sum_{i=1}^{2n} X_{w,i}(x)\frac{\partial}{\partial x_i}.$$

Since the exponential map at p is a quasi-isometry,

$$|X_{w,i}(x)| = 0(|x|^{[n((1/\mu)-1)+(\nu/\mu)+(t/2)](n-1)}),\ |x| = \left(\sum_{i=1}^{2n}|x_i|^2\right)^{1/2}.$$

Let $g(t)$, $t \geq 0$ be the solution of, with initial value $g(0) = 0$,

$$\frac{d}{dt}g(t) = C_1 g(t)^{[n((1/\mu)-1)+(\nu/\mu)+(t/2)](n-1)} + C_2 \quad C_1, C_2 \geq 0.$$

Then, for a unit vector w, $\Phi(tw) = (x_1(t,\ ), \ldots, x_{2n}(t, w))$, $|x_i(t, w)| \leq |g(t)|$.

Write $s = [n((1/\mu) - 1) + (\nu/\mu) + (t/2)](n-1)$, g is an increasing function of t satisfying

$$\frac{dg}{C_1 g^s + C_2} = dt, \quad \text{hence} \quad g(t) \leq C_3 t^{1/1-s} + C_4.$$

Therefore, when A_ϵ is small enough, we can choose ν and t such that $s < 1$. In this case $\Phi(tw)$ is defined for all $t \geq 0$, and moreover, for suitable constants C_5, C_6

$$\Phi(b_1, \ldots, b_{2n}) \leq C_5 \left(\sum |b_i|^2\right)^{1/2(1-s)} + C_6.$$

Since $F(\Phi(b_1, \ldots, b_{2n}) = (b_1 + \sqrt{-1} b_{n+1}, \ldots, b_n + \sqrt{-1} b_{2n})$ for x belonging to the image of Φ, and suitable constants C_7, C_8, C_9, C_{10}

$$C_7 d(p, x)^{1-s} - C_8 \leq |F(x)| \leq C_9 d(p, x)^{1+n((1/\mu)-1)+(\nu/\mu)+(t/2)} + C_{10}.$$

To prove F is a biholomorphism, and at the same time establish the above estimate for all x, it suffices to show that $\Phi(\mathbb{R}^{2n}) = M$. $\Phi(\mathbb{R}^{2n})$ is clearly an open subset of M. To show closedness, let x_σ be a sequence of points in $\Phi(\mathbb{R}^{2n})$ converging to some point $x \in M$. Let $x_\sigma = \Phi(t_\sigma w_\sigma)$, $|w_\sigma| = 1$. Passing to a subsequence, we can assume w_σ converges to w, $|w| = 1$. From the smooth dependence of solutions of ordinary differential equations, $\Phi(\mathbb{R}^{2n})$ would be closed if we show that (t_σ) is bounded. But since $F(\Phi(t_\sigma w_\sigma)) = t_\sigma$ and $|F(x)| \geq C_7 d(p, x)^{1-s} - C_8$, $t_\sigma \leq C_{11} F(x_\sigma)^{1/1-s} + C_{12}$ is bounded, proving that $\Phi(\mathbb{R}^{2n})$ is closed and F is a biholomorphism.

Proof II (that F is a biholomorphism): In the second proof we are going to construct Taylor series expansions of global holomorphic functions f in terms of the holomorphic functions of minimal degree $f_1, \ldots, f_n$. Since s^2 is a smooth strictly plurisubharmonic exhaustion function on M, M is a Stein manifold. Hence, for any divergent sequence (x_σ) of points on M, there exists a holomorphic function f on M such that $\sup_\sigma |f(x_\sigma)| = \infty$. From the Taylor expansion of f, this implies that $\sup_{\sigma,n} |f_n(x_\sigma)| = \infty$ and hence the properness of the map $F = (f_1, \ldots, f_n)$. Since $df_1 \wedge \cdots \wedge df_n$ is nowhere zero, F is a local biholomorphism. Properness of F implies that it is a covering map. Since $\mathbb{C}^n$ is simply connected, F is a biholomorphism.

Let $X_1, \ldots, X_n$ be the holomorphic vector fields obtained by inverting f_i in Lemma 3. Formally, define the Taylor coefficients

$$a_{i_1 \dots i_n} = \frac{X_1^{i_1} X_2^{i_2} \dots X_n^{i_n} f(p)}{i_1! \dots i_n!}, \quad i_k \geq 0$$

Clearly $f = \Sigma_{i_k \geq 0} a_{i_1 \cdots i_n} f_1^{i_1} \dots f_n^{i_n}$ if the right hand side converges uniformly on compact sets. We are going to estimate the coefficients by means of the Cauchy estimate in Lemma 2 and the estimates of the holomorphic vector fields X_i in Lemma 3. Fix $R > 0$. Then

$$\sup_{B(R/2)} |X_i f| = \sup_{\partial B(R/2)} |X_i f| \leq C \sup_{B(R)} |f| R^{(n-1)[((1/\mu)-1)n+(\nu/\mu)+(t/2)]} \left(\frac{R}{2}\right)^{t/2-1}$$

where for R large enough ($\geq p$) we used Lemma 2(1) and for R small ($\leq p$) we used Lemma 2(2). Performing differentiation successively, we obtain

$$|a_{i_1} \dots i_n| \leq \frac{1}{i_1! \dots i_n!} (C')^{i_1 + \cdots + i_n} R^{(i_1 + \cdots + i_n)(\sigma - 1)}$$

where $\sigma = (n-1)[((1/\mu) - 1)n + (\nu/\mu) + (t/2)] + (t/2)$.

Recall that μ is equal to the modulus of quasi-isometry for the model N_ϵ with (radial) curvature $A_\epsilon/(1+r^2)^{1+\epsilon}$. By taking A_ϵ sufficiently small, and choosing ν, t small, we can assume $\sigma < 1$.

Consider the infinite sum $\Sigma_{i_k \geq 0} a_{i_1 \dots i_n} W_1^{i_1} \dots w_n^{i_n}$. It is bounded by

$$\sum_{i_k \geq 0} (C' w_1)^{i_1} \dots (C' w_n)^{i_n} R^{(i_1 + \cdots + i_n)(\sigma - 1)} \frac{\sup_{B(R)} |f|}{i_1! \dots i_n!}$$

Clearly the series converges if $|w_i| \leq (R^{1-\sigma}/C')$. Since $\sigma < 1$, letting R go to infinity, the formal series $\Sigma\, a_{i_1 \dots i_n} f_1^{i_1} \dots f_n^{i_n}$ converges uniformly on compact sets on M. This proves the Proposition and concludes the proof of Theorem 2.2.

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