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AN ALGEBRAIC GROUP ASSOCIATED TO AN ERGODIC DIFFEOMORPHISM*

Robert J. Zimmer**

Introduction

The point of this paper is to observe that one can in a natural way associate to any ergodic diffeomorphism of a manifold (or more generally, to any Lie group acting ergodically on a manifold by diffeomorphisms) an algebraic group that reflects the action of the diffeomorphism (or Lie group) on the tangent bundle. We then relate the structure of this group to a measure theoretic version of a notion suggested by H. Furstenberg and studied by M. Rees [15], namely, that of a tangentially distal diffeomorphism, and to the vanishing of the entropy of the diffeomorphism. More precisely, we prove the following (see the sequel for the definitions involved).

THEOREM: *Let H be a discrete group acting by diffeomorphisms on a manifold M and μ a measure on M quasi-invariant and ergodic under the H-action. Suppose the H-action is essentially free, i.e., almost all stabilizers are trivial, and suppose further that the action is amenable* [19], [20] (*for example, if H itself is amenable*). *Then the following are equivalent.*

1) *The action is tangentially measure distal of all orders.*
2) *Every element of the algebraic group associated to the action has all eigenvalues on the unit circle.*
3) *There is a measurable Riemannian metric on M and H-invariant measurable sub-bundles of the tangent bundle $0 = V_0 \subset V_1 \subset \ldots \subset V_k = T(M)$ such that H is isometric on V_{i+1}/V_i for all i.*

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The algebraic group associated to the action is defined in section 1. Tangential measure distality is described and the above theorem proven in section 2. In section 3 we observe that the entropy of a tangentially measure distal C^2 diffeomorphism of a compact M preserving a finite smooth measure vanishes.

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1. Ergodic actions of algebraic groups

To construct the algebraic group associated to an ergodic diffeomophism we shall find it convenient to discuss first some general features of ergodic group actions. If G is a locally compact second countable group acting on a standard measure space (S, μ), μ is called quasi-invariant if for $A \subset S$ measurable and $g \in G$, $\mu(Ag) = 0$ if and only if $\mu(A) = 0$, and the action is called ergodic if $Ag = A$ for all g implies A is null or conull. (We always assume the action to be such that $S \times G \to S$, $(s, g) \to s \cdot g$ is Borel.) Thus any action by diffeomorphisms of a manifold leaves a smooth measure quasi-invariant. If $H \subset G$ is a closed subgroup and S is an ergodic H-space, then there is a naturally associated induced ergodic G-space X which we denote by $X = \mathrm{ind}_H^G(S)$. This notion is studied in [20] and here we only briefly recall the construction. Let $X = (S \times G)/H$ be the space of H-orbits under the product action $(s, g)h = (sh, gh)$. There is also a G-action on $S \times G$ given by $(s, g)g_0 = (s, g_0^{-1}g)$, and as this action commutes with the H-action G acts on X as well. Furthermore, by projecting a probability measure in the natural measure class on $S \times G$ to X, we obtain a quasi-invariant ergodic measure on X. A natural criterion for determining when an action is induced from that of a subgroup is the following.

Proposition 1.1: *If X is an ergodic G-space and $H \subset G$ is a closed subgroup then $X = \mathrm{ind}_H^G(S)$ for some ergodic H-space S if and only if there is a G-map $X \to G/H$. (As usual, we allow ourselves to discard invariant Borel null sets.)*

Proof: [20, Theorem 2.5].

The following observation is the basic point in the construction of the algebraic group associated to an ergodic diffeomorphism.

PROPOSITION 1.2: *Suppose G is a real algebraic group and X is an ergodic G-space. Then there is an algebraic subgroup $H \subset G$ such that X is induced from an ergodic H-space but not from an ergodic action of any smaller algebraic subgroup. This group H is unique up to conjugacy.*

PROOF: The existence of such an H is clear from the descending chain condition on algebraic subgroups, and so the point of the proposition is the uniqueness. Suppose there exist G-maps $X \to G/H_1$, $X \to G/H_2$ where H_i are algebraic subgroups and satisfy the above minimality property. Then there is a G-map φ: $X \to G/H_1 \times G/H_2$ which, of course, need no longer be measure class preserving. However, $\varphi_*(\mu)$ is a quasi-invariant G-ergodic measure on $G/H_1 \times G/H_2$. Because algebraic actions are "smooth" in that every ergodic measure is supported on an orbit [2, pp. 183–4], [3], $\varphi_*(\mu)$ is supported on a G-orbit in $G/H_1 \times G/H_2$. We can identify the orbit with G/G_0, where G_0 is the stabilizer of a point, and we can then view φ as a G-map $X \to G/G_0$. Since $G_0 = hH_1h^{-1} \cap gH_2g^{-1}$ for some $h, g \in G$, it follows by the minimality of H_i that $H_1 = (h^{-1}g)H_2(h^{-1}g)^{-1}$.

DEFINITION 1.3: We call H the algebraic hull of the ergodic G-space X. It is well-defined up to conjugacy. We call X Zariski dense if the algebraic hull is G itself.

This terminology is justified by the fact that if $X = G/G_0$ for some closed subgroup $G_0 \subset G$ then the algebraic hull of X is just the usual algebraic hull of the subgroup G_0. In other words, H is the algebraic hull of the virtual subgroup [7] of G defined by X.

In this framework, the Borel density theorem [1] has a natural formulation.

THEOREM 1.4: *Let G be a Zariski connected semisimple real algebraic group with no compact factors. Suppose X is an ergodic G-space with finite invariant measure. Then X is Zariski dense in G.*

PROOF: If $X \to G/H$ is a G-map then G/H has a finite invariant measure and the result follows by the theorem of Borel [1].

Suppose S is an ergodic H-space and G is locally compact (second countable) group. A Borel function α: $S \times H \to G$ is called a cocycle if for all $h_1, h_2 \in H$, $\alpha(s, h_1 h_2) = \alpha(s, h_1)\alpha(sh_1, h_2)$ a.e., and the cocycle is called strict if equality holds for all (s, h_1, h_2). There is a natural G-action associated to any cocycle that reduces to the induced action if $H \subset G$ and $\alpha(s, h) = h$. Namely, we let H act on $S \times G$ by $(s, g)h = (sh, g\alpha(s, h))$ and let X be the space of ergodic components. G also acts on $S \times G$ as in the inducing construction and this will factor to an ergodic action of G on X. We call this action the Mackey range of α. (See [7] for discussion and [14], [17] for details of the construction.] Two cocycles α, β are called equivalent if there is a Borel function φ: $S \to G$ such that for all $h \in H$, $\varphi(s)\alpha(s, h)\varphi(sh)^{-1} = \beta(s, h)$ a.e. The Mackey ranges of α and β will then be isomorphic G-spaces.

PROPOSITION 1.5: *If G is a locally compact group and $\alpha: S \times H \to G$ is a cocycle on the ergodic H-space S, then α is equivalent to a cocycle β taking values in a closed subgroup $G_0 \subset G$ if and only if the Mackey range is induced from an action of G_0.*

PROOF: Proposition 1.1 and [17, Theorem 3.5].

Thus, if G is algebraic the algebraic hull of the Mackey range of α is the unique (up to conjugacy) smallest algebraic subgroup which contains the image of a cocycle β which is equivalent to α.

Suppose now that H is a Lie group acting by diffeomorphisms on a smooth manifold M of dimension n. For each $m \in M$ and $h \in H$ we have a linear isomorphism $dh_m : T_mM \to T_{m \cdot h}M$. We can choose global Borel sections of the tangent bundle, so from the Borel point of view the tangent bundle can be trivialized. Under a choice of trivialization, dh_m corresponds to an element of $GL(n, \mathbb{R})$, and we define $d(m, h) = (dh_m)^{-1}$, so that $d: M \times H \to GL(n)$ is a strict cycle. Different trivializations clearly yield equivalent cocycles. Suppose now that the H-action is ergodic with respect to some quasi-invariant measure μ on M.

DEFINITION 1.6: By the algebraic group associated to the differentiable ergodic action of H on (M, μ) we mean the algebraic hull of the Mackey range of d.

We remark that this algebraic group may be trivial without the action being trivial. For example, the group will be trivial for any ergodic translation of a torus. However, the group will be nontrivial for any smooth measure preserving diffeomorphism of a compact manifold with positive entropy.

PROPOSITION 1.7: *Suppose that the differentiable action of H on*

(M, μ) is amenable. (This holds, in particular, if H is amenable; see [19], [20] for the general definition.) Then the algebraic group associated to the action is amenable.

PROOF: By [19, Theorem 3.3], the Mackey range of d is an amenable ergodic action, and by [20, Theorem 5.7] this action is induced from an action of an amenable subgroup. By a result of C. C. Moore [9], every amenable subgroup of $GL(n, \mathbb{R})$ is contained in an amenable algebraic subgroup. [20, Prop. 2.4] then implies that the Mackey range of d is induced from an action of an amenable algebraic subgroup, from which the proposition follows.

2. Measure distal linear extensions

Among the best understood continuous group actions on compact metric spaces are the distal actions, whose structure was described by Furstenberg in [5]. For actions on measure spaces there is a parallel theorem proved by the author in [18]. (See also [10].) For smooth actions there is a natural notion of tangential distality and the problem of investigating the structure of such actions was raised by Furstenberg. These actions were studied by M. Rees in [15] where she shows that one can deduce results about the structure of such actions on the tangent bundle, and the vanishing of entropy for measure preserving tangentially distal diffeomorphisms. The point of the rest of this paper is to define the notion of a tangentially measure distal action (which is weaker than that of a tangentially distal action), to show that one has an analgous description of the structure of such actions on the tangent bundle, and to discuss the relationship with entropy.

Let S be an ergodic H-space. For simplicity, we assume H is a countable discrete group. The results in this section should carry over to a general locally compact H, but we work in the discrete case to avoid some measure theoretic technicalities. Suppose $E \to S$ is a measurable vector bundle such that H acts on E in such a way that the projection to S is an H-map and the induced maps from fiber to fiber are linear. We call E a linear extension of S. For example, if $\alpha : S \times H \to GL(n, \mathbb{R})$ is a cocycle, (which, by discarding an invariant Borel null set we can assume is strict, owing to countability of H), we have an action of H on the product $S \times \mathbb{R}^n$ given by $(s, v) \cdot h = (sh, \alpha(s, h)^{-1} v)$. Clearly, every linear extension is equivalent in the obvious sense to one of this form.

DEFINITION 2.1: We call the linear extension $\varphi : E \to S$ a measure distal linear extension of S if there is a decreasing sequence of measurable sets $U_i \subset E$ such that

(i) $U_i(s) = U_i \cap \varphi^{-1}(s)$ is an open neighborhood of the origin in $\varphi^{-1}(s)$ for almost all s.

(ii) $\bigcap U_i(s) = \{0\}$ a.e.

(iii) for almost all s, if $v \in \varphi^{-1}(s)$ and there is a sequence $h_i \in H$ such that $(s, v) \cdot h_i \in U_i(sh_i)$ for all i, then $v = 0$.

Following [15], we call a linear extension measure distal of order p if the linear extension $\Lambda^p(E) \to S$ is a measure distal, and measure distal of all orders if $\Lambda^p(E) \to S$ is measure distal for all $1 \leq p \leq \dim E$. These conditions are clearly invariant under a change to an equivalent extension. An ergodic action of H by diffeomorphisms on (M, μ) is called tangentially measure distal (of order p, of all orders) if the tangent bundle is a measure distal (of order p, of all orders) linear extension of M. The main result of this section is the following. It is a measure theoretic analogue of the results of M. Rees [15]. It can also be considered as an extension to measurable bundles of the results of C. C. Moore on distal linear actions [8] and we in fact use this latter result.

THEOREM 2.2: *Let S be an essentially free amenable ergodic H-space (in particular, if H is amenable) where H is a countable discrete group and suppose $\alpha : S \times H \to GL(n)$ is a cocycle. Then the following are equivalent.*

(i) *$S \times \mathbb{R}^n$, with the action defined by α, is a measure distal linear extension of all orders.*

(ii) *α is equivalent to a cocycle into an amenable algebraic group G such that all eigenvalues of each element of G are on the unit circle.*

(iii) *There is a measurable assignment of inner products $s \to \langle\, , \rangle_s$ on $\mathbb{R}^n$ and a collection of H-invariant measurable fields of subspaces $s \to V_i(s)$, $\{0\} = V_0(s) \subset \ldots \subset V_k(s) = \mathbb{R}^n$ such that H acts (via the action defined by α) by isometries on the fibers of V_{i+1}/V_i, i.e., $\alpha(s, h) : (V_{i+1}(sh)/V_i(sh), \langle , \rangle_{sh}) \to (V_{i+1}(s)/V_i(s), \langle , \rangle_s)$ is an isometry.*

The theorem stated in the introduction follows by applying Theorem 2.2 to the tangent bundle of M.

PROOF: That (ii) and (iii) are equivalent follows easily from results of Moore [8]; that (ii) and (iii) imply (i) is straightforward; thus the problem is to show (i) implies (ii). Let G be the algebraic hull of the

Mackey range of α so that G is an amenable algebraic group. Then there is a sequence of subspaces $\{0\} = V_0 \subset V_1 \subset \ldots \subset V_k = \mathbb{R}^n$ and an inner product on $\mathbb{R}^n$ such that for $T = \{g \in GL(n, \mathbb{R}) \mid g(V_i) = V_i$ and the induced map on each V_{i+1}/V_i is a similarity of the induced inner product$\}$ we have $G \cap T$ is an algebraic subgroup of finite index in G [6, p. 356], and hence $G_0 = \bigcap_{g \in G} g(G \cap T)_g^{-1}$ is a normal algebraic subgroup of finite index in G. Let $\beta : S \times H \to G$ be a cocycle equivalent (as cocycles into $GL(n, \mathbb{R})$) to α and $q : G \to G/G_0$ the natural projection. The cocycle $q \circ \beta : S \times H \to G/G_0$ cannot be equivalent to a cocycle into a proper subgroup of G/G_0, for then, by [20, Lemma 5.2] β (and hence α) would be equivalent to a cocycle into a proper algebraic subgroup of G which would contradict the definition of G. It follows that the skew product action of H on $X = S \times G/G_0$ given by $(s, [g]) \cdot h = (sh, [g]q(\beta(s, h)))$ is ergodic [17, Cor. 3.8]. Let $\tilde{\beta} : X \times H \to G$ be the cocycle $\tilde{\beta}(x, h) = \beta(p(x), h)$ where $p : X \to S$ is a projection. By [20, Lemma 5.1], $\tilde{\beta}$ is equivalent to a cocycle γ taking values in G_0. Since β defines a measure distal linear extension, it is clear that $\tilde{\beta}$ and hence γ do as well.

Now let T_1 be the subgroup of T consisting of all matrices for which all eigenvalues lie on the unit circle. Since any $g \in T$ is a similarity on V_{i+1}/V_i, $g \in T_1$ if and only if $(\det g \mid (V_{i+1}/V_i))^2 = 1$, and hence T_1 is a normal algebraic subgroup of T. Let $G_1 = G_0 \cap T_1$, so that G_1 is a normal algebraic subgroup of G_0. We claim that to prove (i) implies (ii), it suffices to show that G_1 is of finite index in G. By [8, Theorem 1], to see that all eigenvalues of elements of G are on the unit circle, it suffices to show that G is distal on $\mathbb{R}^n$. However, we know G_1 is distal on $\mathbb{R}^n$ by [8, Theorem 1], and it is straightforward to check that if a subgroup of finite index acts distally, so does the whole group.

To show that G_1 is of finite index in G, we first claim that the fact that $\gamma : X \times H \to G_0$ defines a measure distal linear extension implies that $r \circ \gamma : X \times H \to G_0/G_1$ is equivalent to the identity cocycle where $r : G_0 \to G_0/G_1$ is the natural projection. There is a natural injection $G_0/G_1 \to T/T_1$ and T/T_1 is naturally isomorphic to $\mathbb{R}^k = \log((\mathbb{R}^*)^k/(\pm 1)^k)$ via the map that takes $g \in T$ to the set of determinants of $g \mid (V_i/V_{i-1})$, $i = 1, \ldots, k$. Since G_0/G_1 has only finitely many components, G_0/G_1 is embedded as a vector group in $\mathbb{R}^k$. To see that $r \circ \gamma$ is a trivial cocycle, it suffices to see that it is trivial when viewed as a cocycle into $\mathbb{R}^k$, and to show this it suffices, by an inductive use of [20, Lemma 5.2], to show that the projection of this cocycle onto each factor in $\mathbb{R}^k$ is trivial. We denote these projections by $r_i \circ \gamma$. Since $\mathbb{R}$ has no non-trivial compact subgroups, it follows from [4, Proposition 8.5 and Remark on p. 322] (see also [16]) that $r_i \circ \gamma$ is trivial if ∞ is not contained in the extended asymptotic range (or "extended asymptotic ratio set") [4, p. 317 ff.] of $r_i \circ \gamma$. We show this by induction on i, and let r_0 be the projection to a point, for which the assertion is trivial.

For any subset $Y \subset X$, let $Y * H = \{(x, h) | x, xh \in Y\}$. If ∞ is not in the extended asymptotic range of a real valued cocycle, then every set of positive measure contains a subset Y of positive measure such that the cocycle is bounded on $Y * H$. So if ∞ is not in the extended asymptotic range $r_p \circ \gamma$, $p = 0, \ldots, i-1$, then we can find a subset Y of positive measure such that $r_p \circ \gamma$ is bounded on $Y * H$ for all $p = 0, \ldots, i-1$. Let $U_j \subset X \times \Lambda^i(\mathbb{R}^n)$ be open sets as in Definition 2.1. For $\epsilon > 0$, let $B(\epsilon)$ be the open ball in $\Lambda^i(\mathbb{R}^n)$ (with respect to the inner product on $\mathbb{R}^n$ above) of radius ϵ. We can find $\epsilon_1 > 0$ such that for $A_1 = \{x \in Y | B(\epsilon_1) \subset U_1(x)\}$ we have $\mu(A_1) > 3\mu(Y)/4$. Choose ϵ_j and A_j inductively such that for $A_j = \{x \in A_{j-1} | B(\epsilon_j) \subset U_j(x)\}$ we have $\mu(A_j) > \mu(A_{j-1}) - \mu(Y)/2^{j+1}$. Thus for $A = \bigcap A_j$, we have $\mu(A) > \mu(Y)/2$. If ∞ is in the extended asymptotic range of $r_i \circ \gamma$, then by discarding a null set in A we have for each $x \in A$ and all N, there is $h_N \in H$ such that $xh_N \in A$ and $|r_i \circ \gamma(x, h_N)| > N$. Let $B_N(A) = \{x \in A | \text{there is } h_N \text{ with } xh_N \in A \text{ and } r_i \circ \gamma(x, h_N) > N\}$ and $C_N(A) = \{x \in A | \text{there is } h_N \text{ with } xh_N \in A \text{ and } r_i \circ \gamma(x, h_N) < -N\}$. Thus for each N, $B_N(A) \cup C_N(A) = A$ and $B_{N+1}(A) \subset B_N(A)$, $C_{N+1}(A) \subset C_N(A)$. We claim that both $\bigcap B_N(A)$ and $\bigcap C_N(A)$ are of positive measure. If one of them, say $\bigcap B_N(A)$, is null, it follows that $\lim \mu(C_N(A)) = \mu(A)$, and so in fact all $C_N(A) = A$. But $r_i \circ \gamma(xh_N, h_N^{-1}) = -r_i \circ \gamma(x, h_N)$. We claim this implies $B_N(A)$ is also equal to A from which our assertion will follow. If $D = A - B_N(A)$ is of positive measure, we can form $B_N(D)$ and $C_N(D)$ as above, so $B_N(D) \cup C_N(D) = D$. We cannot have $C_N(D)$ of positive measure, for if x, $xh \in D$ with $r_i \circ \gamma(x, xh) < -N$, then $xh \in B_N(D) \subset B_N(A)$. Since xh is also in D, this is impossible. On the other hand, $B_N(D) \subset B_N(A) \cap D$ which implies that it too is null. Thus D is null and $B_N(A) = A$.

Let $x \in B_N(A)$. Then the definition of $r_i \circ \gamma$ implies that for any unit vectors $v_p \in V_p \ominus V_{p-1}$, $p = 1, \ldots, i$, we have

$$\|\Lambda^i(\gamma(x, h)^{-1})v_1 \wedge \cdots \wedge v_i\| = \prod_{p=1}^{i} \exp(-r_p \circ \gamma(x, h)).$$

But by the definition of Y, $r_p \circ \gamma$ is bounded on $Y * H$ for $p < i$ and by the choice of A and $B_N(A)$, for any j we can find $h \in H$ such that

$(x, \Lambda^i(\gamma(x,h)^{-1})v_1 \wedge \cdots \wedge v_i) \in U_j$. This contradicts the condition that $\Lambda^i(\gamma)$ defines a measure distal linear extension. This verifies our assertion that ∞ is not in the extended asymptotic range of $r_i \circ \gamma$ and with it the assertion that $r \circ \gamma : X \times H \to G_0/G_1$ is trivial.

It follows from [20, Lemma 5.2] that γ, and hence $\tilde{\beta}$, is equivalent to a cocycle into G_1. Let $\lambda : X \to G$ be a Borel map such that (recalling that $X = S \times G/G_0$) for each h, $\lambda(s,[g])\beta(s,h)\lambda((s,[g]) \cdot h)^{-1} \in G_1$ a.e. Thus, letting $\omega : G \to G/G_1$ the projection, we have

$$(*) \qquad (\omega \circ \lambda)(s,[g]) \cdot \beta(s,g) = (\omega \circ \lambda)(sh,[g]\beta(s,h)).$$

Let $\mathcal{F}$ be the space of functions from G/G_0 to G/G_1. Then G acts on $\Phi \in \mathcal{F}$ by $(\Phi \cdot g)(y) = \Phi(yg^{-1}) \cdot g$. By identifying a function with its graph, $\mathcal{F}$ can be considered as a subset of the space $\mathcal{S}$ of finite subsets of $G/G_0 \times G/G_1$. Furthermore, G acts naturally on $\mathcal{S}$ and the restriction to $\mathcal{F}$ is just the action described above. As in the proof of [20, Theorem 5.5], $\mathcal{S}$ has a natural standard Borel structure, and the action of G is smooth on $\mathcal{S}$ because G, G_0, and G_1 are algebraic groups. Thus, G acts smoothly on $\mathcal{F}$. Define a map $\Phi : S \to \mathcal{F}$ by $\Phi(s)(y) = \omega(\lambda(s,y))$. Equation $(*)$ then implies that for each h and almost all s,

$$(**) \qquad \Phi(s) \cdot \beta(s,h) = \Phi(sh).$$

In particular, $\Phi(s)$ and $\Phi(sh)$ are in the same G-orbit in $\mathcal{F}$, and since the action of G on $\mathcal{F}$ is smooth, ergodicity of G on S implies that almost all $\Phi(s)$ are in the single G-orbit in $\mathcal{F}$. Let Φ_0 be an element in this orbit. We can then find a Borel map $\theta : \text{Orbit}\ (\Phi_0) \to G$ such that $\Phi_0 \cdot \theta(z) = z$ for all z. Let $f : S \to G$ be given by $f = \theta \circ \Phi$. Then $(**)$ implies $\Phi_0 \cdot f(s)\beta(s,h)f(sh)^{-1} = \Phi_0$. Thus β is equivalent to a cocycle β' taking values in a subgroup of G leaving Φ_0 fixed, and in particular into the subgroup $\tilde{G} \subset G$ leaving $\Phi_0(G/G_0) \subset G/G_1$ invariant. However, since $\Phi_0(G/G_0)$ is a finite set, $\tilde{G}$ is an algebraic subgroup of G and by the definition of G it follows that $\tilde{G} = G$. From this it follows that Φ_0 must be surjective and so G/G_1 is finite. (This in fact shows $G_1 = G_0$.) This completes the proof.

3. Vanishing entropy

Our aim in this section is to observe the following result.

THEOREM 3.1: *Let $T : M \to M$ be a C^2 ergodic diffeomorphism of a compact manifold M that preserves a finite smooth measure μ. If T is tangentially measure distal then the entropy $h(T,\mu) = 0$.*

PROOF: It suffices to show that all characteristic exponents of T are 0 [12, Theorem 5.1]. (See also [11], [13] for the theory of characteristic exponents.) If λ is a characteristic exponent, then for almost all $x \in M$ there is a non-zero $v \in T_xM$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|d(f^n)_x v\| = \lambda.$$

Thus, if $\lambda < 0$, $\lim \|d(f^n)_x v\| = 0$ and it is easy to see that this contradicts tangential measure distality. On the other hand, if $\lambda > 0$, then $-\lambda$ is a characteristic exponent of f^{-1}, which in a similar manner is impossible.

We remark that for tangentially distal diffeomorphisms, vanishing of topological entropy was demonstrated by M. Rees [15].

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