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# ON THE BERNSTEIN-GELFAND-GELFAND RESOLUTION AND THE DUFLO SUM FORMULA

O. Gabber and A. Joseph

**Abstract**

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. In ([8], Prop. 12) Duflo gave a remarkable sum formula interrelating induced ideals. The main result of this paper provides a natural generalization of this formula and more precisely gives a resolution for certain primitive quotients of the enveloping algebra $U(\mathfrak{g})$. The proof has three distinct steps. One, the extension of the Bernstein-Gelfand-Gelfand (in short, B.G.G.) resolution of a simple finite dimensional $U(\mathfrak{g})$ module to certain simple highest weight modules. Two, the description of the so-called $\mathfrak{k}$-finite part of the space of homomorphisms of any one Verma module to any other. Three, the proof of exactness of a certain functor. The last can be viewed as a non-trivial generalization of the fact that a Verma module with dominant highest weight is projective in the so-called $\mathcal{O}$ category. A by-product gives some results on a problem of Kostant relating $U(\mathfrak{g})$ to the $\mathfrak{k}$-finite part of the space of endomorphisms of a simple highest weight module.

## 1. Preliminaries

1.1: Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra for $\mathfrak{g}$, $R$ the set of non-zero roots, $R^+ \subset R$ a system of positive roots, $B \subset R^+$ the set of simple roots, $\rho$ the half sum of the positive roots, $s_\alpha \in \mathrm{Aut}(\mathfrak{h}^*)$ the reflection corresponding to the root $\alpha \in R$, and $W$ the group generated by the $s_\alpha : \alpha \in B$. Let $X_\alpha$ be the element of a

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Chevalley basis for $\mathfrak{g}$ corresponding to the root $\alpha$ and set

$$\mathfrak{n}^+ = \sum_{\alpha \in R^+} \mathbb{C}X_\alpha, \quad \mathfrak{n}^- = \sum_{\alpha \in R^+} \mathbb{C}X_{-\alpha}, \quad \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+.$$

1.2: For each $\lambda \in \mathfrak{h}^*$, set $R_\lambda = \{\alpha \in R : 2(\lambda, \alpha)/(\alpha, \alpha) \in \mathbb{Z}\}$ (which is itself a root system) and $R_\lambda^+ = R_\lambda \cap R^+$, with $B_\lambda \subset R_\lambda^+$ the corresponding set of simple roots. Call $\lambda$ regular (resp. dominant) if $(\lambda, \alpha) \neq 0$ (resp. $(\lambda, \alpha) \geq 0$) for all $\alpha \in R^+$. For each $B' \subset B_\lambda$, let $W_{B'}$ be the subgroup of $W$ generated by the $s_\alpha : \alpha \in B'$ and $w_{B'}$ the largest element of $W_{B'}$ with respect to its Bruhat order $\leq$ (as defined in [7], 7.7.3). If $B' = B_\lambda$ we write $W_{B'} = W_\lambda$, $w_{B'} = w_\lambda$. Let $M(\lambda)$ denote the Verma module with highest weight $\lambda$-$\rho$ associated to the quadruplet $\mathfrak{g}$, $\mathfrak{h}$, $B$, $\lambda$ (see [7], 7.1.4), $\overline{M(\lambda)}$ the unique maximal submodule of $M(\lambda)$, and set $L(\lambda) = M(\lambda)/\overline{M(\lambda)}$, $J(\lambda) = \operatorname{Ann} L(\lambda)$. For each $\mathfrak{h}$ module $V$ we set $V_\lambda = \{v \in V : Hv = (\lambda, H)v$, for all $H \in \mathfrak{h}\}$. Let $e_\lambda$ denote the canonical generator of $M(\lambda)$ (which has weight $\lambda - \rho$). Set $R_\lambda^0 = \{\alpha \in R : (\alpha, \lambda) = 0\}$.

1.3: Let $u \mapsto \check{u}$ (resp. $u \mapsto {}^t u$) denote the involutory antiautomorphism of $U(\mathfrak{g})$ defined by $\check{X} = -X : X \in \mathfrak{g}$ (resp. ${}^t X_\alpha = X_{-\alpha} : \alpha \in R$, ${}^t H = H : H \in \mathfrak{h}$). Identify $U := U(\mathfrak{g}) \otimes U(\mathfrak{g})$ canonically with $U(\mathfrak{g} \times \mathfrak{g})$. Define $j : \mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$ through $j(X) = (X, -{}^t X)$, set $\mathfrak{k} = j(\mathfrak{g})$, so $U(\mathfrak{k})$ may be regarded as a subalgebra of $U$. Let $\mathfrak{k}^\wedge$ denote the set of equivalence classes of finite dimensional irreducible representations of $\mathfrak{k}$. For each locally finite $\mathfrak{k}$ module $L$ and each $\sigma \in \mathfrak{k}^\wedge$, we let $L_\sigma$ denote the isotypical component of type $\sigma$ of $L$. Let $\iota : U(\mathfrak{k}) \to U(\mathfrak{g})$ be the $\mathbb{C}$ algebra isomorphism sending $(-{}^t X, X)$ to $X$ for every $X \in \mathfrak{g}$. If $R \xrightarrow[\varphi]{} S$ is a ring homomorphism and $M$ is a left $S$ module, we let $M^\varphi$ denote the left $R$ module which consists of the underlying abelian group $|M|$ of $M$ together with the operation $(r, m) \mapsto \varphi(r) \cdot m$ of $R$ on $|M|$.

1.4: Let $\mathcal{O}$ denote the category of finitely generated $U(\mathfrak{g})$ modules which are $\mathfrak{h}$ semisimple and $\mathfrak{b}$ locally finite (see [1-3, 6]). Each $M \in 0b\mathcal{O}$ has finite length [2]. This category has enough projectives and so the extension groups $\operatorname{Ext}^k(\cdot\,,\cdot)$ relative to $\mathcal{O}$ are thereby defined. Let $Z(\mathfrak{g})$ denote the centre of $U(\mathfrak{g})$. Then $\operatorname{Max} Z(\mathfrak{g})$ is isomorphic to $\mathfrak{h}^*/W$ such that for each $\lambda \in \mathfrak{h}^*$, $\hat{\lambda} := W\lambda$ corresponds to the element $Z(\mathfrak{g}) \cap J(\lambda)$ of $\operatorname{Max} Z(\mathfrak{g})$. Let $\mathcal{O}_{\hat{\lambda}}$ denote the subcategory of $\mathcal{O}$ of all modules annihilated by a power of this maximal ideal. Each $M \in 0b\mathcal{O}$ admits a primary decomposition and we denote by $p_{\hat{\lambda}} : 0b\mathcal{O} \to 0b\mathcal{O}_{\hat{\lambda}}$ the projection onto the primary component defined by $\hat{\lambda}$. It is an exact functor on $\mathcal{O}$.

1.5: Given $M, N \in Ob\,\mathcal{O}$, consider $\operatorname{Hom}_{\mathbb{C}}(M, N)$ (resp. $(M \otimes N)^*$) as a $U$ module through $((a \otimes b) \cdot x)m = ({}^t\check{a}x\check{b})m$ (resp. $((a \otimes b) \cdot y,$

$m \otimes n) = (y,\ \check{a}m \otimes \check{b}n))$ where $a,\ b \in U(g),\ m \in M,\ n \in N,\ x \in \mathrm{Hom}(M, N),\ y \in (M \otimes N)^*$. We remark that $(M(-\lambda) \otimes M(-\mu))^*$ is isomorphic to the $\mathfrak{g} \times \mathfrak{g}$ module co-induced from the $\mathfrak{b} \times \mathfrak{b}$ module $\mathbb{C}_{\lambda+\rho,\mu+\rho}$. Let $L(M, N)$ (resp. $L(M \otimes N)^*$) denote the set of all $\mathfrak{k}$-finite elements of $\mathrm{Hom}(M, N)$ (resp. $(M \otimes N)^*$) which we remark is again a $U$ module. For $\lambda, \mu \in \mathfrak{h}^*$, we set $L(\lambda, \mu) = L(M(-\lambda) \otimes M(-\mu))^*$.

1.6: Let $E$ be a finite dimensional $U(\mathfrak{g})$ module and given $M \in 0b\mathcal{O}$, consider $E \otimes M$ as a $U(\mathfrak{g})$ module through the diagonal action. One has $E \otimes M \in 0b\mathcal{O}$ and the functor $M \mapsto E \otimes M$ is exact. Again one has the natural isomorphisms

$$\mathrm{Hom}_{\mathfrak{g}}(E, \mathrm{Hom}_{\mathbb{C}}(M, N)) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{g}}(E \otimes M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{g}}(M, E^* \otimes N).$$

The latter gives on taking projective resolutions natural isomorphisms

$$\mathrm{Ext}^k(E \otimes M, N) \xrightarrow{\sim} \mathrm{Ext}^k(M, E^* \otimes N) : k \in \mathbb{N}.$$

1.7: Let $\mathcal{H}$ denote the category of all $U$ modules which satisfy the following properties. One, each $L \in 0b\mathcal{H}$ is locally finite as a $\mathfrak{k}$ module. Two, $\dim L_\sigma < \infty$ for each $\sigma \in \mathfrak{k}^\wedge$. Three, each $L \in 0b\mathcal{H}$ admits a finite filtration such that the centre of $U$ acts by scalars on each subquotient. Clearly $\mathcal{H}$ is stable under tensoring with finite dimensional $U$ modules. It follows from the classification ([21], I, Sect. 4) of the simple modules in $\mathcal{H}$ that each $L \in 0b\mathcal{H}$ has finite length (for example, as shown in ([2], 4.2)). For each $M, N \in 0b\mathcal{O}$, one has $L(M, N) \in \mathcal{H}$. Indeed, the first property holds by construction. The second obtains from the isomorphism valid for any simple finite dimensional $U(\mathfrak{g})$ module $E$, namely $\mathrm{Hom}_{\mathfrak{k}}(E^\iota, L(M, N)) \xleftarrow{\sim} \mathrm{Hom}_{\mathfrak{g}}(E \otimes M, N)$, the last space being finite dimensional (since $E \otimes M, N$ have finite length). The third obtains by taking composition series for $M, N$. We have shown that

LEMMA: *For each $M, N \in 0b\mathcal{O}$, the $U$ module $L(M, N)$ has finite length.*

1.8: Observe that $\tau : a \mapsto {}^t\check{a}$ is an involutory automorphism of $U(\mathfrak{g})$. Given $M \in 0b\mathcal{O}$, we let $\delta(M)$ denote the submodule of $(M^*)^\tau$ of all $\mathfrak{h}$ finite elements. Through the existence of a non-degenerate contravariant form on $L(\lambda)$ (see [11], 1.6), one has $L(\lambda) \cong \delta(L(\lambda))$. In particular $E^* \cong E^\tau$ for any finite dimensional module $E$. Again each $M \in 0b\mathcal{O}$ has finite length, so $\delta(M) \in 0b\mathcal{O}$ and $\delta(M)$ has the same composition factors as $M$ (with the same multiplicities).

1.9: For each $M, N \in 0b\mathcal{O}$, define $\sigma : \mathrm{Hom}_{\mathbb{C}}(M, (N^*)^\tau) \to (N \otimes M)^*$ through $(\sigma(x), m \otimes n) = (xm, n)$. From $(\sigma((a \otimes b) \cdot x), m \otimes n) = (((a \otimes b) \cdot x)m, n) = ({}^t\check{a}x\check{b}m, n) = (x\check{b}m, \check{a}n) = (\sigma(x), \check{a}n \otimes \check{b}m) = ((a \otimes b) \cdot \sigma(x), n \otimes m)$, it follows that $\sigma$ is a $U$ module homomorphism. Again $\sigma$ is obviously injective. Given $y \in (N \otimes M)^*$, then for each $m \in M$ the map $g(y, m) : n \mapsto (y, n \otimes m)$ of $N$ to $\mathbb{C}$ is $\mathbb{C}$-linear. It follows that the map $\eta(y) : m \mapsto g(y, m)$ of $M$ to $(N^*)^\tau$ is $\mathbb{C}$-linear and the map $\eta : y \mapsto \eta(y)$ is inverse to $\sigma$.

LEMMA: *The map $\sigma$ restricts to a $U$ module isomorphism of $L(M, \delta(N))$ onto $L(N \otimes M)^*$. In particular $L(N \otimes M)^*$ has finite length as a $U$ module.*

If $x \in L(M, \delta(N))$, then $\sigma(x)$ is obviously $\mathfrak{k}$-finite. Conversely for each $y \in L(N \otimes M)^*$, $m \in M$, $X \in \mathfrak{g}$, we have $X(\eta(y)m) = \eta(j({}^t\check{X})y)m + \eta(y)Xm$, and so the local finiteness of $\mathfrak{h}$ on $M$ implies that $\eta(y)m \in \delta(N)$. Hence the surjectivity of the restriction of $\sigma$. The last part follows from 1.7.

1.10: Define an ordering on $\mathbb{Z}B$ through $\mu \geq \nu$ if $\mu - \nu \in \mathbb{N}B$. Given $M \in 0b\mathcal{O}$, set $\Omega(M) = \{\lambda \in \mathfrak{h}^* : M_\lambda \neq 0\}$. If $M \neq 0$, then $\Omega(M)$ admits at least one maximal element. Note that $H_0(\mathfrak{n}^-, M) = M/\mathfrak{n}^- M$ is a locally finite semisimple $\mathfrak{h}$ module.

LEMMA: *Suppose $M, N \in 0b\mathcal{O}$ with $N$ a submodule of $M$. If $H_0(\mathfrak{n}^-, M)$, $H_0(\mathfrak{n}^-, N)$ are isomorphic as $\mathfrak{h}$ modules, then $M = N$.*

Assume $Q := M/N \neq 0$. Let $\mu \in \Omega(Q)$ be maximal. Through the maximality of $\mu$ one has $(\mathfrak{n}^- M)_\mu = \Sigma X_{-\alpha} M_{\mu+\alpha} = \Sigma X_{-\alpha} N_{\mu+\alpha} = (\mathfrak{n}^- N)_\mu$. Yet $\dim N_\mu/(\mathfrak{n}^- N)_\mu = \dim M_\mu/(\mathfrak{n}^- M)_\mu$, by hypothesis. This gives $M_\mu = N_\mu$, which is a contradiction.

1.11: For each $M \in 0b\mathcal{O}$, let $[M]$ denote the corresponding element in the Grothendieck group $\mathcal{G}$ of $\mathcal{O}$. For each $\hat{\lambda} \in \mathfrak{h}^*/W$, let $\mathcal{G}_{\hat{\lambda}}$ denote the subgroup of $\mathcal{G}$ corresponding to $\mathcal{O}_{\hat{\lambda}}$. It is well-known that $\{[L(\mu)] : \mu \in \hat{\lambda}\}$ is a basis for $\mathcal{G}_{\hat{\lambda}}$. Again each $M(\lambda) : \lambda \in \mathfrak{h}^*$ has finite length with simple factors amongst the $L(\mu) : \mu \in \hat{\lambda}$ and we denote by $[M(\lambda) : L(\mu)]$ the number of times $L(\mu)$ occurs in $M(\lambda)$. The resulting matrix is invertible (by [7], 7.6.23) and (by [7], 7.6.14) one has

$$[E \otimes M(\lambda)] = \sum_{\nu \in \Omega(E)} [M(\lambda + \nu)] \dim E_\nu$$

for any finite dimensional $U(\mathfrak{g})$ module $E$.

1.12: Let $P(R)$ denote the lattice of integral weights. Let $P(R)^+$ (resp. $P(R)^{++}$) denote the dominant (resp. dominant and regular) elements of $P(R)$. For each $\nu \in P(R)$, let $E(\nu)$ denote a (unique up to isomorphism) simple finite dimensional $U(\mathfrak{g})$ module with extreme weight $\nu$. The map $\nu \mapsto E(\nu)^\iota$ identifies the $\mathfrak{k}^\wedge$ of classes of finite dimensional simple $U(\mathfrak{k})$ modules with $P(R)/W$ and hence with $P(R)^+$. Frobenius reciprocity gives $\dim \operatorname{Hom}_{\mathfrak{k}}(E(\nu)^\iota, L(\lambda, \mu)) = \dim E(\nu)_{\mu-\lambda}$ for all $\lambda, \mu \in \mathfrak{h}^*$, $\nu \in P(R)$. In particular, $L(\lambda, \mu) \neq 0$ if and only if $\lambda - \mu \in P(R)$. Now assume $\lambda - \mu \in P(R)$. Then by 1.9, $L(\lambda, \mu)$ has finite length. Since $\dim E(\lambda - \mu)_{\lambda-\mu} = 1$, it follows that $L(\lambda, \mu)$ admits a unique simple subquotient, which we denote by $V(\lambda, \mu)$, satisfying $\dim \operatorname{Hom}_{\mathfrak{k}}(E(\lambda - \mu), V(\lambda, \mu)) = 1$. We shall need the following

THEOREM:

(i) *Every simple module in $\mathcal{H}$ is isomorphic to some $V(\lambda, \mu)$.*

(ii) *$V(\lambda, \mu)$ is isomorphic to $V(\lambda', \mu')$ if and only if there exists $w \in W$ such that $\lambda' = w\lambda$, $\mu' = w\mu$.*

(iii) *Suppose $\lambda \in \mathfrak{h}^*$ is dominant. Then if $L(M(\lambda), L(\mu)) \neq 0$ (which holds in particular if $\lambda$ is regular), it is isomorphic to $V(-\mu, -\lambda)$. Furthermore every simple $V \in 0b\mathcal{H}$ is so obtained.*

(i), (ii) are just ([9], I, 4.1, 4.5) and (iii) follows from ([14], 4.7) and (i), (ii).

1.13: Given $-\lambda \in \mathfrak{h}^*$ dominant, then for each $-\mu \in -\lambda + P(R)$ dominant we define following Jantzen ([11], Sect. 2) a translation operator $T_\lambda^\mu : \mathcal{O} \to \mathcal{O}$ through $T_\lambda^\mu M = p_{\hat{\mu}}(E(\mu - \lambda) \otimes p_{\hat{\lambda}}(M))$. If $R_\lambda^0 \subset R_\mu^0$, then for all $w \in W_\lambda$ we have $T_\lambda^\mu M(w\lambda) \cong M(w\mu)$ (see [10], 2.10). Let $E$ be a finite dimensional $U(\mathfrak{g})$ module. Through the natural $U$ module isomorphisms $L(M, N) \otimes (\mathbb{C} \otimes E) \xrightarrow{\sim} L(M \otimes E^*, N)$, $L(M, N) \otimes (E \otimes \mathbb{C}) \xrightarrow{\sim} L(M, N \otimes E^\tau)$, it is obvious how to define exact functors on $\mathcal{H}$ satisfying $R_\lambda^\mu L(M, N) \cong L(T_\lambda^\mu M, N)$, $S_\lambda^\theta L(M, N) \cong L(M, T_\lambda^\mu N)$ for $M, N \in 0b\mathcal{O}$. Again by 1.6, $T_\lambda^\mu$ is both left and right adjoint to $T_\mu^\lambda$.

1.14: For each $j \in \mathbb{N}$, $\mu \in \mathfrak{h}^*$, $N \in 0b\mathcal{O}$, one has $\operatorname{Ext}^j(M(\mu), N) \cong H^j(\mathfrak{n}^+, N)_{\mu-\rho} \cong (H_j(\mathfrak{n}^-, \delta(N))_{\mu-\rho})^*$, the first isomorphism being due to Delorme ([6], Thm. 2), the second a formal consequence of the appropriate standard complexes.

1.15: Take $\lambda, \mu \in \mathfrak{h}^*$ and let us note the almost obvious fact that $L(M(\lambda), M(\mu)) = 0$ unless $\lambda - \mu \in P(R)$. This latter condition further implies that $W_\lambda = W_\mu$.

LEMMA: *Fix* $-\lambda, -\mu \in \mathfrak{h}^*$ *dominant with* $\lambda - \mu \in P(R)$. *Then for each* $w \in W_\lambda$ *and each finite dimensional* $U(\mathfrak{g})$ *module* $E$ *one has*

$$\dim \operatorname{Hom}_{\mathfrak{k}}(E^\iota, L(M(w_\lambda\lambda), M(w\mu))) = \dim \operatorname{Hom}_{\mathfrak{k}}(E^\iota, L(M(w^{-1}w_\lambda\lambda), M(\mu))).$$

We show that both sides equal $\dim E_{w\mu - w_\lambda\lambda}$. For the right hand side this follows from the fact that $M(\mu)$ is simple (and so isomorphic to $\delta M(\mu)$), 1.9 and 1.12, noting that $\Omega(E)$ is $W$ stable. The left hand side equals (by 1.7) $\dim \operatorname{Hom}_{\mathfrak{g}}(M(w_\lambda\lambda),\ E^* \otimes M(w\mu))$; since $M(w_\lambda\lambda)$ is projective in $\mathcal{O}$, we have by 1.11 that the latter equals $\dim(E^*)_{w_\lambda\lambda - w\mu} = \dim E_{w\mu - w_\lambda\lambda}$.

*Remarks*: Although this also follows from ([14], 4.10) the above proof is much simpler. It is not difficult to extend the above to a further proof of ([14], 4.3) and hence of Duflo's theorem ([8], Thm. 1); but then this becomes essentially the proof given in ([3], 4.4).

1.16: Take $\lambda \in \mathfrak{h}^*$ dominant. $Z(\mathfrak{g})$ acts on $M(\lambda)$ by a homomorphism $\chi_\lambda : Z(\mathfrak{g}) \to \mathbb{C}$. Let $C = \lambda + P(R)$, and let $\mathcal{O}_C$ be the full subcategory of $\mathcal{O}$ consisting of those modules $M$ that satisfy $\Omega(M) \subset C$. Define a functor $T : \mathcal{O}_C \to \mathcal{H}$ by $T(N) = L(M(\lambda), N)$ (cf. 1.7). $T$ is exact since any $M(\lambda) \otimes E$ ($E$ being a finite dimensional $U(\mathfrak{g})$ module) is projective in $\mathcal{O}$. Let $\mathcal{H}$ consisting of those $M \in \mathrm{Ob}(\mathcal{H})$ on which $1 \otimes Z(\mathfrak{g})$ acts through $1 \otimes z \mapsto \chi_\lambda(\check{z})$. The image of $T$ lies in $\mathcal{H}_{\hat{\lambda}}$, and in the following theorem we view $\mathcal{H}_{\hat{\lambda}}$ as the target category of $T$.

THEOREM:

(i) *$T$ has a left adjoint $T'$.*

(ii) *The unit map $\eta : Id_{\mathcal{H}_{\hat{\lambda}}} \to TT'$ is an isomorphism of functors.*

(iii) *If $\lambda$ is regular, then $T$ is an equivalence of categories.*

We indicate a proof for the theorem, which has also been proved by Bernstein and Gelfand ([3], 6.3, 6.1 (ii), 5.9 (i)).

(i). If $M \in \mathrm{Ob}(\mathcal{H}_{\hat{\lambda}})$, we make $M$ into a two-sided $U(\mathfrak{g})$ module by $\mathrm{amb} = ({}^\iota\check{a} \otimes \check{b}) \cdot m$ for all $m \in M$, $a, b \in U(g)$. Define $T'(M) = M \otimes_A M(\lambda)$, where $A = U(\mathfrak{g})/U(\mathfrak{g})\ker(\chi_\lambda)$. Now $T'(M) \in \mathrm{Ob}(\mathcal{O}_C)$ because if $E \subset M$ is a finite dimensional $\mathfrak{k}$ stable generating subspace (so $M = EU(\mathfrak{g})$), then we get a surjective $\mathfrak{g}$ linear map $E^{\iota^{-1}} \otimes M(\lambda) \to T'(M)$. If $M \in \mathrm{Ob}(\mathcal{H}_{\hat{\lambda}})$ and $N \in \mathrm{Ob}(\mathcal{O}_C)$, one defines an isomorphism $\zeta(M, N) : (\operatorname{Hom}_{\mathfrak{g}}(M \otimes_A M(\lambda), N) \to \operatorname{Hom}_U(M, L(M(\lambda), N))$ by $\zeta(\varphi) = (m \mapsto \varphi(m \otimes u)))$. This makes $T'$ a left adjoint to $T$.

(ii). We have to show that for any $M \in \mathrm{Ob}(\mathcal{H}_{\hat{\lambda}})$ the map $\eta(M) : M \to$

$L(M(\lambda), M \otimes_A M(\lambda))$ (given by $m \mapsto (n \mapsto m \otimes n)$)) is bijective. We make $A$ into a $U$ module by $(a \otimes b) \cdot x = {}^t\check{a}x\check{b}$, for all $x \in A$, $a$, $b \in U(g)$. Then $\eta(A)$ is an isomorphism by ([13], 6.4).

If $E$ is a finite dimensional $U(\mathfrak{g})$ module we have natural isomorphisms

$$T(E \otimes N) \xleftarrow{\sim} (E^\tau \otimes \mathbb{C}) \otimes T(N), N \in 0b(\mathcal{O}_C)$$

$$T'((E^\tau \otimes \mathbb{C}) \otimes M) \xleftarrow{\sim} E \otimes T'(M), M \in 0b(\mathcal{H}_{\hat{\lambda}}).$$

Using these isomorphisms, one shows that if $\eta(M)$ is an isomorphism then so is $\eta((E^\tau \otimes \mathbb{C}) \otimes M)$. In particular, $\eta((E^\tau \otimes \mathbb{C}) \otimes A)$ is an isomorphism. This implies that $\eta(M)$ is an isomorphism for any $M \in 0b(\mathcal{H}_{\hat{\lambda}})$, by observing that $TT'$ is right exact and that for suitable finite dimensional $U(\mathfrak{g})$ modules $E_1, E_2$ there exists an exact sequence $(E_1 \otimes \mathbb{C}) \otimes A \to (E_2 \otimes \mathbb{C}) \otimes A \to M \to 0$ in $\mathcal{H}_{\hat{\lambda}}$.

(iii). We have to show that the counit map $\epsilon : T'T \to Id_{\mathcal{O}_C}$ is also an isomorphism of functors. The composition $T \xrightarrow{\eta T} TT'T \xrightarrow{T\epsilon} T$ is $Id_T$, so by (ii) $T\epsilon$ is an isomorphism. Thus, as $T$ is exact, $0 = T(\ker(\epsilon(N))) = T(\operatorname{coker}(\epsilon(N)))$ for any $N \in 0b(\mathcal{O}_C)$. So it remains to show that if $N \in 0b(\mathcal{O}_C)$ and $TN = 0$ then $N = 0$. Indeed, if $N \neq 0$, then $N$ contains a simple submodule $L(\mu) : \mu \in C$, so $TN \supset TL(\mu)$; but by ([14], 4.7) $TL(\mu) \neq 0$, and we get a contradiction.

## 2. The generalized B.G.G. resolution

Throughout this section we fix $-\lambda \in \mathfrak{h}^*$ dominant and regular.

2.1: Given $\alpha \in B_\lambda$, one can choose $\nu_\alpha \in P(R)$ such that $-\lambda_\alpha := -\lambda + \nu_\alpha$ is dominant and $(\beta, \lambda_\alpha) = 0 : \beta \in R^+$ is equivalent to $\beta = \alpha$. Following Vogan ([22]) we set $\theta_\alpha = T^\lambda_{\lambda_\alpha} \circ T^{\lambda_\alpha}_\lambda : \mathcal{O} \to \mathcal{O}$. Using 1.13, $\theta_\alpha$ is left adjoint to $\theta_\alpha$. So we obtain natural isomorphisms

$$\operatorname{Ext}^j(\theta_\alpha M, N) \xrightarrow{\sim} \operatorname{Ext}^j(M, \theta_\alpha N) : j \in \mathbb{N}, M, N \in 0b\,\mathcal{O}.$$

2.2: For each $w \in W_\lambda$, let $l(w)$ denote the reduced length of $w$ with respect to $B_\lambda$. For each $w, w' \in W_\lambda$, we define an expression $P_{w,w'}$ in the indeterminate $q$ through

$$P_{w,w'}(q) = \sum_{k=0}^{\infty} q^{(l(w')-l(w)-k)/2} \dim \operatorname{Ext}^k(M(w\lambda), L(w'\lambda)).$$

A result of Casselman and Schmid (proved also in [6], Thm. 4)

implies that $P_{w,w'}(q)$ is polynomial in $q^{1/2}$. Kazhdan and Lusztig ([19], Conj. 1.5) have further conjectured that $P_{w,w'}(q)$ is polynomial in $q$ and that this polynomial is determined by a particular purely combinatorial procedure which uses only the description of $W_\lambda$ as a Coxeter group. This has been shown to follow from certain other conjectures ([10], [23]); but for the moment remains an open problem. Here we just establish one of the identities which would follow from the Kazhdan-Lusztig conjecture.

LEMMA: *For each $w, w' \in W_\lambda$, $\alpha \in B_\lambda$, such that $w's_\alpha < w'$, one has that $P_{ws_\alpha,w'}(q) = P_{w,w'}(q)$. In particular, for each $B' \subset B_\lambda$, $w \in W_{B'}$, one has that $P_{w,w_{B'}}(q) = 1$.*

We can assume $ws_\alpha < w$, without loss of generality. Then the conclusion of the lemma is equivalent to the identity

$$(*) \qquad \dim \operatorname{Ext}^{j+1}(M(ws_\alpha\lambda), L(w'\lambda)) = \dim \operatorname{Ext}^j(M(w\lambda)), L(w'\lambda)), j \in \mathbb{N}.$$

Under the hypothesis $w's_\alpha < w'$, it follows that $M(w's_\alpha\lambda)$ is a submodule of $M(w'\lambda)$ and by ([10], 2.10a) that $\theta_\alpha M(w'\lambda) \xrightarrow{\sim} \theta_\alpha M(w's_\alpha\lambda)$. Hence $\theta_\alpha(M(w'\lambda)/M(w's_\alpha\lambda)) = 0$. Since $L(w'\lambda)$ is a quotient of $M(w'\lambda)/M(w's_\alpha\lambda)$, it follows that $\theta_\alpha L(w'\lambda) = 0$, and so $\operatorname{Ext}^j(\theta_\alpha M(w\lambda), L(w'\lambda)) = 0$ by 2.1. In particular $L(w\lambda)$ is not a quotient of $\theta_\alpha M(w\lambda)$. Then from ([11], 2.17) we obtain an exact sequence

$$0 \to M(w\lambda) \to \theta_\alpha M(w\lambda) \to M(ws_\alpha\lambda) \to 0$$

from which the corresponding long exact sequence for $\operatorname{Ext}^*(\cdot\,, L(w'\lambda))$ gives $(*)$.

2.3: From 1.8, 1.14 and 2.2 we obtain

COROLLARY: *For each $B' \subset B_\lambda$, $w \in W_\lambda$, one has*

$$\dim H_j(\mathfrak{n}^-, L(w_{B'}\lambda))_{w\lambda-\rho} = \begin{cases} 1: w \in W_{B'}, j = l(w_{B'}) - l(w), \\ 0: \textit{otherwise.} \end{cases}$$

*Remarks.* As is well-known the remaining weight spaces of $H_j(\mathfrak{n}^-, L(w_{B'}\lambda))$ are null. This follows from the action of $Z(\mathfrak{g})$ and the fact that $w\lambda - \lambda \in \mathbb{Z}B$ implies $w \in W_\lambda$. This result then generalizes the Bott-Kostant formula established for finite dimensional simple modules (i.e. when $-\lambda \in P(R)^{++}$and $B' = B$).

2.4: Fix $B' \subset B_\lambda$ and set $s = l(w_{B'})$. Then for each $j \in \mathbb{N}$, set $W^j_{B'} =$

$\{w \in W_{B'} : l(w) = j\}$, and

$$C_j = \bigoplus_{w \in W_{B'}^j} M(w\lambda).$$

As $M(w\lambda)$ is $U(\mathfrak{n}^-)$ free, we have for each $y \in W_{B'}$ that

$$\dim H_t(\mathfrak{n}^-, C_j)_{y\lambda-\rho} = \begin{cases} 1 : t = 0, j = l(y), \\ 0 : \text{otherwise.} \end{cases}$$

2.5: For each $w \in W_{B'}$, fix a $U(\mathfrak{g})$ module embedding $i_w : M(w\lambda) \hookrightarrow M(w_{B'}\lambda)$. For $w, w' \in W_{B'}$ such that $w \leq w'$, let $i_{w,w'} : M(w\lambda) \to M(w'\lambda)$ be the embedding such that $i_{w'} \circ i_{w,w'} = i_w$.

Fix $j \in \{1, 2, \ldots, s\}$ and consider a $U(\mathfrak{g})$ module map $\partial_j : C_{j-1} \to C_j$ defined by $(x_w)_{w \in W_B^{j-1}} \overset{\partial_j}{\mapsto} (y_{w'})_{w' \in W_{B'}^j}$ when

$$y_{w'} = \sum_{w \leq w'} c^j_{w,w'} i_{w,w'}(x_w), \quad x_w \in M(w\lambda),$$

where $c^j_{w,w'} \in \mathbb{Z}$ is non-zero and defined whenever $w \leq w'$, $w \in W_{B'}^{j-1}$, $w' \in W_{B'}^j$.

LEMMA: *The natural surjection* $H_0(\mathfrak{n}^-, C_{j-1}) \to H_0(\mathfrak{n}^-, \operatorname{Im} \partial_j)$ *is bijective.*

Set $K = \ker \partial_j$, $V = W_{B'}^{j-1}$. We have an exact sequence $0 \to K/K \cap \mathfrak{n}^- C_{j-1} \to C_{j-1}/\mathfrak{n}^- C_{j-1} \to \partial_j C_{j-1}/\mathfrak{n}^-(\partial_j C_{j-1}) \to 0$, so the lemma is equivalent to $K \subset \bigoplus_{w \in V} \mathfrak{n}^- M(w\lambda)$, or to $K \subset \bigoplus_{w \in V} \overline{M(w\lambda)}$, that is to

$$\overline{K} := \operatorname{Im}(K \to \bigoplus_{w \in V} L(w\lambda)) = 0.$$

If $\overline{K} \neq 0$, there exists $w \in V$ such that $[\overline{K} : L(w\lambda)] > 0$, and so $[K : L(w\lambda)] > 0$. Yet equality of lengths in $V$ implies through ([7], 7.6.23) that $[C_j : L(w\lambda)] = 1$, so $0 = [C_j/K : L(w\lambda)] = [\partial_j C_{j-1} : L(w\lambda)]$. On the other hand since there exists $w' \in W_{B'}^j$ such that $w \leq w'$ and by hypothesis we then have $c^j_{w,w'} \neq 0$, it follows that $\partial_j$ is injective on the summand $M(w\lambda)$ of $C_{j-1}$. Thus $\partial_j C_{j-1}$ contains a copy of $M(w\lambda)$, which implies $[\partial_j C_{j-1} : L(w\lambda)] \geqslant 1$. This contradiction proves the lemma.

2.6: An appropriate combinatorial property of the Bruhat ordering enables one to choose the $c^j_{w,w'}$ of 2.5 such that $\partial_j \partial_{j-1} = 0$, for all $j = 2, \ldots, s$. (See [2], Sect. 11 or [7], 7.8.14). Furthermore

PROPOSITION: *The sequence*

$$0 \to C_0 \to C_1 \to \cdots C_{j-1} \xrightarrow{\partial_j} C_j \to \cdots \to C_s \to L(w_{B'}\lambda) \to 0$$

*is exact.*

Set $X_{s+1} = Y_{s+1} = L(w_{B'}\lambda)$, $Y_s = \ker(C_s \to X_{s+1})$, and for each $j \in \{1, 2, \ldots, s\}$, set $X_j = \operatorname{Im} \partial_j$, $Y_{j-1} = \ker \partial_j$. For each $j \in \{1, 2, \ldots, s+1\}$, $X_j$ is a submodule of $Y_j$ and we show that $X_j = Y_j$. Fix $r \geq 1$ and assume that this has been established for all $j > r$. This means that we have the short exact sequences

$$0 \to X_j \to C_j \to X_{j+1} \to 0 : r < j \leq s.$$

By 2.4, the associated long exact sequence for homology implies for all $\mu \in \mathfrak{h}^*$ and $r < j \leq s$ that

$$\dim H_t(\mathfrak{n}^-, X_j)_\mu = \begin{cases} \dim H_{t+1}(\mathfrak{n}^-, X_{j+1})_\mu & : t > 0 \\ \dim H_1(\mathfrak{n}^-, X_{j+1})_\mu - \dim H_0(\mathfrak{n}^-, X_{j+1})_\mu \\ \quad + \dim H_0(\mathfrak{n}^-, C_j)_\mu & : t = 0. \end{cases}$$

Then from 2.3 and 2.4 we obtain

$$\dim H_t(\mathfrak{n}^-, X_j)_\mu = \begin{cases} 1 : \mu = w\lambda,\ w \in W_{B'},\ l(w) = j - t - 1, \\ 0 : \text{otherwise.} \end{cases}$$

for all $j > r$ and in particular for $j = r + 1$.

Finally from the long exact sequence associated to $0 \to Y_r \to C_r \to X_{r+1} \to 0$, 2.4 and the above we eventually obtain

$$\dim H_0(\mathfrak{n}^-, Y_r)_\mu = \dim H_0(\mathfrak{n}^-, C_{r-1})_\mu$$

for all $\mu \in \mathfrak{h}^*$. Then by 2.5, $H_0(\mathfrak{n}^-, Y_r)$ and $H_0(\mathfrak{n}^-, X_r)$ are isomorphic as $\mathfrak{h}$-modules and so $X_r = Y_r$ by 1.10. Noting that $\partial_1$ is injective completes the proof of the proposition.

*Remark.* This generalizes the B.G.G. resolution originally established [2] for the case $-\lambda \in P(R)^{++}$, $B' = B$. The original proof is different to ours and can only be generalized to the case when $B' \subset B$ (see [20] for this). The present proof was found following conversations with M. Duflo and P. Delorme.

## 3. Mappings of Verma modules

3.1: Take $-\lambda \in \mathfrak{h}^*$ dominant. Then $M(\lambda)$ is a simple module and so isomorphic to $\delta(M(\lambda))$. Then by 1.9 one has for all $\mu \in \mathfrak{h}^*$ that

$$L(M(\mu), M(\lambda)) = L(M(\mu), \delta M(\lambda)) = L(-\lambda, -\mu),$$

up to isomorphisms. This relationship of mappings of Verma modules to the principal series has been known for some time. Here we consider the most general form this takes when $-\lambda$ is not necessarily dominant. Some results in this direction were already obtained in ([5], 5.5) and in ([14], 4.10).

3.2: Fix $-\lambda$, $-\mu \in \mathfrak{h}^*$ dominant with $\lambda - \mu \in P(R)$ (recall 1.15). Choose $w_1$, $w_2 \in W_\lambda$ and $\alpha \in B_\lambda$ such that $s_\alpha w_1 > w_1$, $s_\alpha w_2 < w_2$. The second relation implies that $M(s_\alpha w_2 \lambda)$ is a submodule of $M(w_2\lambda)$.

LEMMA: *Under the above hypotheses, one has* $L(M(w_1\mu), M(w_2\lambda)/M(s_\alpha w_2\lambda)) = 0$.

Equivalently for any finite dimensional $U(\mathfrak{g})$ module $E$ one has $\mathrm{Hom}_{\mathfrak{g}}(M(w_1\mu), (E \otimes (M(w_2\lambda)/M(s_\alpha w_2\lambda)))) = 0$. To establish this it is enough to show that $L(w_1\mu)$ is not a subquotient of $p_{\hat{\mu}}(E \otimes M(w_2\lambda)/M(s_\alpha w_2\lambda))$. Now by 1.11 and the invariance of $\Omega(E)$ under $W$ one has

$$[E \otimes (M(w_2\lambda)/M(s_\alpha w_2\lambda))] = \sum_{\nu \in \Omega(E)} ([M(w_2(\lambda+\nu))] - [M(s_\alpha w_2(\lambda+\nu))])(\dim E_\nu),$$

and so

$$(*) \qquad [p_{\hat{\mu}}(E \otimes M(w_2\lambda)/M(s_\alpha w_2\lambda))] =$$

$$= \sum_{\substack{w \in W_\lambda: \\ w\mu \in \lambda + \Omega(E)}} (\dim E_{w\mu-\lambda})([M(w_2 w\mu)] - [M(s_\alpha w_2 w\mu)]).$$

Through the hypothesis $s_\alpha w_1 > w_1$, one has by ([10], 5.19) that

$$(**) \qquad [M(w_2 w\mu) : L(w_1\mu)] = [M(s_\alpha w_2 w\mu) : L(w_1\mu)].$$

Combined with (*) this establishes the assertion of the lemma.

*Remarks.* A technically easier proof of (**) follows from ([11],

2.16) and ([3], 4.5 (6)). Again the analysis of ([14], 5.4) can be combined with the operators of coherent continuation to give an alternative proof of the fact that $L(w_1\mu)$ is not a subquotient of $E \otimes (M(w_2\lambda)/M(s_\alpha w_2\lambda))$.

3.3: Let $W$ be a Coxeter group with $S$ the corresponding set of simple reflections and length function $l(\cdot)$. It is well-known that there exists an associative product $*$ on $W$ uniquely defined through

$$w_* w' = ww' \quad \text{if} \quad l(w) + l(w') = l(ww'),$$

$$s_* s = s \quad \text{if} \quad s \in S.$$

(Up to a sign, these are the defining relations for the generators of the "singular Hecke algebra" obtained say from ([19], Sect. 1) by putting $q = 0$.)

LEMMA: *For all $w, y \in W$, $s \in S$, one has*

(i)
$$s_* w = \begin{cases} sw : sw > w, \\ w \ : sw < w. \end{cases}$$

(ii)
$$w_* s = \begin{cases} ws : ws > w, \\ w \ : ws < w. \end{cases}$$

(iii)
$$(w_* y)^{-1} = y^{-1} * w^{-1}.$$

(iv)
$$w_* w' \geq w', w.$$

The top lines of (i), (ii) are immediate from the definition of $*$. For the bottom line in say (ii), set $w' = sw$. Then $sw' > w'$ and so $s_* w = s_*(s_* w') = (s_* s)_* w' = s_* w' = w$.

We prove (iii) by induction on $l(w)$. For $l(w) = 0, 1$, it follows from (i), (ii). Otherwise write $w = s_* z : l(z) < l(w)$. Then $(w_* y)^{-1} = ((s_* z)_* y)^{-1} = (s_*(z_* y))^{-1} = ((z_* y)^{-1}{}_* s) = (y^{-1}{}_* z^{-1})_* s = y^{-1}{}_*(z^{-1}{}_* s) = y^{-1}{}_* w^{-1}$. (iv) follows from (i), (ii).

3.4: Fix $-\lambda \in \mathfrak{h}^*$ dominant. For all $w_1, w_2 \in W_\lambda$, one has from 3.3 (iv) that $w_2^{-1}{}_* w_1 w_\lambda \geq w_1 w_\lambda$ and so $w_3 := (w_2^{-1}{}_* w_1 w_\lambda) w_\lambda \leq w_1$.

PROPOSITION: *Assume $-\lambda, -\mu \in \mathfrak{h}^*$ dominant with $\lambda - \mu \in P(R)$. Given $w_1, w_2 \in W_\lambda$, define $w_3 \in W_\lambda$ as above. Then the $U$-module*

*homomorphism of* $L(M(w_1\lambda), M(w_2\mu))$ *into* $L(M(w_3\lambda), M(w_2\mu))$ *defined by restriction is injective with image* $L(M(w_3\lambda), M(\mu))$.

The assertion is clear for $w_2 = 1$. If $w_2 \neq 1$, choose $\alpha \in B_\lambda$ such that $s_\alpha w_2 < w_2$. If $s_\alpha w_1 > w_1$, then by 3.2 the natural embedding $L(M(w_1\lambda), M(s_\alpha w_2\mu)) \hookrightarrow L(M(w_1\lambda), M(w_2\mu))$ is surjective. If $s_\alpha w_1 < w_1$, then by ([13], 6.1) the map of $L(M(w_1\lambda), M(w_2\mu))$ into $L(M(s_\alpha w_1\lambda), M(w_2\mu))$ defined by restriction is injective and so, by 3.2 again, we obtain an embedding of $L(M(w_1\lambda), M(w_2\mu))$ into $L(M(s_\alpha w_1\lambda), M(s_\alpha w_2\mu))$. In either case we obtain an embedding of $L(M(w_1\lambda), M(w_2\mu))$ into $L(M((s_{\alpha *} w_1 w_\lambda)w_\lambda\lambda), M(s_\alpha w_2\mu))$, and so by induction an embedding into $L(M(w_3\lambda), M(\mu))$. On the other hand we can take $\alpha \in B_\lambda$ such that $s_\alpha w_1 < w_1$. Then a similar argument gives an embedding of $L(M(s_\alpha w_1\lambda), M((s_{\alpha *} w_2)\mu))$ into $L(M(w_1\lambda), M(w_2\mu))$. By induction this gives an embedding of $L(M(w_\lambda\lambda), M((w_\lambda w_1^{-1}{}_* w_2)\mu))$ into $L(M(w_1\lambda), M(w_2\mu))$ which we saw above further embeds in $L(M(w_3\lambda), M(\mu))$, both maps having been defined by restriction. Now by 3.3, we have $(w_2^{-1} * w_1 w_\lambda)^{-1} = w_\lambda w_1^{-1}{}_* w_2$ and so by 1.15 the combined map is surjective. Consequently the second map must also be surjective, proving the assertion.

3.5: Assume $-\lambda, -\mu \in \mathfrak{h}^*$ dominant with $\lambda - \mu \in P(R)$ and fix $B' \subset B_\lambda$.

COROLLARY: *For each* $w \in W_{B'}$ *and each finite dimensional* $U(\mathfrak{g})$ *module* $E$, *one has*

$$\dim \operatorname{Hom}_{\mathfrak{g}}(Q, M(w\mu)) = \dim \operatorname{Hom}_{\mathfrak{g}}(Q, \delta M(w\mu)),$$

*where* $Q = E \otimes M(w_{B'}\lambda)$.

From $l(ww_\lambda) = l(w_\lambda) - l(w)$ for all $w \in W_\lambda$ and an analogous assertion for $W_{B'}$, we obtain $l(w^{-1}w_{B'}w_\lambda) = l(w^{-1}) + l(w_{B'}w_\lambda)$. Since $w_\lambda^2 = 1$, it follows from the definition of $*$ that $(w^{-1}{}_* w_{B'}w_\lambda)w_\lambda = w^{-1}w_{B'}$, so by 3.4, 3.1 one has the isomorphisms $L(M(w_{B'}\lambda), M(w\mu)) \xrightarrow{\sim} L(M(w^{-1}w_{B'}\lambda), M(\mu)) \xrightarrow{\sim} L(-\mu, -w^{-1}w_{B'}\lambda)$. On the other hand, by 1.9 we have $L(M(w_{B'}\lambda), \delta(M(w\mu))) \cong L(-w\mu, -w_{B'}\lambda)$. Combined with 1.7 and 1.12, these isomorphisms imply the assertion of the corollary.

3.6: Take $\lambda$, $\mu$, $w_1$, $w_2$, $\alpha$ as in 3.2.

LEMMA:

(i) $L(M(w_2\lambda)/M(s_\alpha w_2\lambda), \delta M(w_1\mu)) = 0$.

(ii) $L(L(w_2\lambda), L(w_1\mu)) = 0$.
(iii) *The map of* $L(-w_1\mu, -w_2\lambda)$ *into* $L(-w_1\mu, -s_\alpha w_2\lambda)$ *defined by restriction is injective.*

For (i), observe that $L(w_1\mu)$ is the unique simple submodule of $\delta M(w_1\mu)$, so it suffices to show for any finite dimensional module $E$ that

$$(*) \qquad [E \otimes (M(w_2\lambda)/M(s_\alpha w_2\lambda)) : L(w_1\mu)] = 0.$$

This obtains by an argument parallel to 3.2. Hence (i). Through the embedding $\mathrm{Hom}_{\mathfrak{g}}(E \otimes L(w_2\lambda), L(w_1\mu)) \hookrightarrow \mathrm{Hom}_{\mathfrak{g}}(E \otimes (M(w_2\lambda)/M(s_\alpha w_2\lambda)), L(w_1\mu))$ and $(*)$ we obtain (ii). Recalling 1.9, (i) gives (iii).

*Remark.* When $\alpha \in B$, the result in (iii) is due to Zelobenko (see [8], Lemmes 4, 5).

3.7: We conclude this section with a result of obvious importance which by virtue of ([4], 2.14) is a far reaching generalization of 3.6 (ii). We start with the following

LEMMA: *For all* $\lambda, \mu, \nu \in \mathfrak{h}^*$ *one has*

(i) $L(L(\mu), L(\lambda)) \neq 0 \Leftrightarrow L(L(\lambda), L(\mu)) \neq 0$.
(ii) $L(L(\mu), L(\lambda))\, L(L(\nu), L(\mu)) = 0$ *implies that one of these modules must vanish.*

(i) follows from the isomorphism $\delta(L\mu) \xleftarrow{\sim} L(\mu)$. (ii) follows from the simplicity of $L(\mu)$.

3.8: PROPOSITION: *Let* $\lambda \in \mathfrak{h}^*$ *be dominant and regular. Then for each* $w, y \in W_\lambda$, *one has*

$$L(L(w\lambda), L(y\lambda)) \neq 0 \Leftrightarrow J(w^{-1}\lambda) = J(y^{-1}\lambda).$$

Suppose $L(L(w\lambda), L(y\lambda)) \neq 0$. Then there exists a finite dimensional $U(\mathfrak{g})$ module $E$ such that $\mathrm{Hom}_{\mathfrak{g}}(L(w\lambda), L(y\lambda) \otimes E) \neq 0$ and so $L(w\lambda)$ is a submodule of $L(y\lambda) \otimes E$. It follows that $L(M(\lambda), L(w\lambda))$ is a submodule of $L(M(\lambda), L(y\lambda) \otimes E)$. Hence the right annihilator of $L(M(\lambda), L(w\lambda))$ contains the right annihilator $J$ of $L := L(M(\lambda), L(y\lambda) \otimes E)$. Since $L$ is isomorphic to $L(M(\lambda), L(y\lambda)) \otimes (E^\tau \otimes \mathbb{C})$, it

follows that $J$ coincides with the right annihilator of $L(M(\lambda), L(y\lambda))$. By ([14], 4.7, 4.12) this gives $J(w^{-1}\lambda) \supset J(y^{-1}\lambda)$. By 3.7 (i), interchange of $w, y$ gives the reverse inclusion.

Suppose $J(w^{-1}\lambda) = J(y^{-1}\lambda)$. By ([8], Prop. 8) $U(\mathfrak{g})/J(w^{-1}\lambda)$ has a unique $U$ submodule which is furthermore isomorphic to some $V(-\sigma\lambda, -\lambda)$ with $\sigma$ an involution of $W_\lambda$. By ([14], 4.12) it is clear that $J(\sigma\lambda) = J(w^{-1}\lambda)$. After Vogan ([24], 3.5) there exists a finite dimensional $U(\mathfrak{g})$ module $E$ such that $U(\mathfrak{g})/J(w^{-1}\lambda)$ (and hence $V(-\sigma\lambda, -\lambda)$) is a submodule of $V(-w^{-1}\lambda, -\lambda) \otimes (\mathbb{C} \otimes E)$. From 1.12(i), we have $V(-w^{-1}\lambda, -\lambda) \cong V(-\lambda, -w\lambda)$, and so $V(-\sigma\lambda, -\lambda)$ is a submodule of $V(-w\lambda, -\lambda) \otimes (E \otimes \mathbb{C})$. Then by 1.12 (iii), $L(M(\lambda), L(\sigma\lambda))$ is a submodule of $L(M(\lambda), L(w\lambda)) \otimes (E \otimes \mathbb{C}))$ which is isomorphic to $L(M(\lambda), L(w\lambda) \otimes E^\tau)$. The resulting injection $i: L(M(\lambda), L(\sigma\lambda)) \to L(M(\lambda), L(w\lambda) \otimes E^\tau)$ must come by 1.16(iii) by applying $T$ to an injection $L(\sigma\lambda) \to L(w\lambda) \otimes E^\tau$. Hence $L(L(\sigma\lambda), L(w\lambda)) \neq 0$. Interchanging $w, y$ and using 3.7 gives $L(L(w\lambda), L(y\lambda)) \neq 0$, as required.

## 4. Exactness of the functor $L(M(w_{B'}\lambda), \cdot)$.

In this section we fix $-\lambda \in \mathfrak{h}^*$ dominant and $B' \subset B_\lambda$. Set $\Lambda = \{\mu \in \lambda + P(R) : -\mu \text{ is dominant}\}$.

4.1: Let $\mathcal{O}_\Lambda^{B'}$ denote the subcategory of $\mathcal{O}$ consisting of all those modules (necessarily of finite length) whose simple factors are amongst the $L(w\mu) : \mu \in \Lambda,\ w \in W_{B'}$. By ([6], Thm. 4(iv)) it follows that the $M(w_{B'}\mu) : \mu \in \Lambda$ are projective in $\mathcal{O}_\Lambda^{B'}$. On the other hand $\mathcal{O}_\Lambda^{B'}$ is not closed under tensoring with finite dimensional $U(\mathfrak{g})$ modules. Nevertheless we have the

PROPOSITION: *Suppose $M_1, M_2, M_3 \in 0b\mathcal{O}_\Lambda^{B'}$, with*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

*exact. Then*

$$0 \to L(M(w_{B'}\lambda), M_1) \to L(M(w_{B'}\lambda), M_2) \to L(M(w_{B'}\lambda), M_3) \to 0$$

*is exact.*

This is proved in sections 4.2, 4.3.

4.2: A module $M \in 0b\mathcal{O}$ is said to admit a $p$-filtration if it has a finite filtration with factors isomorphic to Verma modules. For

example, by ([7], 7.6.14) $E \otimes M(\mu)$ ($E$ finite dimensional, $\mu \in \mathfrak{h}^*$) has a $p$-filtration.

LEMMA: *Suppose $Q \in 0b\mathcal{O}$ admits a $p$-filtration. Then for all $\mu \in \mathfrak{h}^*$, $k > 0$, one has*

$$\mathrm{Ext}^k(Q, \delta M(\mu)) = 0.$$

It is enough to prove the assertion for $Q$ a Verma module, say $M(\nu): \nu \in \mathfrak{h}^*$. By 1.14, $\mathrm{Ext}^k(M(\nu), \delta M(\mu)) = (H_k(\mathfrak{n}^-, M(\mu))_{\nu-\rho})^*$, up to isomorphism, so the assertion follows from the fact that $M(\mu)$ is $U(\mathfrak{n}^-)$ free.

4.3: Let $E$ be a finite dimensional $U(\mathfrak{g})$ module and set $Q = E \otimes M(w_{B'}\lambda)$ and fix $\mu \in \Lambda$. We show that $\mathrm{Ext}^1(Q, L(y\mu)) = 0 : y \in W_{B'}$ by induction on $l(y)$. This will establish 4.1. When $l(y) = 0$, that is $y = 1$, we have $L(\mu) \cong M(\mu) \cong \delta M(\mu)$ and so the assertion follows from 4.2. Now fix $w \in W_{B'}$ and suppose the assertion proved for all $y \in W_{B'}$ such that $l(y) < l(w)$. In particular this gives

$$\mathrm{Ext}^1(Q, \overline{M(w\mu)}) = 0. \tag{1}$$

From the exact sequence

$$0 \to L(w\mu) \to \delta M(w\mu) \to \delta\overline{M(w\mu)} \to 0$$

and 4.2 we obtain an exact sequence

$$0 \to \mathrm{Hom}(Q, L(w\mu)) \to \mathrm{Hom}(Q, \delta M(w\mu)) \to \mathrm{Hom}(Q, \delta\overline{M(w\mu)}) \to \tag{2}$$

$$\to \mathrm{Ext}^1(Q, L(w\mu)) \to 0.$$

From the exact sequence

$$0 \to \overline{M(w\mu)} \to M(w\mu) \to L(w\mu) \to 0,$$

and (1) we obtain an exact sequence

$$0 \to \mathrm{Hom}(Q, \overline{M(w\mu)}) \to \mathrm{Hom}(Q, M(w\mu)) \to \mathrm{Hom}(Q, L(w\mu)) \to 0. \tag{3}$$

Combining (2) and (3) gives

$$\begin{aligned}\dim \operatorname{Ext}^1(Q, L(w\mu)) = \{&\dim \operatorname{Hom}(Q, \delta\overline{M(w\mu)}) - \\ &- \dim \operatorname{Hom}(Q, \overline{M(w\mu)})\} \\ &- \{\dim \operatorname{Hom}(Q, \delta M(w\mu)) \\ &- \dim \operatorname{Hom}(Q, M(w\mu))\}.\end{aligned}$$

The first term in curly brackets vanishes by the induction hypothesis and the fact that $\delta\overline{M(w\mu)}$ and $\overline{M(w\mu)}$ have the same composition factors which are amongst the $L(y\mu): y < w$. The second term vanishes by 3.5.

4.4: Let $M$ be a simple $U(\mathfrak{g})$ module. The natural action of $U(\mathfrak{g})$ in $M$ defines an embedding of $U(\mathfrak{g})/\operatorname{Ann}M$ into $\operatorname{Hom}(M, M)$ and in fact the image lies in the $\mathfrak{k}$-finite part $L(M, M)$. Kostant has asked if the image is exactly $L(M, M)$. This is generally false ([5], 6.5; [13], 9.3, 9.4); yet it is quite important to ascertain when it does hold, especially for highest weight modules.

THEOREM: *For each $-\lambda \in \mathfrak{h}^*$ dominant and $B' \subset B$, one has*

$$U(g)/J(w_{B'}\lambda) = L(L(w_{B'}\lambda), L(w_{B'}\lambda)).$$

By 4.1, $L(M(w_{B'}\lambda), L(w_{B'}\lambda))$ is a quotient of $L(M(w_{B'}\lambda), M(w_{B'}\lambda))$ and the latter by ([14], 3.6) identifies with $U(\mathfrak{g})/\operatorname{Ann}M(w_{B'}\lambda)$. Since $L(L(w_{B'}\lambda), L(w_{B'}\lambda))$ is a submodule of $L(M(w_{B'}\lambda), L(w_{B'}\lambda))$, this proves the theorem.

*Remark.* In the special case when $B' \subset B$ the above result is due to Conze-Berline and Duflo ([5], 2.12, 6.3). Their proof does not admit further generalization since it uses induction from the parabolic subalgebra defined by $B'$. When $B' = B_\lambda$ with $\lambda$ regular, the result is noted in ([12], 5.7).

4.5: For $\mu \in \mathfrak{h}^*$, we write $A_\mu := U(\mathfrak{g})/J(\mu)$, $A'_\mu := L(L(\mu), L(\mu))$. The embedding of $A_\mu$ into $A'_\mu$ extends ([13], 4.3) to an embedding of Fract $A_\mu$ into Fract $A'_\mu$. In order to compute the scale factors in the Goldie polynomial defined by the Goldie rank of $A_\mu$ (see [15], 5.12) it is useful to know when Fract $A_\mu$ = Fract $A'_\mu$.

Since $J(\mu)$ is a prime ideal, $A_\mu$ admits a unique simple submodule $V_\mu$ which furthermore ([8], Prop. 4) has annihilator $J_\mu := \check{J}(\mu) \otimes U(g) + U(g) \otimes \check{J}(\mu)$. We let $l_0(A'_\mu)$ denote the number of factors in a $U$ composition series of $A'_\mu$ having annihilator $J_\mu$.

LEMMA: $l_0(A'_\mu) = 1 \Leftrightarrow$ Fract $A_\mu$ = Fract $A'_\mu$.

If $M$ is a finitely generated left $U(\mathfrak{g})$ module, let Dim $M$ denote its Gelfand-Kirillov dimension over $U(\mathfrak{g})$ as defined in ([17], 2.1). Now let $M$ be a simple $U$ subquotient of $A'_\mu$, which by $k$-finiteness is a finitely generated left $U(\mathfrak{g})$ module. By ([17], 1.4, 3.1 and 3.3 Remark) we have $\operatorname{Dim} M = \operatorname{Dim}(U(\mathfrak{g})/\operatorname{Ann}_{U(\mathfrak{g})}M)$. Since $\operatorname{Ann}_{U(\mathfrak{g})}M \supset \check{J}(\mu)$, it follows from the primeness of $\check{J}(\mu)$ that $\operatorname{Ann}_{U(\mathfrak{g})}M = \check{J}(\mu)$ if and only if $\operatorname{Dim} M = \operatorname{Dim} V_\mu$. A similar argument on the right, taking account of ([8], Prop. 4), shows that $\operatorname{Ann} M = J_\mu$ if and only if $\operatorname{Dim} M = \dim V_\mu$. Let $S$ denote the set of regular elements of $A_\mu$. Since $A'_\mu$ is $\mathfrak{k}$-finite and has finite length as a $U$ module, it follows from ([18], 3.7) that $S$ is an Ore subset of the regular elements of $A'_\mu$ and $S^{-1}A'_\mu = \operatorname{Fract} A'_\mu$. Hence it remains to show that $S^{-1}M = 0$ if and only if $\operatorname{Dim} M < \operatorname{Dim} V_\mu = \operatorname{Dim} U(\mathfrak{g})/J(\mu)$. This follows from ([16], 5.1, 5.2(i)).

4.6: Retain the above notation and take $\nu \in \mu + P(R)$ in the upper closure of the $W_\mu$ facette containing $\mu$ (for this see [11], 2.6).

LEMMA: *Set* $H^\nu_\mu = R^\nu_\mu S^\nu_\mu$ (*notation* 1.13). *Then*

(i) $H^\nu_\mu A'_\mu = A'_\mu$.
(ii) $H^\nu_\mu A_\nu = A_\nu$.
(iii) $l_0(A'_\mu) = l_0(A'_\nu)$.
(iv) $H^\nu_\mu V_\mu = V_\nu$.
(v) $\operatorname{Fract} A_\mu = \operatorname{Fract} A'_\mu \Leftrightarrow \operatorname{Fract} A_\nu = \operatorname{Fract} A'_\nu$.

By ([11], 2.10, 2.11) we have under the hypothesis of the lemma the isomorphisms $T^\nu_\mu L(\mu) \cong L(\nu)$ (resp. $T^\nu_\mu M(\mu) \cong M(\nu)$) and so by 1.13 the isomorphisms $H^\nu_\mu A'_\mu = A'_\nu$ (resp. $H^\nu_\mu L(M(\mu), M(\mu)) = L(M(\nu), M(\nu))$). Hence (i). Since $L(M(\mu), M(\mu)) \rightleftarrows U(\mathfrak{g})/\operatorname{Ann} M(\mu)$ by ([13], 6.4) and $A_\mu$ is the image of $U(\mathfrak{g})/\operatorname{Ann} M(\mu)$ in $A'_\mu$, exactness of $H^\nu_\mu$ gives (ii). Now let $K$ be a simple $U$ subquotient of $A'_\mu$. Then by 1.12, $K$ is isomorphic to some $L(M(\lambda_1), L(\lambda_2)) : \lambda_1, \lambda_2 \in \mathfrak{h}^*$ with $\lambda_1$ dominant. Furthermore from the action of the centre of $U$ it easily follows that $\lambda_1, \lambda_2 \in W\mu$. Then from ([11], 2.10, 2.11) and 1.12, 1.13, it follows that either $H^\nu_\mu K = 0$ or is a simple subquotient of $A_\nu$; then, by an argument similar to that given in ([4], 2.11), $H^\nu_\mu K$ has the same Gelfand-Kirillov dimension as $K$. Moreover by a trivial extension of ([4], 2.4), whether or not $H^\nu_\mu K = 0$ depends only on Ann $K$. Hence (iii), (iv). Finally (v) follows from (iii).

4.7: COROLLARY: *Fix* $-\lambda \in \mathfrak{h}^*$ *dominant, regular and take* $B' \subset B_\lambda$. *Then for each* $\alpha \in B'$, *one has*

$$\operatorname{Fract} U(g)/J(w_{B'}s_\alpha\lambda) = \operatorname{Fract} L(L(w_{B'}s_\alpha\lambda), L(w_{B'}s_\alpha\lambda)).$$

With respect to $\lambda$, $\alpha$ define $\nu_\alpha$ as in 2.1. Then apply 4.6(v) to 4.4 with $\mu = w_{B'}s_\alpha\lambda$, $\nu = w_{B'}s_\alpha(\lambda - \nu_\alpha) = w_{B'}(\lambda - \nu_\alpha)$.

4.8: For each $w \in W_\lambda$, set $S(w) = \{\alpha \in R_\lambda^+ : w\alpha \in R_\lambda^-\}$. Define an ordering $\subseteq$ on $W_\lambda$ through $y \subseteq w$ given $S(y^{-1}) \subseteq S(w^{-1})$. One checks that $y \subseteq w$ implies $y \leq w$ and that $y \subseteq w \Leftrightarrow (y_*^{-1}ww_\lambda)w_\lambda = y^{-1}w$. Thus the obvious generalization of 4.1 shows that $L(M(w\lambda), \cdot)$ is exact when restricted to the subcategory of $\mathcal{O}_{\hat{\lambda}}$ of all modules with simple factors $L(y\lambda): y \in W_\lambda$ where $y$ satisfies $y' \leq y \Rightarrow y' \subseteq w$. Since $s_\alpha \leq y$, $\forall \alpha \in$ supp $y$ it follows that supp $y \subset S(w^{-1})$, that is $y \in W_{B'}$ where $B' = B_\lambda \cap S(w^{-1})$. Though this rather weak generalization is probably not the best the corresponding assertion with $\subseteq$ replaced by $\leq$ is false for it implies that Kostant's problem has always a positive answer (which is false by ([5], 6.5)) for simple highest weight modules. This is in spite of the fact that $\mathrm{Ext}^1(M(w\lambda), L(y\lambda)) = 0$ if $w \geq y$.

## 5. Main theorem

5.1: Fix $-\lambda$, $-\mu \in \mathfrak{h}^*$ dominant, $\mu$ regular, with $\lambda - \mu \in P(R)$. Take $B' \subset B_\lambda$. Let $s = l(w_{B'})$, and for each $j \in \{0, 1, 2, \ldots, s\}$ set $D_j = \bigoplus_{w \in W_{B'}^j} L(M(w_{B'}\lambda), M(w\mu))$. Finally put

$$L = L(M(w_{B'}\lambda), L(w_{B'}\mu)).$$

THEOREM: *There is a long exact sequence*

$$0 \to D_0 \to D_1 \to \cdots \to D_s \to L \to 0.$$

Apply 4.1 to 2.6.

5.2: When $\lambda = \mu$ in 5.1, we have that $L = U(\mathfrak{g})/J(w_{B'}\lambda)$ by 4.4. Again by 3.4 one has that

$$D_s = L(M(w_{B'}\lambda), M(w_{B'}\lambda)) = U(\mathfrak{g})/J(\lambda)$$

$$D_{s-1} = \bigoplus_{\alpha \in B'} L(M(w_{B'}\lambda), M(w_{B'}s_\alpha\lambda)) = \bigoplus_{\alpha \in B'} L(M(s_\alpha\lambda), M(\lambda))$$
$$= \bigoplus_{\alpha \in B'} J(s_\alpha\lambda).$$

In view of the definition of the maps in 5.1 this gives the

COROLLARY: *For each $B' \subset B$, one has*

$$\sum_{\alpha \in B'} J(s_\alpha \lambda) = J(w_{B'}\lambda).$$

*Remark.* When $B' \subset B$, this result is due to Duflo ([8], Prop. 12). When $B' = B_\lambda$, it is just ([12], 4.4, 4.5). By ([12], 4.5) it implies that $J(w_{B'}\lambda)/J(\lambda)$ is an idempotent ideal and has exactly card $B'$ distinct maximal submodules.

5.3: Again take $\lambda = \mu$ in 5.1. Then by 3.1, 3.4

$$D_j = \bigoplus_{w \in W_{B'}^j} L(M(w_{B'}\lambda), M(w\lambda)) = \bigoplus_{w \in W_{B'}^j} L(-\lambda, -w^{-1} w_{B'}\lambda).$$

Combined with 1.12 this gives the following multiplicity formula for simple $\mathfrak{k}$ submodules of $U(\mathfrak{g})/J(w_{B'}\lambda)$.

COROLLARY: *Fix $-\lambda \in \mathfrak{h}^*$ dominant and regular. Then for each $\nu \in P(R)$ one has*

$$\dim \operatorname{Hom}_{\mathfrak{k}}(E(\nu), U(\mathfrak{g})/J(w_{B'}\lambda)) = \sum_{w \in W_{B'}} (\det w) \dim E(\nu)_{\lambda - w\lambda}.$$

*Remarks.* When $B' \subset B$, Conze-Berline and Duflo ([5], 2.12, 6.3) gave a formula for the left hand side above. Their formula obtains from 4.4 and Frobenius reciprocity with respect to induction from the parabolic subalgebra defined by $B'$. The equivalence of these two formulae imply a combinatorial statement concerning weight subspaces of finite dimensional $U(\mathfrak{g})$ modules.

## 6. Duality

6.1: Some of our results can be given a dual form with the help of the following. Fix $\lambda, \mu \in \mathfrak{h}^*$ with $\lambda - \mu \in P(R)$. Then (see 6.3) $L(\lambda, \mu) \times L(-\lambda, -\mu)$ admits a bilinear form $\langle , \rangle$ satisfying $\langle (a \otimes b)x, y\rangle = \langle x, (\check{a} \otimes \check{b})y\rangle$, for all $x \in L(\lambda, \mu)$, $y \in L(-\lambda, -\mu)$, $a, b \in U(\mathfrak{g})$. For each $\sigma, \tau \in k^\wedge$, $\langle , \rangle$ restricts to a $\mathfrak{k}$-invariant bilinear form on $L(\lambda, \mu)_\sigma \times L(-\lambda, -\mu)_\tau$ which is non-degenerate if $\tau$ is contragradient to $\sigma$ and zero otherwise.

6.2: To apply 6.1 to the comparison of mappings of principal series modules we start with the following observation. Suppose $\lambda$, $\lambda' \in \mathfrak{h}^*$ are chosen so that we have an embedding of $M(\lambda')$ into $M(\lambda)$. Then there exists $a \in U(\mathfrak{n}^-)_{\lambda'-\lambda}$ such that $ae_\lambda = e_{\lambda'}$. (Furthermore $a$ is unique up to a non-zero scalar which can be fixed canonically as follows. First, under the above hypothesis, $\lambda - \lambda'$ is a non-negative integral linear combination of the $\alpha \in B$ (with say coefficients $k_\alpha$) and second, with respect to the canonical filtration of $U(\mathfrak{n}^-)$, the leading term of $a$ is just

$$\prod_{\alpha \in B} X_{-\alpha}^{k_\alpha}$$

up to a non-zero scalar ([21], Lemma 1). Fix this scalar to be one.)

LEMMA: *There exists an embedding of $M(-\lambda)$ into $M(-\lambda')$ and $\check{a}e_{-\lambda'} = se_{-\lambda}$, with $s = \pm 1$.*

Fix $\alpha \in B$. Then $[X_\alpha, a]e_\lambda = 0$ and so $[X_\alpha, a] \in \operatorname{Ann} e_\lambda = U(\mathfrak{g})\mathfrak{n}^+ + \Sigma_{\beta \in B}\, U(g)(H_\beta - (\lambda - \rho, H_\beta))$. Since $a \in U(\mathfrak{n}^-)_{\lambda'-\lambda}$ and $\alpha$ is simple, we have in fact the more precise result, namely

$$[X_\alpha, a] \in U(\mathfrak{g})_\eta (H_\alpha - (\lambda - \rho, H_\alpha)),$$

where $\eta = \lambda' - \lambda + \alpha$. Hence

$$[X_\alpha, \check{a}] \in (H_\alpha + (\lambda - \rho, H_\alpha))U(\mathfrak{g})_\eta = U(\mathfrak{g})_\eta (H_\alpha + (\lambda + \eta - \rho, H_\alpha)).$$

Yet $-(\lambda + \eta - \rho, H_\alpha) = -(\lambda' + \rho, H_\alpha)$, and so $X_\alpha \check{a}e_{-\lambda'} = [X_\alpha, \check{a}]e_{-\lambda'} = 0$. Since $\alpha$ was arbitrary, it follows that $\check{a}e_{-\lambda'}$ is a highest weight vector (necessarily non-zero) of weight $(\lambda' - \lambda) - (\lambda' + \rho) = -(\lambda + \rho)$ and hence proportional to the canonical generator $e_{-\lambda}$ of $M(-\lambda)$ embedded in $M(-\lambda')$ "canonically" as above. Comparison of leading terms shows that the constant of proportionality is just $(-1)^{\Sigma k_\alpha}$.

*Remark.* Of course the first part also obtains from ([7], 7.6.23). When $B_\lambda \subset B$, the second part can also be derived from ([7], 7.8.8).

6.3: The bilinear form referred to in 6.1 has been defined purely algebraically in ([7], 9.6.9) for the case $\lambda = \mu$. We describe the modifications needed in the general case. In this we denote by $t, u, v$ elements of $U(\mathfrak{k})$, $a, b$ elements of $U(\mathfrak{g})$, $\theta$ an element of $U(\mathfrak{k})^*$, $f$ an element of $L := L(M(\lambda) \otimes M(\mu))^*$.

Define an action of $U(\mathfrak{k})\otimes U(\mathfrak{k})$ on $U(\mathfrak{k})^*$ through $((u\otimes v)\cdot\theta)(t)=\theta(\check{u}tv)$ and set

$$U(\mathfrak{k})_l = U(\mathfrak{k})\otimes\mathbb{C},\quad U(\mathfrak{k})_r=\mathbb{C}\otimes U(\mathfrak{k}).$$

By ([7], 2.7.12) the sum of the simple finite dimensional $U(\mathfrak{k})_l$ submodules of $U(\mathfrak{k})^*$ coincides with the sum of the simple finite dimensional $U(\mathfrak{k})_r$ submodules of $U(\mathfrak{k})^*$, and we denote this subspace by $L(U(\mathfrak{k})^*)$. Let $\epsilon: U(\mathfrak{k})\to\mathbb{C}$ be the augmentation. $\mathbb{C}\epsilon$ occurs as the unique one dimensional subrepresentation of $L(U(\mathfrak{k})^*)$. Let $\varphi_0: L(U(\mathfrak{k})^*)\to\mathbb{C}$ be the linear form on $L(U(\mathfrak{k})^*)$ which takes the value 1 on $\epsilon$ and zero on the $U(\mathfrak{k})\otimes U(\mathfrak{k})$ stable complement of $\mathbb{C}\epsilon$ in $L(U(\mathfrak{k})^*)$.

Now for each $\nu\in\mathfrak{h}^*$, define $T_\nu=\{\theta\in U(\mathfrak{k})^*: \theta(uj(H))=(\nu,H)\theta(u)$, for all $H\in\mathfrak{h}$, $u\in U(\mathfrak{k})\}$ which is a $U(\mathfrak{k})_l$ module. With $\nu=\lambda-\mu$, $f\in L$, we define $\theta_f\in T_\nu$ through $\theta_f(u)=f(u(e_\lambda\otimes e_u))$. Then the map $f\mapsto\theta_f$ is a $U(\mathfrak{k})$ module isomorphism of $L$ onto the $U(\mathfrak{k})_l$ finite part $L(T_\nu)$ of $T_\nu$. (For this see [7], 5.5.8 or [8], Sect. I,2). Now take $\lambda'\in\mathfrak{h}^*$ such that $M(\lambda')\subset M(\lambda)$ and $a\in U(\mathfrak{n}^-)_{\lambda'-\lambda}$ as in 6.2. Then for all $f\in L$, we have $((1\otimes j(a))\cdot\theta_f)(u)=\theta_f(uj(a))=f(uj(a)(e_\lambda\otimes e_\mu))=f(u(ae_\lambda\otimes e_\mu))$, since $a\in U(\mathfrak{n}^-)$ and ${}^tXe_\mu=0$ for all $X\in\mathfrak{n}^-$.

Let $\psi: L\to L':=L(M(\lambda')\otimes M(\mu))$ be defined by restriction. Set $\nu'=\lambda'-\mu$, and define for any $f'\in L'$ the element $\theta_{f'}\in T_{\nu'}$ as above. Then for all $f\in L$, we have $\theta_{\psi(f)}(u)=\psi(f)(u(e_{\lambda'}\otimes e_\mu))=f(u(e_{\lambda'}\otimes e_\mu))$. Since $ae_\lambda=e_{\lambda'}$, this gives

$$(*)\qquad (1\otimes j(a))\cdot\theta_f=\theta_{\psi(f)}.$$

Similarly let $\psi': L(M(-\lambda')\otimes M(-\mu))^*\to L(M(-\lambda)\otimes M(-\mu))^*$ be defined by restriction. Then for each $g'\in L(M(-\lambda')\otimes M(-\mu))^*$, we have $\theta_{g'}\in T_{-\nu'}$, $\theta_{\psi'(g')}\in T_{-\nu}$, and by $(*)$ and 6.2 we get

$$(1\otimes j(\check{a}))\cdot\theta_{g'}=s\theta_{\psi'(g')}.$$

Using ([7], 2.7.7) we have $T_{-\nu}T_\nu\subset T_0$ and so $L(T_{-\nu})L(T_\nu)\subset L(T_0)$. Similarly $L(T_{-\nu'})L(T_{\nu'})\subset L(T_0)$. By ([7], 2.7.7), the invariance of $\varphi_0$ under $U(\mathfrak{k})_r$ gives (noting $j(\check{a})=j(a)^\vee$) that

$$\varphi_0(((1\otimes j(\check{a}))\cdot\theta_{g'})\theta_f)=\varphi_0(\theta_{g'}((1\otimes j(a))\cdot\theta_f)).$$

Just as in ([7], 9.6.9) using ([7], 2.7.15, 9.6.8) and the reductivity of $\mathfrak{k}$,

one checks that the form $\langle g, f\rangle \mapsto \varphi_0(\theta_g\theta_f)$ on $L(\lambda, \mu)\times L(-\lambda, -\mu)$ has the properties claimed in 6.1. Furthermore with respect to the above maps we have the

LEMMA: *The diagram*

$$\begin{array}{ccc} L(\lambda', \mu)\times L(-\lambda, -\mu) & \xrightarrow{1\times\psi} & L(\lambda', \mu)\times L(-\lambda', -\mu) \\ \downarrow s\psi'\times 1 & & \downarrow \\ L(\lambda, \mu)\times L(-\lambda, -\mu) & \longrightarrow & \mathbb{C} \end{array}$$

*commutes. That is* $s\langle\psi'(g'), f\rangle = \langle g', \psi(f)\rangle$.

Indeed

$$s\langle\psi'(g'), f\rangle = \varphi_0(s\theta_{\psi'(g')}\theta_f) =$$

$$\varphi_0(((1\otimes j(\check{a}))\cdot\theta_{g'})\theta_f) = \varphi_0(\theta_{g'}(1\otimes j(a))\cdot\theta_f)) =$$

$$= \varphi_0(\theta_{g'}\theta_{\psi(f)}) = \langle g', \psi(f)\rangle.$$

*Remark.* A similar result holds for the second variable.

6.4: Take $\lambda$, $\mu$, $w_1$, $w_2$, $\alpha$ as in 3.2. Under the hypothesis of 3.2, it follows that $M(-w_2\lambda)$ is a submodule of $M(-s_\alpha w_2\lambda)$. Applying the analogue of 6.3 with respect to second variable to 3.6 we obtain

COROLLARY: *The map* $L(w_1\mu, s_\alpha w_2\lambda)\to L(w_1\mu, w_2\lambda)$ *defined by restriction is surjective.*

*Remark.* When $\alpha\in B$, this was given in ([4], V, 1.11).

6.5: Both 3.6 and 6.4 admit analogous assertions for the first variable. This gives the commutative diagram of restriction maps

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \uparrow & & & & \\ 0 & \to & L(-w_1\mu, -w_2\lambda) & \to & L(-w_1\mu, -s_\alpha w_2\lambda) & & \\ & & \uparrow & & \uparrow & & \\ & & L(-s_\alpha w_1\mu, -w_2\lambda) & \to & L(-s_\alpha w_1\mu, -s_\alpha w_2\lambda) & \longrightarrow & 0 \\ & & & & \uparrow & & \\ & & & & 0 & & \end{array}$$

which implies an isomorphism of $L(-w_1\mu, -w_2\lambda)$ with $L(-s_\alpha w_1\mu, -s_\alpha w_2\lambda)$. The intertwining operators of ([8], Sect. I, 2) also give an isomorphism between those modules.

## *Index of notation*

Symbols frequently used are given below in order of appearance.

1.1 $\mathfrak{g}$, $\mathfrak{h}$, $R$, $R^+$, $B$, $\rho$, $s_\alpha$, $W$, $X_\alpha$, $\mathfrak{n}^+$, $\mathfrak{n}^-$, $\mathfrak{b}$.
1.2 $R_\lambda$, $R_\lambda^+$, $B_\lambda$, $W_{B'}$, $w_{B'}$, $W_\lambda$, $w_\lambda$, $M(\lambda)$, $\overline{M(\lambda)}$, $L(\lambda)$, $J(\lambda)$, $e_\lambda$.
1.3 $\check{u}$, ${}^t u$, $U$, $j$, $\mathfrak{k}$, $\mathfrak{k}^\wedge$.
1.4 $\mathcal{O}$, $Z(\mathfrak{g})$, $\hat{\lambda}$, $\mathcal{O}_{\hat{\lambda}}$, $p_{\hat{\lambda}}$.
1.5 $L(M, N)$, $L(M \otimes N)^*$, $L(\lambda, \mu)$.
1.7 $\mathcal{H}$.
1.8 $\tau$, $M^\tau$, $\delta(M)$.
1.11 $[M]$, $[M(\lambda): L(\mu)]$.
1.12 $P(R)$, $P(R)^+$, $P(R)^{++}$, $E(\nu)$.

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