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Abstract

In the theory of abstract convexity, as introduced in 1951 by F.W. Levi and further developed by D.C. Kay and E.W. Womble in 1971, the relationships between the classical Carathéodory, Helly, and Radon numbers have stimulated much of the work. In this paper the various numbers for the so-called convex sum space, which is – roughly speaking – a convexity structure on the union of sets, are determined, and the sharpness of several relationships between the various numbers is studied.

1. Introduction

The results in this paper are given, in general, in a *convexity space* $(X, \mathfrak{C})$, as introduced by Levi [7], and used by Eckhoff [1], Jamison [5], Kay and Womble [6], Sierksma [9], and others. Here X is a set and $\mathfrak{C}$ is a family of subsets of X, called *convex sets*, satisfying (a) $\emptyset, X \in \mathfrak{C}$; and (b) $\cap F \in \mathfrak{C}$ whenever the family $\mathfrak{F} \subset \mathfrak{C}$. If, moreover, (c) $\cup T \in \mathfrak{C}$ for each chain $\mathfrak{T} \subset \mathfrak{C}$, $(X, \mathfrak{C})$ is called an *aligned space*; see Jamison [5]. We also use $\mathfrak{C}$ to denote the *convex hull operator* on subsets of X; that is, if $S \subset X$, then $\mathfrak{C}(S) = \cap\{A \in \mathfrak{C} \mid S \subset A\}$. Historically, many of the concepts used here were first given for the special case where X is a vector space over a totally ordered field K, for example $\mathbb{R}^d$, and the convex sets in $\mathfrak{C}$ are determined by the order, i.e. $A \in \mathfrak{C}$ provided $\alpha x + (1-\alpha)y \in A$ for each $x, y \in A$ and each $\alpha \in K$ with $0 \leqq \alpha \leqq 1$. When this is the case we will denote the convex hull operator by conv and call (X, conv) an *ordinary convexity space.*

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In 1968 Eckhoff [1] introduced the so-called *convex product space.* The classical numbers of Carathéodory, Helly, and Radon, together with the Exchange number – introduced by Sierksma [8] – for the convex product space are determined and studied extensively in [8] and [9]. For definitions of the various numbers we refer to [10].

A problem related to that of studying convexity on the product of sets is that of defining convexity on the *union* of a collection of sets and of investigating what the Carathéodory, Helly, Radon and Exchange number of such a convexity space are.

It is well-known that the four numbers of the ordinary convexity space $(\mathbb{R}^d, \text{conv})$ are dependent on d. However, in general, there is not such a close connection between the numbers. On the other hand, the close relationship between the various numbers in $(\mathbb{R}^d, \text{conv})$ has stimulated much of the work in the general case of convexity spaces; see e.g. Eckhoff [3], Hammer [4], Kay and Womble [6], and Sierksma [10]. For a survey of relationships we refer to [10]. One of the main problems is to show the *sharpness* of all those relationships. In this paper both the convex product and sum space are used to study the sharpness of some of the well-known relationships.

In section 2 we derive the various numbers for the convex sum space and in section 3 the sharpness of relationships between the numbers if studied.

Let $(X_1, \mathfrak{C}_1)$ and $(X_2, \mathfrak{C}_2)$ be convexity spaces. The *convex sum space* is the pair $(X_1 \cup X_2, \mathfrak{C}_1 + \mathfrak{C}_2)$, with

$$\mathfrak{C}_1 + \mathfrak{C}_2 = \{(A \backslash X_2) \cup (B \backslash X_1) \cup (A \cap B) \mid A \in \mathfrak{C}_1, B \in \mathfrak{C}_2\}.$$

It is clear that $(X_1 \cup X_2, \mathfrak{C}_1 + \mathfrak{C}_2)$ is again a convexity space, and that $(X_1 \cup X_2, \mathfrak{C}_1 + \mathfrak{C}_2)$ is an aligned space provided $(X_1, \mathfrak{C}_1)$ and $(X_2, \mathfrak{C}_2)$ are aligned spaces. The $(\mathfrak{C}_1 + \mathfrak{C}_2)$-hull of any set $S \subset X_1 \cup X_2$ is given by $(\mathfrak{C}_1 + \mathfrak{C}_2)(S) = [\mathfrak{C}_1(S \cap X_1) \backslash X_2] \cup [\mathfrak{C}_2(S \cap X_2) \backslash X_1] \cup [\mathfrak{C}_1(S \cap X_1) \cap \mathfrak{C}_2(S \cap X_2)]$.

The following three cases can be distinguished $(S \subset X_1 \cup X_2)$:

(1) $X_1 \cap X_2 = \emptyset$. Then $\mathfrak{C}_1 + \mathfrak{C}_2 = \{A \cup B \mid A \in \mathfrak{C}_1, B \in \mathfrak{C}_2\}$ and $(\mathfrak{C}_1 + \mathfrak{C}_2)(S) = \mathfrak{C}_1(S \cap X_1) \cup \mathfrak{C}_2(S \cap X_2)$.

(2) $X_1 = X_2$. Then $\mathfrak{C}_1 + \mathfrak{C}_2 = \{A \cap B \mid A \in \mathfrak{C}_1, B \in \mathfrak{C}_2\}$ and $(\mathfrak{C}_1 + \mathfrak{C}2)(S) = \mathfrak{C}_1(S) \cap \mathfrak{C}_2(S)$. Note that in this case $\mathfrak{C}_1 + \mathfrak{C}_2 = \mathfrak{C}_1 \vee \mathfrak{C}_2$, which is the so-called *convex join structure* on $X_1 (= X_2)$, see [9] p. 11.

(3) $X_1 \subset X_2$. Then $\mathfrak{C}_1 + \mathfrak{C}_2 = \{(B \backslash X_1) \cup (A \cap B) \mid A \in \mathfrak{C}_1, B \in \mathfrak{C}_2\}$ and $(\mathfrak{C}_1 + \mathfrak{C}_2)(S) = [\mathfrak{C}_2(S) \backslash X_1] \cup [\mathfrak{C}_1(S \cap X_1) \cap \mathfrak{C}_2(S)]$.

Throughout this paper we shall only deal with convex sums in case the universal sets are *disjoint*, hence case (1) in the above paragraph.[1]

2. Carathéodory, Helly, Radon, and Exchange numbers for convex sum spaces

THEOREM 1: *Let $X_1 \cap X_2 = \emptyset$ and let $(X_i, \mathfrak{C}_i)$ be a convexity space with Carathéodory number c_i, Exchange number e_i, Helly number h_i, and Radon number r_i; $i = 1, 2$. Then the respective numbers c, e, h, and r for the convex sum space $(X_1 \cup X_2, \mathfrak{C}_1 + \mathfrak{C}_2)$* satisfy:

$$c = \max\{c_1, c_2\}$$
$$e = 1 + \max\{c_1, c_2\}$$
$$h = h_1 + h_2$$
$$r = r_1 + r_2 - 1.$$

PROOF: (a) Take any $S \subset X$ and define $S \cap X_1 = S_1$, $S \cap X_2 = S_2$. Then, according to the definition of the Carathéodory number, we find that $(\mathfrak{C}_1 + \mathfrak{C}_2)(S) = \mathfrak{C}_1(S_1) \cup \mathfrak{C}_2(S_2) = [\cup\{\mathfrak{C}_1(U) \mid U \subset S_1, \ |U| \leqq c_1\}] \cup [\cup\{\mathfrak{C}(V) \mid V \subset S_2, |V| \leqq c_2\}] = \cup\{\mathfrak{C}_1(U) \cup \mathfrak{C}_2(V) \mid U \subset S_1, V \subset S_2, |U| \leqq c_1, \ |V| \leqq c_2\} = \cup\{(\mathfrak{C}_1 + \mathfrak{C}_2)(T) \mid T \subset S, \ |T| \leqq \max\{c_1, c_2\}\}$. Hence, $c \leqq \max\{c_1, c_2\}$. On the other hand, it is clear that $c \geqq \max\{c_1, c_2\}$. Therefore, we have in fact that $c = \max\{c_1, c_2\}$.

(b) We first show that $e \leqq 1 + \max\{c_1, c_2\}$. To that end, take any $A \subset X_1 \cup X_2$ with $1 + \max\{c_1, c_2\} \leqq |A| < \infty$ and any $p \in X_1 \cup X_2$. Further take any $x \in (\mathfrak{C}_1 + \mathfrak{C}_2)(A) = \mathfrak{C}_1(X_1 \cap A) \cup \mathfrak{C}_2(X_2 \cap A)$, and assume that $x \in \mathfrak{C}_1(X_1 \cap A)$.

We must show that $x \in (\mathfrak{C}_1 + \mathfrak{C}_2)(p \cup (A \backslash a))$ for some $a \in A$.

If $A \backslash X_1 \neq \emptyset$, take some $a_0 \in A \backslash X_1$, and we have that $x \in \mathfrak{C}_1(X_1 \cap A) = \mathfrak{C}_1(X_1 \cap (A \backslash a_0)) \subset \mathfrak{C}_1(X_1 \cap (p \cup (A \backslash a_0))) \cup \mathfrak{C}_2(X_2 \cap (p \cup (A \backslash a_0))) = (\mathfrak{C}_1 + \mathfrak{C}_2)(p \cup (A \backslash a_0))$.

If $A \backslash X_1 = \emptyset$, we find that $A \subset X_1$. As $|A| \geqq 1 + \max\{c_1, c_2\} \geqq 1 + c_1$, it follows that $\mathfrak{C}_1(A) = \cup\{\mathfrak{C}_1(A \backslash a) \mid a \in A\}$.

So, $(\mathfrak{C}_1 + \mathfrak{C}_2)(A) = \mathfrak{C}_1(A) \cup \mathfrak{C}_2(X_2 \cap A) = [\cup\{\mathfrak{C}_1(A \backslash a) \mid a \in A\}] \cup \mathfrak{C}_2(X_2 \cap A) \subset \cup\{\mathfrak{C}_1(X_1 \cap (p \cup (A \backslash a))) \cup \mathfrak{C}_2(X_2 \cap (p \cup (A \backslash a))) \mid a \in A\} = \cup\{(\mathfrak{C}_1 + \mathfrak{C}_2)(p \cup (A \backslash a)) \mid a \in A\}$. Hence, $1 \leqq e \leqq 1 + \max\{c_1, c_2\}$. We now show that $e \geqq 1 + \max\{c_1, c_2\}$. Let $c_2 \geqq c_1$ and take any $p \in X_1$. As the

[1] In two recent papers by E. Degreef, Free Univ. Brussels, the convex sum space is studied in case the universal sets are *not* disjoint.

Carathéodory number of $(X_2, \mathfrak{C}_2)$ equals c_2, there is a set $A \subset X_2$ with $|A| = c_2$ and such that $\mathfrak{C}_2(A) \not\subset \cup\{\mathfrak{C}_2(A\backslash a) \mid a \in A\}$. Hence, $(\mathfrak{C}_1 + \mathfrak{C}_2)(A) = \mathfrak{C}_2(A) \not\subset \mathfrak{C}_1(p) \cup [\cup\{\mathfrak{C}_2(A\backslash a) \mid a \in A\}] = \cup\{(\mathfrak{C}_1 + \mathfrak{C}_2)(p \cup (A\backslash a)) \mid a \in A\}$, and it follows that $e \geqq 1 + c_2 \geqq 1 + \max\{c_1, c_2\}$. Therefore, we have that $e = 1 + \max\{c_1, c_2\}$.

(c) We first show that $h \leqq h_1 + h_2$. Take any $S \subset X_1 \cup X_2$ with $|S| = h_1 + h_2 + 1$. Obviously, $|X_1 \cap S| \geqq h_1 + 1$ or $|X_2 \cap S| \geqq h_2 + 1$.

Assume that $|X_1 \cap S| \geqq h_1 + 1$. Then it follows that

$$\begin{aligned}\cap\{(\mathfrak{C}_1 + \mathfrak{C}_2)(S\backslash x) \mid x \in S\} &= \cap\{\mathfrak{C}_1(X_1 \cap (S\backslash x)) \cup \mathfrak{C}_2(X_2 \cap (S\backslash x)) \mid x \in S\}\\ &= [\cap\{\mathfrak{C}_1(X_1 \cap (S\backslash x)) \mid x \in S\}] \cup \times\\ &\quad [\cap\{\mathfrak{C}_2(X_2 \cap (S\backslash x)) \mid x \in S\}]\\ &= [\cap\{\mathfrak{C}_1((X_1 \cap S)\backslash x) \mid x \in X_1 \cap S\}] \cup \times\\ &\quad [\cap\{\mathfrak{C}_2((X_2 \cap S)\backslash x) \mid x \in X_2 \cap S\}] \neq \emptyset.\end{aligned}$$

Hence, $h \leqq h_1 + h_2$.

We now show that $h \geqq h_1 + h_2$. Take some $S \subset X_1 \cup X_2$ with $|S \cap X_1| = h_1$ and $|S \cap X_2| = h_2$, and such that $\cap\{\mathfrak{C}_i((S \cap X_i)\backslash x) \mid x \in S \cap X_i\} = \emptyset$ for each $i \in \{1, 2\}$. Then it follows that

$$\begin{aligned}\cap\{(\mathfrak{C}_1 + \mathfrak{C}_2)(S\backslash x) \mid x \in S\} &= [\cap\{\mathfrak{C}_1((S \cap X_1)\backslash x) \mid x \in S \cap X_1\}] \cup \times\\ &\quad [\cap\{\mathfrak{C}_2((S \cap X_2)\backslash x) \mid x \in S \cap X_2\}] = \emptyset.\end{aligned}$$

Hence, $h \geqq h_1 + h_2$, so that in fact $h = h_1 + h_2$.

Note that if $|X_1\backslash X_2| = h_1$ and $|X_2\backslash X_1| = h_2$ in the above theorem, then we also have that $h = h_1 + h_2$.

(d) We first show that $r \leqq r_1 + r_2 - 1$. Take any $S \subset X_1 \cup X_2$ with $|S| \geqq r_1 + r_2 - 1$. Obviously, $|X_1 \cap S| \geqq r_1$ or $|X_2 \cap S| \geqq r_2$. We may assume that $|X_1 \cap S| \geqq r_1$. This means that $X_1 \cap S$ has a $\mathfrak{C}_1$-Radon partition, say $\{S_1, S_2\}$. Hence, $S_1 \cup S_2 = X_1 \cup S$, $S_1 \cap S_2 = \emptyset$, and $\mathfrak{C}_1(S_1) \cap \mathfrak{C}_2(S_2) \neq \emptyset$. As

$$\begin{aligned}(\mathfrak{C}_1 + \mathfrak{C}_2)(S_1 \cup (X_2 \cap S)) \cap (\mathfrak{C}_1 + \mathfrak{C}_2)(S_2) &=\\ [\mathfrak{C}_1(S_1) \cup \mathfrak{C}_2(X_2 \cap S)] \cap \mathfrak{C}_1(S_2) &=\\ [\mathfrak{C}_1(S_1) \cap \mathfrak{C}_1(S_2)] \cup \mathfrak{C}_2(X_2 \cap S) &\neq \emptyset,\end{aligned}$$

it follows that $\{S_1 \cup (X_2 \cap S), S_2\}$ is a $(\mathfrak{C}_1 + \mathfrak{C}_2)$-Radon partition of S. Hence, $r \leqq r_1 + r_2 - 1$.

We now show that $r \geqq r_1 + r_2 - 1$. Take some $S_i \subset X_i$ with $|S_i| = r_i - 1$ but without $\mathfrak{C}_i$-Radon partition; $i = 1, 2$. Then $|S_1 \cup S_2| = r_1 + r_2 - 2$, and clearly, $S_1 \cup S_2$ has no $(\mathfrak{C}_1 + \mathfrak{C}_2)$-Radon partition. Therefore we find that $r > r_1 + r_2 - 2$. Hence, $r = r_1 + r_2 - 1$.

EXAMPLE 1: Take $X_1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 < 0\}$ and $X_2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geqq 0\}$ and $\mathfrak{C}_1 = \mathfrak{C}_2 = \text{conv}$. Then $c_1 = c_2 = 3$ and, hence, $c = 3$. Note that conv + conv consists of all convex sets in the ordinary sense, together with all unions of two convex sets.

EXAMPLE 2: Take $X_1 = \{(x_1, x_2) \mid x_1 < 0\} \subset \mathbb{R}^2$ and $X_2 = \{(x_1, x_2) \mid x_1 \geqq 0\} \subset \mathbb{R}^2$. Further, let $k_1, k_2 \in \mathbb{N}$ and define $\mathfrak{C}_1 = \{X_1\} \cup \{A \mid A \subset X_1, |A| \leqq k_1\}$, $\mathfrak{C}_2 = \{X_2\} \cup \{A \mid A \subset X_2, |A| \leqq k_2\}$. Then it follows that $e_1 = e_2 = 2$, $c_1 = k_1 + 1$, and $c_2 = k_2 + 1$. Note that $X_1 \cap X_2 = \emptyset$. Let $k_2 \geqq k_1$. We shall show that $e = k_2 + 2$. Take any set $A \subset X_2$ with $|A| = k_2 + 1$, and take any $p \in X_1$. Then $(\mathfrak{C}_1 + \mathfrak{C}_2)(A) = \mathfrak{C}_2(A) = X_2$, and $(\mathfrak{C}_1 + \mathfrak{C}_2)(p \cup (A \backslash a)) = \mathfrak{C}_1(p) \cup \mathfrak{C}_2(A \backslash a) = \{p\} \cup A \backslash \{a\}$ for each $a \in A$. So, $(\mathfrak{C}_1 + \mathfrak{C}_2)(A) \not\subset \cup\{(\mathfrak{C}_1 + \mathfrak{C}_2)(p \cup (A \backslash a)) \mid a \in A\}$, and it follows that $e \geqq k_2 + 2$. Clearly, $(\mathfrak{C}_1 + \mathfrak{C}_2)(A) \subset \cup\{(\mathfrak{C}_1 + \mathfrak{C}_2)(p \cup (A \backslash a)) \mid a \in A\}$ for each $A \subset X_1 \cup X_2$ with $k_2 + 2 \leqq |A| < \infty$, so that $e \leqq k_2 + 2$. Hence, $e = k_2 + 2 = 1 + \max\{c_1, c_2\}$.

The following example shows that $e_1 < \infty$ and $e_2 < \infty$ does not imply that $e < \infty$.

EXAMPLE 3: Consider the convexity spaces $(X_1, \mathfrak{C}_1)$ and $(X_2, \mathfrak{C}_2)$ with $X_1 = \{(x_1, x_2) \mid x_1 < 0\} \subset \mathbb{R}^2$, $\mathfrak{C}_1 = \text{conv}$, $X_2 = \{(x_1, x_2) \mid x_1 \geqq 0\} \subset \mathbb{R}^2$, and $\mathfrak{C}_2 = \{\emptyset, X_2\} \cup \{C \mid (\exists n)[n \in \mathbb{N},\ C \subsetneqq C_n]\}$ with $C_1 = \{(1, 0)\}$, $C_2 = \{(2, 0), (3, 0)\}$, $C_3 = \{(4, 0), (5, 0), (6, 0)\}$, $C_4 = \{(7, 0), (8, 0), (9, 0), (10, 0)\}$, etc. (see [9] 5.5 Ex. 8). Clearly, $e_1 = 3$. In [9] Ch. 6.3 it is shown that $e_2 = 2$. We now show that the Exchange number e of $(\mathbb{R}^2, \mathfrak{C}_1 + \mathfrak{C}_2)$ is infinite. Take any $n \in \mathbb{N}$, and let $p = (-1, 0)$. Then $(\mathfrak{C}_1 + \mathfrak{C}_2)(C_n) = \mathfrak{C}_2(C_n) = X_2$. On the other hand,

$$\begin{aligned} \cup\{(\mathfrak{C}_1 + \mathfrak{C}_2)(p \cup (C_n \backslash a)) \mid a \in C_n\} &= \cup\{\mathfrak{C}_1(p) \cup \mathfrak{C}_2(C_n \backslash a) \mid a \in C_n\} \\ &= \{p\} \cup \{[\cup \mathfrak{C}_2(C_n \backslash a) \mid a \in C_n\}] \\ &= \{p\} \cup C_n. \end{aligned}$$

Hence, $(\mathfrak{C}_1 + \mathfrak{C}_2)(C_n) = X_2 \not\subset \{p\} \cup C_n = \cup\{(\mathfrak{C}_1 + \mathfrak{C}_2)(p \cup (C_n \backslash a)) \mid a \in C_n\}$. Therefore we find that $e = \infty$.

EXAMPLE 4: Take $X_1 = \{(x_1, \ldots, x_n) \mid x_1 < 0\} \subset \mathbb{R}^n$, $X_2 = \{(x_1, \ldots, x_n) \mid x_1 \geqq 0\} \subset \mathbb{R}^n$ with $n \in \mathbb{N}$, and let $\mathfrak{C}_1 = \text{conv}$ and $\mathfrak{C}_2 = \text{conv}$, the ordinary convexity structures on X_1 and X_2, respectively. Then it follows that $h_1 = h_2 = n+1$, and that $h = h_1 + h_2 = 2n+2$. Also note that $r = r_1 + r_2 - 1 = 2n+3$.

The concept of convex sum space can be generalized to sums of *finitely* many convexity spaces. For instance the convex sum space with basic spaces $(X_1, \mathfrak{C}_1)$, $(X_2, \mathfrak{C}_2)$, and $(X_3, \mathfrak{C}_3)$, denoted by $(X_1 \cup X_2 \cup X_3, \mathfrak{C}_1 + \mathfrak{C}_2 + \mathfrak{C}_3)$ is defined by

$$\begin{aligned}\mathfrak{C}_1 + \mathfrak{C}_2 + \mathfrak{C}_3 = \{&[A \backslash (X_2 \cup X_3)] \cup [B \backslash (X_1 \cup X_3)] \cup \times \\ &[C \backslash (X_1 \cup X_2)] \cup [(A \cap B) \backslash X_3] \cup [(A \cap C) \backslash X_2] \cup \times \\ &[(B \cap C) \backslash X_1] \cup [A \cap B \cap C] \mid A \in \mathfrak{C}_1, B \in \mathfrak{C}_2, C \in \mathfrak{C}_3\}.\end{aligned}$$

Note that $\mathfrak{C}_1 + \mathfrak{C}_2 + \mathfrak{C}_3 = (\mathfrak{C}_1 + \mathfrak{C}_2) + \mathfrak{C}_3 = \mathfrak{C}_1 + (\mathfrak{C}_2 + \mathfrak{C}_3)$. In case $X_i \cap X_j = \emptyset$ for each $i, j = 1, \ldots, n$ with $i \neq j$ we have:

$$\mathop{+}_{i=1}^{n} \mathfrak{C}_i = \mathfrak{C}_1 + \cdots + \mathfrak{C}_n = \left\{ \bigcup_{i=1}^{n} A_i \mid A_i \in \mathfrak{C}_i \text{ for each } i = 1, \ldots, n \right\}.$$

Note that $\mathfrak{C}_1, \ldots, \mathfrak{C}_n \subset +_{i=1}^{n} \mathfrak{C}_i$ in case the universal sets are disjoint.

Let $X_i \cap X_j = \emptyset$ for each $i, j = 1, \ldots, n$ with $i \neq j$, and let $(X_i, \mathfrak{C}_i)$ be a convexity space with Carathéodory number c_i, Exchange number e_i, Helly number h_i, and Radon number r_i; $i = 1, \ldots, n$. Then it can be shown by induction on n that the respective numbers $c^{(n)}$, $e^{(n)}$, $h^{(n)}$, and $r^{(n)}$ of the convex sum space $(\bigcup_{i=1}^{n} X_i, +_{i=1}^{n} \mathfrak{C}_i)$ satisfy:

$$c^{(n)} = \max_{1 \leqq i \leqq n} c_i$$

$$e^{(n)} = 1 + \max_{1 \leqq i \leqq n} c_i \quad (n \geqq 2)$$

$$h^{(n)} = \sum_{i=1}^{n} h_i$$

$$r^{(n)} = \sum_{i=1}^{n} r_i - n + 1.$$

By a *copy* of the convexity space $(X, \mathfrak{C})$ we mean a convexity space $(X \times \{i\}, \mathfrak{C} \times \{i\})$ for some $i \in \{1, \ldots, n\}$, with $\mathfrak{C} \times \{i\} = \{A \times \{i\} \mid A \in \mathfrak{C}\}$. The $(\mathfrak{C} \times \{i\})$-convex hull of any set $S \subset X \times \{i\}$ is given by $(\mathfrak{C} \times \{i\})(S) = \mathfrak{C}(\pi_i A) \times \{i\}$, where π_i is the projection of

$X \times \{i\}$ onto X. Furthermore, we define

$$\underset{n}{+}\,\mathfrak{C} = \overset{n}{\underset{i=1}{+}} (\mathfrak{C} \times \{i\})$$

and

$$\bigcup_n X = \bigcup_{i=1}^{n} (X \times \{i\}) = X \times \{1, \ldots, n\}.$$

THEOREM 2: *Let c, e, h, and r be the Carathéodory, Exchange, Helly, and Radon numbers, respectively, of the convexity space $(X, \mathfrak{C})$. Let n be an integer $\geqq 2$. Then, $(\bigcup_n X, +_n \mathfrak{C})$ is a convexity space and the $(+_n \mathfrak{C})$-hull of any set $S \subset \bigcup_n X$ satisfies*

$$(\underset{n}{+}\,\mathfrak{C})(A) = \bigcup_{i=1}^{n} [\mathfrak{C}_i(\pi_i A) \times \{i\}].$$

Moreover, the respective numbers $c^{(n)}$, $e^{(n)}$, $h^{(n)}$, and $r^{(n)}$ of $(\bigcup_n X, +_n \mathfrak{C})$ satisfy:

$$\begin{aligned} c^{(n)} &= c \\ e^{(n)} &= 1 + c \\ h^{(n)} &= nh \\ r^{(n)} &= n(r-1) + 1. \end{aligned}$$

PROOF: As $(X \times \{i\}) \cap (X \times \{j\}) = \emptyset$ for each $i, j = 1, \ldots, n$ with $i \neq j$, and $(\bigcup_n X, +_n \mathfrak{C})$ is an n-convex sum space, the theorem follows almost directly from the above remarks.

3. Sharpness of relationships between the various numbers

For relationships between the Carathéodory, Exchange, Helly, and Radon numbers c, e, h, and r, respectively, we refer to Sierksma [10]. One of the interesting problems is the sharpness of those relationships. It is well-known that the inequality of Levi, namely $h \leqq r-1$, is sharp. By being *sharp* we mean that for each two integers h and r with $h = r-1$ there exists a convexity space such that h is the Helly number and r the Radon number. Sharpness of other inequalities can be defined in the same way. So it follows from Theorem 2 that the relation $e \leqq 1 + c$ is also sharp. In the remaining part of this paper we study the sharpness of the 'special' Eckhoff and Jamison inequality, namely

$$r \leqq (c-1)(h-1)+3$$

provided $e \leqq c$, and the inequality

$$c \leqq \max\{h, e-1\}$$

which holds in case $(X, \mathfrak{C})$ is an aligned space. To that end we note that for $(\mathbb{R}^d, \bigoplus_d \text{conv})$, where $\bigoplus_d \text{conv} = \text{conv} \oplus \cdots \oplus \text{conv}$ (d times the convex product of conv), the following holds: $c = d$, $e = d+1$, $h = 2$, and $r = \min\{k \in \mathbb{N} \mid \binom{k}{[k/2]} > 2d\}$. This follows directly from Sierksma [9, Theorem 5.5, 6.10, 7.6II, and 8.3] and Eckhoff [2].

THEOREM 3: *Let m and n be integers* $\geqq 3$. *Then the Carathéodory, Helly, and Radon numbers c, h, and r, respectively, of the m-convex sum space* $(\bigcup_m \mathbb{R}^n, +_m \bigoplus_n \text{conv})$ *satisfy*:

$$r < (c-1)(h-1)+3 \text{ if } n > 3$$
$$r = (c-1)(h-1)+3 \text{ if } n = 3.$$

PROOF: Theorem 2 implies that $c = n$, $h = 2m$, and $r = m(s-1)+1$ with $s = \min\{k \in \mathbb{N} \mid \binom{k}{[k/2]} > 2n\}$. The proof now follows directly from the above remark for the convex product space $(\mathbb{R}^n, \bigoplus_n \text{conv})$. Note that if $n = 3$, then $c = 3$, $e = 4$, $h = 2m$, and $r = 4m+1$, hence that $(c-1)(h-1)+3 = 4m+1 = r$.

Note that it follows from the above theorems that $e \leqq c$ is not necessary condition for $r \leqq (c-1)(h-1)+3$ to hold. Also note that the convexity space $(\bigcup_m \mathbb{R}^3, +_m \bigoplus_3 \text{conv})$ is a non-trivial example such that equality holds in the 'special' Eckhoff and Jamison inequality.

THEOREM 4: *The inequality* $c \leqq \max\{h, e-1\}$ *is sharp in case* $h \leqq e-1$.

PROOF: First note that $(\bigcup_m \mathbb{R}^n, +_m \bigoplus_n \text{conv})$ is an aligned space $(m, n \in \mathbb{N})$. Choosing m and n such that $2m \leqq n$, it follows that

$h \leqq e - 1$ and that $c = \max\{h, e-1\}$ for *even* Helly number h. Now consider the 'sum' of $(\mathbb{R}^n, \mathfrak{C})$ with $\mathfrak{C} = \{\emptyset, \mathbb{R}^n\}$ and $(\bigcup_m \mathbb{R}^n, +_m \bigoplus_n \text{conv})$, i.e. $(\bigcup_{m+1} \mathbb{R}^n, +_m \bigoplus_n \text{conv} + \mathfrak{C})$. Clearly, this is again an aligned space. As the Carathéodory, Exchange, and Helly numbers of $(\mathbb{R}^n, \mathfrak{C})$ are equal to 1, it follows for the respective numbers of $(\bigcup_{m+1} \mathbb{R}^n, +_m \bigoplus_n \text{conv} + \mathfrak{C})$ that $c = n$, $e = n + 1$, and $h = 2m + 1$. Choosing $2m + 1 \leqq n$, it follows that $h \leqq e - 1$, and that $c = \max\{h, e-1\}$ for *odd* Helly number h. Therefore, we have in fact that the inequality $c \leqq \max\{h, e-1\}$ is sharp for $h \leqq e - 1$. Sharpness of the inequality in case $h \geqq e - 1$ is still an open problem.

The next theorem enables us to construct convexity spaces with Carathéodory, Helly, and Radon numbers with no 'close' connection.

THEOREM 5: *Let k, m, n be integers $\geqq 1$. Then for the convexity space $(\bigcup_{m+k} \mathbb{R}^n, (+_m \bigoplus_n \text{conv}) + (+_k \mathfrak{C})$ with $\mathfrak{C} = \{\emptyset, \mathbb{R}^n\}$ the Carathéodory, Exchange, Helly, and Radon numbers, c, e, h, and r, respectively, the following holds*:

$$c = n$$
$$e = n + 1$$
$$h = 2m + k$$
$$r = m(s-1) + k + 1 \text{ with } s = \min\{a \in \mathbb{N} \mid \tbinom{a}{[a/2]} > 2n\}.$$

PROOF: The proof can be given easily by induction on k.

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