PIERRETTE CASSOU-NOGUÈS

*p-adic* *L*-functions for elliptic curves with complex multiplication I

*Compositio Mathematica*, tome 42, n° 1 (1980), p. 31-56

<http://www.numdam.org/item?id=CM_1980__42_1_31_0>
1. Introduction

Let $K$ be an imaginary quadratic field with class number one, lying inside the complex field $C$, and $\mathcal{O}$ the ring of integers of $K$. Let $E$ be an elliptic curve defined over $K$, whose ring of endomorphisms is isomorphic to $\mathcal{O}$. Since $K$ has class number one, we can choose a Weierstraß model for $E$

$$y^2 = 4x^3 - g_2x - g_3$$

where $g_2$ and $g_3$ belong to $\mathcal{O}$, and where the discriminant of (1) is divisible only by the primes of $K$ where $E$ has a bad reduction, and possibly by the primes of $K$ above 2 and 3. Let $p(z)$ be the associated Weierstraß function and $L$ its period lattice. As $K$ has class number one, we can choose $\Omega \subseteq L$ such that $L = \Omega \mathcal{O}$. We fix, once and for all, an algebraic closure $\bar{K}$ of $K$, which we suppose lies inside the complex field $C$.

Let $S$ be the set of rational primes consisting of 2, 3, and all $q$ such that $E$ has a bad reduction at at least one prime of $K$ above $q$. For the rest of the paper, we shall assume that $p$ is a rational prime, not in $S$, which splits in $K$, say $(p) = p\mathfrak{p}$. We write $K_p$ for the completion of $K$ at $p$, $\mathcal{O}_p$ the ring of integers of $K_p$, and $C_p$ for the completion of an algebraic closure

* This paper was begun while the author was at the Department of Pure Mathematics and Mathematical Statistics, University of Cambridge and the author would like to thank the Department for its hospitality. During the preparation of this paper the author was partially supported by a grant from the British Council.
of $K_p$. We assume that we are given a fixed prime $\mathfrak{p}$ of $\bar{K}$ lying above $p$, or, what amounts to the same thing, an embedding $\tau$ of $\bar{K}$ into $\mathbb{C}_p$.

The aim of the present paper is to prove the existence of $\mathfrak{p}$-adic $L$-functions attached to $E$ and certain abelian extensions of $K$, and to give several arithmetic applications of these. Functions of this type have already been constructed by Katz [9], [10], Manin-Vishik [15], and Lichtenbaum [12]. In fact, much of our construction has been based on an earlier version of Lichtenbaum's paper [12], and we wish to make quite clear our indebtedness to his work. We do, however, go further than [12] both in defining $\mathfrak{p}$-adic $L$-functions for a wider class of abelian extensions of $K$, and in the arithmetic applications we give. Also, we shall treat the case in which the class number of $K$ is greater than 1 by similar methods in a later paper. The present paper should be viewed as an introduction to our later work.

Finally, I wish to thank J. Coates for helpful suggestions on this work.

1. Results used from elsewhere

In this section we summarize, without proofs, a number of results from related papers, which will be used in our construction of the $\mathfrak{p}$-adic $L$-functions. We use the notation in the introduction.

Let $\mathcal{E}$ be the formal group giving the kernel of reduction modulo $\mathfrak{p}$ on the curve $E$; for a detailed discussion of this, see [19], p. 42. A local parameter for $\mathcal{E}$ is given by $t = -2x/y$, where $x$ and $y$ are the coordinates of the model (1) of $E$. Since $p$ splits in $K$, it is easy to see that $\mathcal{E}$ has height one. Let $T$ be the completion of the maximal unramified extension of $K_p$, and $\mathcal{O}_T$ the ring of integers of $T$. It is shown in [13] that every formal group of height 1 defined over $\mathcal{O}_T$ is isomorphic over $\mathcal{O}_T$ to the formal multiplicative group $G_m$. From this fact, it is easy to deduce the following lemma. Let $z$ be given by $t = -2p(z)/p'(z)$. Thus we can view $z$ as the parameter of the formal additive group $G_a$.

**Lemma 1:** There exists $g(X) \in \mathcal{O}_T[[X]]$, and $\gamma \in \mathcal{O}_T^\times$, such that $t = g(e^{\gamma z} - 1)$.

Here $\mathcal{O}_T[[X]]$ denotes the ring of formal power series in $X$ with coefficients in $\mathcal{O}_T$.

If $\mathcal{L}$ is any lattice in the complex plane, we define, as usual
\[ \sigma(z, \mathcal{L}) = z \prod_{\omega \in \mathcal{L}} \left( 1 - \frac{z}{\omega} \right) e^{(z/\omega + (1/2)(z/\omega)^2} \]

and put

\[ \theta(z, \mathcal{L}) = \Delta(\mathcal{L}) e^{-6s_2(\mathcal{L})z^2} \sigma(z, \mathcal{L})^{12} \]

where \( \Delta(\mathcal{L}) \) is the discriminant function of \( \mathcal{L} \), and

\[ s_2(\mathcal{L}) = \lim_{z \to 0} \sum_{\omega \in \mathcal{L}} \omega^{-2} |\omega|^{-2z}. \]

If \( a \) is any integral ideal of \( K \), we define

\[ (2) \quad \Theta(z, a) = \theta(z, L)^{Na/g} \theta(z, a^{-1}L) \]

where \( Na \) is the absolute norm of \( a \). In fact, as is shown in Robert [16], \( \Theta(z, a) \) is an elliptic function for the lattice \( L \).

Assume now that \( H \) is an arbitrary finite abelian extension of \( K \). Let \( \psi \) be the Grössencharacter of \( E \) over \( K \). We define \( \mathfrak{h} \) to be the least common multiple of the conductor of \( \psi \) and the conductor of \( H/K \). Let \( h \) be a generator of the ideal \( \mathfrak{h} \) and define \( \rho = \Omega/h \). Let \( E_{\mathfrak{h}} \) be the group of \( \mathfrak{h} \)-division-points on \( E \). By Lemma 2 of [1], \( K(E_{\mathfrak{h}}) \) is the ray class field of \( K \) modulo \( \mathfrak{h} \). We now choose and fix a set \( B \) of integral ideals of \( K \), which are prime to \( \mathfrak{h} \), and which are such that \( \{(b, K(E_{\mathfrak{h}})/K); b \in B\} \) is precisely the Galois group of \( K(E_{\mathfrak{h}})/H \); here \( (b, K(E_{\mathfrak{h}})/K) \) denotes the Artin symbol of \( b \) for \( K(E_{\mathfrak{h}})/K \). If \( a \) is an integral ideal of \( K \), we define

\[ \Lambda(z, a) = \prod_{b \in B} \Theta(z + \psi(b)\rho, a) \]

It is shown in [1] (cf. Lemma 7) that \( \Lambda(z, a) \) is a rational function of \( \psi(z) \) and \( \psi'(z) \) with coefficients in \( H \). If \( \sigma \) is an element of the Galois group of \( H \) over \( K \), we write \( \Lambda_\sigma(z, a) \) for the rational function of \( \psi(z) \) and \( \psi'(z) \), which is obtained by letting \( \sigma \) act on the coefficients of \( \Lambda(z, a) \).

If \( c \) is an integral ideal of \( K \), prime to the conductor of \( H/K \), we write \( \sigma_c \) for the Artin symbol \( (c, H/K) \). Let \( k \) be an integer \( \geq 1 \). We introduce the partial Hecke \( L \)-function, for each \( \sigma \) in the Galois group of \( H \) over \( K \),
where the summation is over all integral ideals \(a\) of \(K\), prime to \(b\), such that \(\sigma_a = \sigma\). It can be shown that \(\zeta_H(\sigma, k ; s)\) can be analytically continued over the whole complex plane, and we write \(\zeta_H(\sigma, k)\) for its value at \(s = k\). The following lemma is proven in [1]:

**Lemma 2:** For each \(\sigma \in G(H|K)\), we have

\[
\zeta_H(\sigma, k ; s) = \sum_{\substack{(a, b) = 1 \\ \sigma_a = \sigma}} \frac{\tilde{\psi}(a)}{(Na)^s},
\]

where, for \(k \geq 1\)

\[
c_k(a, \sigma) = 12(-1)^{k-1} \rho^{-k}(Na \zeta_H(\sigma, k) - \psi^k(a)\zeta_H(\sigma \sigma, k)).
\]

Here \(a\) is any integral ideal of \(K\), prime to \(b\).

Finally we recall some basic facts about Leopoldt's \(\Gamma\)-transform (see [12]). Let \(M\) be any complete subfield of \(\mathbb{C}_p\). Let \(Q_M\) be the set of power series \(\sum_{n=0}^\infty a_n x^n\) in \(M[[x]]\) such that \(\lim_{n \to \infty} |a_n| = 0\), where \(|\cdot|\) denotes the valuation of \(\mathbb{C}_p\). Let \(C_M\) be the set of continuous functions from \(\mathbb{Z}_p\) to \(M\). Then both \(Q_M\) and \(C_M\) are Banach algebras with the norms \(\sup_n |n! a_n|\) and \(\max_{z \in \mathbb{Z}_p} |f(z)|\), respectively. Let \(\alpha\) be a residue class \(n \equiv z \mod (p-1)\). Following Leopoldt [11], Lichtenbaum has shown in [12] that one can define the \(\Gamma^\alpha\)-transform. For the precise definition, see [12]. We simply note that \(\Gamma^\alpha\) is a bounded linear map from \(Q_M\) to \(C_M\). The following is a key lemma about \(\Gamma^\alpha\).

**Lemma 3:** Given \(A(X) \in Q_M\), define

\[
\tilde{A}(X) = A(X) - \frac{1}{p} \sum_\zeta A(\zeta(X + 1) - 1)
\]

where \(\zeta\) ranges over all \(p\)-th roots of unity. If \(k\) is an integer \(\geq 0\) with \(k = \alpha \mod p - 1\), then

\[
\Gamma^\alpha(A)(k) = \frac{d^k}{dz^k} \tilde{A}(e^z - 1) \big|_{z = 0}.
\]
Let $\mathcal{O}_M$ be the ring of integers of $M$. Given a power series $f(X) \in \mathcal{O}_M[[X]]$, we can obtain a function $f^* \in C_M$ by $f^*(s) = f((1 + p)^s - 1)$. We call $f^*$ an Iwasawa function in $C_M$. Another basic result about $\Gamma^\alpha$ is the following (see [12]). If $A(X) \in \mathcal{O}_M[[X]]$, then $\Gamma^\alpha(A)(s)$ is an Iwasawa function.

II. $\beta$-adic $L$-functions

As before, let $M$ denote a complete subfield of $\mathbb{C}_p$, and $\mathcal{O}_M$ the ring of integers of $M$. We suppose, for simplicity, that $M$ contains the field $T$, which is the completion of the maximal unramified extension of $K_p$. By Lemma 1, there exists a power series $g(X) \in \mathcal{O}_T[[X]]$, and $\gamma \in \mathcal{O}_T^\times$, such that $t = g(e^{\gamma X} - 1)$. In fact, $g(X)$ defines an isomorphism from $G_m$ to $\hat{E}$. Let $\hat{E}_\pi$, where $\pi = \psi(p)$ be the kernel of the endomorphism $[\pi]$ of $\hat{E}$. Given $A(t) \in \mathcal{O}_M[[t]]$, we define, as before,

$$\hat{A}(t) = A(t) - \frac{1}{p} \sum \zeta A(\zeta(t + 1) - 1),$$

where $\zeta$ runs over all $p$-th roots of unity in $\mathbb{C}_p$.

**Lemma 4:** Let $B(t) \in \mathcal{O}_M[[t]]$, and define $A(X) = B(g(X))$. Then, for each integer $k \geq 0$, we have

$$\left(\frac{d}{dz}\right)^k \hat{A}(e^z - 1) \bigg|_{z=0} = \gamma^{-k} \left(\frac{d}{dz}\right)^k \left\{ \left( B(t) - \frac{1}{p} \sum_{\eta \in \hat{E}_\pi} B(t * \eta) \right) \right\}_{t=0},$$

here $t * \eta$ denotes the sum of $t$ and $\eta$ on $\hat{E}$.

**Proof:** Since $t = g(e^{\gamma z} - 1)$ and $\eta = g(\zeta - 1)$, it follows from the fact that $g$ is an isomorphism from $G_m$ to $\hat{E}$ that $t * \eta = g(\zeta e^{\gamma z} - 1)$ (note that $\zeta e^{\gamma z} - 1$ is the product of $\zeta - 1$ and $e^{\gamma z} - 1$ on $G_m$). Hence

$$\left(\frac{d}{dz}\right)^k B(g(\zeta e^{\gamma z} - 1)) = \gamma^{-k} \left(\frac{d}{dz}\right)^k B(g(\zeta e^{\gamma z} - 1))$$

$$= \gamma^{-k} \left(\frac{d}{dz}\right)^k B(t * \eta).$$

It is clear that $\eta$ ranges over $\hat{E}_\pi$ as $\zeta$ runs over the $p$-th roots of unity. Then the assertion of the lemma is clear.

As in §1, let $H$ be an arbitrary finite abelian extension of $K$ and
write $G = G(H/K)$. We assume now that $p$ is prime to 2, 3 and $\mathfrak{h}$, where $\mathfrak{h}$ is the least common multiple of the conductor of $H/K$ and the conductor of $\psi$. Let $a$ be an integral ideal of $K$, which is prime to $\mathfrak{h}$, and let $A(z, a)$ be as defined in §.1. The prime $\mathfrak{p}$ of $\bar{K}$ determines a prime $\mathfrak{B}$ of $H$ lying above $\mathfrak{p}$.

**Lemma 5:** Let $\sigma \in G$. In terms of the parameter $t = -2p(z)/p'(z)$, the function

$$\frac{d}{dz} \log A_\sigma(z, a)$$

has an expansion whose coefficients all belong to $\mathcal{O}_\mathfrak{B}$, the ring of integers of the completion of $H$ at $\mathfrak{B}$.

**Proof:** By Lemma 11 of [1], $A_\sigma(z, a)$ has a power series expansion $\sum_{k=0}^\infty h_k(a, \sigma)t^k$, where the $h_k(a, \sigma)$ belong to $\mathcal{O}_\mathfrak{B}$, and $h_0(a, \sigma)$ is a unit in $\mathcal{O}_\mathfrak{B}$. It follows that the logarithmic derivative, with respect to $t$, of this power series also belongs to $\mathcal{O}_\mathfrak{B}[[t]]$. Now we can write $z = \lambda(t)$, where $\lambda$ is the logarithm map of $\hat{E}$. It is well known that $\lambda'(t)$ is a power series with coefficients in $\mathbb{Z}_p$ and leading coefficient 1. Thus $1/(\lambda'(t))$ also belongs to $\mathbb{Z}_p[[t]]$, and the assertion of Lemma 5 follows by the chain rule for differentiation.

**Lemma 6:** Let $n$ be an integer $\geq 0$. There exists $c \in \mathbb{C}$ such that

$$\prod_q \Theta(z + q, a) = c\Theta(\pi^n z, a),$$

where the product on the left is taken over a set of representatives modulo $L$ of the $\pi^n$-division points of $L$.

**Proof:** Both sides of (3) are elliptic functions for the lattice $L$, and so it suffices to verify that the two sides have the same zeros and poles. The zeros of $\Theta(z, a)$ occur precisely at the elements of $L$ each with the multiplicity $12(Na - 1)$. Similarly, the poles of $\Theta(z, a)$ are each of order 12, and occur precisely at the elements of $a^{-1}L$ which are not in $L$. Using these remarks, one immediately concludes that the right and left sides of (3) have the same zeros and poles, as required.

**Lemma 7:** Let $\sigma \in G$, and let $n$ be an integer $\geq 0$. There exists $C \in \mathbb{C}$ such that
where the product on the left is taken over a set of representatives modulo L of the \(\pi^n\)-division points of L.

**Proof:** Let \(\sigma = \sigma_c\), where \(c\) is an integral ideal of \(K\) prime to \(\mathfrak{p}\). Then it is shown in the proof of Lemma 8 of [1] that

\[
\Lambda_\sigma(z, a) = \prod_{b \in B} \Theta(z + \psi(bc)\rho, a).
\]

On the other hand, recalling that \(\pi = \psi(p)\), it follows from (3) that

\[
\prod_q \Theta(z + q + \psi(bc)\rho, a) = \Theta(\pi^n z + \psi(b\mathfrak{p}^n c)\rho, a).
\]

Taking the product of both sides of this equation over the \(b \in B\), and using (5) with \(c\) replaced by \(c\mathfrak{p}^n\), the assertion of Lemma 7 follows.

We now apply Lemma 4 with \(B_\sigma(t)\) given by the expansion in \(t\) of \(d\log \Lambda_\sigma(z, a)\). By Lemma 5, this expansion does, in fact, belong to \(O_T[[t]]\). Taking the logarithm derivative with respect to \(z\) of both sides of (4), we conclude that

\[
\left(\frac{d}{dz}\right)^k \left(\sum_{\eta \in E_\sigma} B_\sigma(t^\ast \eta)\right)_{z=0} = \left(\frac{d}{dz}\right)^{k+1} \log \Lambda_{\sigma_\mathfrak{p}}(\pi z, a)_{z=0}
\]

Hence, if \(A_\sigma(X) = B_\sigma(g(X))\), Lemma 4 implies that

\[
\left(\frac{d}{dz}\right)^k \tilde{A}_\sigma(e^z - 1)_{z=0} = \gamma^{-k} \left(\frac{d}{dz}\right)^{k+1} \left\{\left(\log \Lambda_\sigma(z, a) - \frac{1}{p} \log \Lambda_{\sigma_\mathfrak{p}}(z, a)\right)_{z=0}\right\}.
\]

Thus, in view of Lemma 2 and 3, we have established the following result. Write \(\lambda_k = 12(-1)^{k-1} \rho^{-k}(k - 1)\). Let \(\alpha\) fixed be a residue class mod \((p - 1)\). We define

\[
\zeta_{\mathcal{H}, \alpha}(\sigma, k) = \zeta_{\mathcal{H}}(\sigma, k) - \frac{\psi(k(p))}{N_\mathfrak{p}} \zeta_{\mathcal{H}}(\sigma\sigma_\mathfrak{p}, k).
\]

**Theorem 8:** Let \(B_\sigma(t) = B(t, \sigma, a)\) be the expansion in \(t\) of \(d\log \Lambda_\sigma(z, a)\). Put \(A_\sigma(t) = B_\sigma(g(t))\). Then for all integers \(k \geq 0\) with \(k = \alpha \mod (p - 1)\), we have
We now use Theorem 8 to construct \( p \)-adic L-functions. Suppose \( \chi \) is a homomorphism of \( G \) into \( \mathbb{K} \). Replacing \( H \) by the fixed field of the kernel of \( \chi \) if necessary, we can assume that the kernel of \( \chi \) is trivial.

Let us denote also by \( \chi \) the homomorphism of \( G \) into \( \mathbb{C}^*_p \) given by \( \tau \circ \chi \). For each integer \( k \geq 1 \), we define the number \( \Omega^{-k}L(\tilde{\psi}^{-k}\chi^{-1}, k) \) in \( \mathbb{C}_p \) by

\[
\Omega^{-k}L(\tilde{\psi}^{-k}\chi^{-1}, k) = \sum_{\sigma \in G} \chi^{-1}(\sigma) \tilde{\psi}_H(\sigma, k) \Omega^{-k}
\]

(7)

Let \( \mathcal{O}_{T,\chi} \) be the ring of integers of the field obtained by adjoining the values of \( \chi \) to \( T \), and write \( \mathcal{O}_{T,\chi}[X] \).

Now take \( \alpha \) an integral ideal in \( K \), prime to \( b \) and \( p \), and let \( A_\alpha(t) = A_\alpha(t, \alpha) \) be the power series in \( t \), which is defined in Theorem 8. Let \( \alpha \) be an arbitrary residue class modulo \( (p - 1) \). It follows from Lemma 5 that there is a power series \( r_\alpha(X; \chi, \alpha) \) in \( \Lambda_\chi \) such that

\[
r_\alpha((1 + p)^r - 1; \chi, \alpha) = \sum_{\sigma \in G} \chi^{-1}(\sigma) \Gamma^{\alpha - 1}(A_\sigma)(-s)
\]

(8)

for all \( s \) in \( \mathbb{Z}_p \).

**Lemma 9:** For all integers \( k \geq 0 \), with \( k \equiv \alpha - 1 \mod (p - 1) \), we have

\[
r_\alpha((1 + p)^k - 1; \chi, \alpha) = \gamma^{-k} \lambda_{k+1}(N\alpha - \psi^{k+1}(\alpha)) \chi(\alpha) \\
\times \left( 1 - \frac{\chi(p)\psi^{k+1}(p)}{Np} \right) L(\chi^{-1}\psi^{-k+1}, k + 1).
\]

**Proof:** This is immediate from Theorem 8 and the definitions (7) and (8).

If \( x \) is a unit in \( K_p \), we write as usual \( x = \omega(x)(x) \), where \( \omega(x) \) is a \( (p - 1) \)-th root of unity, and \( \langle x \rangle \equiv 1 \mod p \). Since \( \psi(a) \) generates the ideal \( \alpha \), and \( \alpha \) is prime to \( p \) by hypothesis, the number \( \psi(a) \) is a unit in \( K_p \) when viewed under the canonical inclusion of \( K \) in \( K_p \). Define \( \beta(\alpha) \) in \( \mathbb{Z}_p \) by the equation

\[
\langle \psi(\alpha) \rangle = (1 + p)^{\beta(\alpha)}
\]

and \( a_\alpha(X; \chi, \alpha) \) by
It is clear that for all integers $k \geq 0$ with $k \equiv \alpha - 1 \mod(p - 1)$, we have

$$a_\alpha((1 + p)^k - 1; \chi, a) = N a - \psi^{k+1}(a)\chi(a).$$

Since $a \neq 1$ and $\psi(a)$ generates $a$, it is easy to see that $a_\alpha(X; \chi, a)$ is not identically zero.

Define

$$f_\alpha(X; \chi, a) = \frac{r_\alpha(X; \chi, a)}{a_\alpha(X; \chi, a)}.$$

For $\lambda \in K$, let $S(\lambda)$ denote the trace, from $K$ to $\mathbb{Q}$, of $\alpha$. Let $\mathcal{D}$ be the different of $K$ and $d$ its discriminant. Let $h_0$ be the conductor of $\chi$ and $h_0^{-1}\mathcal{D}^{-1} = (\delta_0)$. We choose once for all $\delta_0$ so that $\delta_0\sqrt{d}$ has exact denominator $h_0$. Put, [18], when $\chi$ is a proper character

$$T(\bar{\chi}) = \sum_{\lambda \mod h_0} \bar{\chi}(\lambda)e^{2\pi i S(\lambda \delta_0)}$$

where $\lambda$ runs through a full system of representatives of residue classes mod $h_0$. $T(\bar{\chi})$ is different from zero.

Let $w_1$ be the number of roots of unity in $K$ congruent to 1 mod $h$. Let $\theta$ be the canonical character giving the action of $G(H(E_p)/K)$ on the group $E_p$ of $p$-division points on $E$. We define the $p$-adic $L$-functions $L_p(\chi_0, s)$ by

$$L_p(\chi_0, s) = \frac{1}{T(\chi)}\sqrt{d}w_1 \frac{1}{T(\bar{\chi})\sqrt{d}w_1} f_\alpha((1 + p)^s - 1; \chi, a).$$

(Here $\eta$ is the least common multiple of the conductor of $\chi_0$ and $\eta$.) Now if $H = K$, $\chi = \chi_0$ is the trivial character with conductor (1). We take $T(\chi_0) = 1$ we consider as before $r_\alpha(X; \chi_0, a)$, $a_\alpha(X; \chi_0, a)$, $f_\alpha(X; \chi_0, a)$ and we define

$$L_p(\eta, s) = \frac{1}{\sqrt{d}w_1} \frac{1}{T(\chi)\sqrt{d}w_1} f_\alpha((1 + p)^s - 1; \chi_0, a).$$

**Theorem 10:** For all integers $k \geq 0$, $k \equiv \alpha - 1 \mod(p - 1)$ we have
The functions $L_p(\chi^s, s)$ have been also constructed in [5].

The factor $1 - \frac{\chi(p)\psi^{k+1}(p)}{Np}$ is the Euler factor of $p$ in the Euler product of $L(\chi^{k+1}, 1)$. In fact $L(\chi^{-1}\psi^{k+1}, k + 1)$ and $L(\chi^{k+1}, 1)$ are linked by the functional equation of $L(\chi^{-1}\psi^{k+1}, s)$ [7].

We have chosen this normalisation of $L(\chi^s, s)$ because in §.III, we want to give a formula for $L_p(\chi^s, 1)$, which will be an analogue of the classical complex formula for $L(\chi^s, 1)$ (see the above remark), arising from Kronecker’s limit formula [18].

We can choose an $a$ such that $a(X; \chi, a)$ is a unit in $\Lambda_r$. Let $e$ denote a generator of the ideal $12b \cap \mathbb{Z}$. Choose $n$ to be a rational integer, prime to $p$, such that $(1 + ne_1 $) is not divisible by $p$ and take $a = (1 + ne_1)$. Then $Na \neq 1 \mod p$; also $a = 1$ because the conductor of $H/K$ divides $e$, and $\psi^k(a) = (1 + ne_1)^k$. Then $\psi^k(a) \equiv 1 \mod p$ because the conductor of $\psi$ divides $e$. Then $f_a(X; \chi, a)$ belongs to $\Lambda_x$ even when $\chi = \chi_0$ is trivial. Moreover as the right hand side of (13) and (14) is independent of the choice of $a$, and $f_a((1 + p)^s - 1; \chi, a)$ is a continuous function, it follows that $L_p(\chi^s, s)$ and $L_p(\theta^s, s)$ are Iwasawa functions independent of $a$.

III. Leopoldt’s formula

Now we will compute the value $L_p(\chi^s, 1)$ to get an analogue of Leopoldt’s formula and we will see that it is a $p$-adic analogue of the complex formula for $L(\chi^s, 1)$.

An important role here is played by the elliptic units of Robert [16]. Let $b$ be an arbitrary integral ideal of $K$. We denote by $\mathcal{P}$ a pair $(\mathcal{A}, \mathcal{N})$ where $\mathcal{A} = \{\alpha_j, j \in J\}$ and $\mathcal{N} = \{n_j, j \in J\}$; here $J$ is an arbitrary finite index set and $\alpha_j$ are integral ideals of $K$, prime to $S$ and $(p)b$, and the $n_j$ are rational integers satisfying $\Sigma_{i \in J} n_j(N\alpha_j - 1) = 0$. Given such a pair $\mathcal{P}$, we put

$$\Theta(z, \mathcal{P}) = \prod_{j \in J} \Theta(z, \alpha_j)^{n_j}$$
where $\Theta(z, a_j)$ is defined in the first part. Let $\rho$ be a $\mathfrak{h}$-division point on $E$. Then Robert has shown that $\Theta(\rho, \mathcal{P})$ is a unit in $K(E_\mathfrak{h})$.

1) Leopoldt's formula

Recall that we have defined

$$L_{s}(\chi^\alpha, s) = \frac{1}{T(\chi)w_\mathfrak{h}\sqrt{d}} f_\alpha((1 + p)^s - 1; \chi, a)$$

and

$$L_{s}(\theta^\alpha, s) = \frac{1}{w_\mathfrak{h}\sqrt{d}} f_\alpha((1 + p)^s - 1; \chi, a)$$

where $a$ is the least common multiple of the conductor of $\chi^\alpha$ (resp. $\theta^\alpha$) and $f$.

This formula is not convenient for studying the value $L_{s}(\chi^\alpha, 1)$. We will find another one.

Let $\mathcal{P}$ a pair as before (for the ideal $\mathfrak{h}$ least common multiple of the conductor of $\chi$ and $f$). For each $\sigma \in G(H/K)$, let:

$$A_\sigma(z, \mathcal{P}) = \prod_{j \in J} A_\sigma(z, a_j)^{\eta_j}.$$ 

In terms of the parameter $t = -2\psi(z)/\psi'(z)$ of $E$, $A_\sigma(z, \mathcal{P})$ has an expansion whose coefficients all belong to $\mathcal{O}_p$. Moreover

$$A_\sigma(0, \mathcal{P}) = N_{K(E_\mathfrak{h})/H} \Theta(\rho, \mathcal{P}).$$

Thus $A_\sigma(0, \mathcal{P})$ is a unit in $\mathcal{O}_p$. Hence $\Log \frac{A_\sigma(z, \mathcal{P})}{A_\sigma(0, \mathcal{P})}$ has an expansion in $t$, whose coefficients all belong to $H_\mathfrak{p}$.

**Lemma 11:** Let $B_{\sigma}(t, \mathcal{P})$ be given by the expansion in $t$ of $\Log \frac{A_\sigma(z, \mathcal{P})}{A_\sigma(0, \mathcal{P})}$ and $A_\sigma(t, \mathcal{P}) = B_{\sigma}(g(t), \mathcal{P})$.

Then for all integers $k \geq 1$, with $k \equiv \alpha \mod (p - 1)$,

$$(15) \quad \Gamma^\alpha(A_\sigma(t, \mathcal{P}))(k) = \gamma^{-k_\lambda} \sum_{j \in J} \eta_j(Na_j\xi_{H_\mathfrak{p}}(\sigma, k) - \psi^k(a_j)\xi_{H_\mathfrak{p}}(\sigma\sigma_\alpha, k)).$$

**Proof:** Let

$$B_{\sigma}(t, \mathcal{P}) = \sum_{n=0}^{\infty} a_n t^n$$
Define
\[ B'_0(t, \mathcal{P}) = \sum_{n=1}^{\infty} n a_n t^n \]
and
\[ DB_\sigma(t, \mathcal{P}) = (1 + t) \log(1 + t) B'_0(t, \mathcal{P}). \]
It is easy to see that \([12]\), for all \( s \in \mathbb{Z}_p \)
\[ \Gamma^\alpha(DB_\sigma(t, \mathcal{P}))(s) = s \Gamma^\alpha(B_\sigma(t, \mathcal{P}))(s). \]

But
\[ DB_\sigma(e^z - 1, \mathcal{P}) = z \frac{d}{dz} B_\sigma(e^z - 1, \mathcal{P}). \]
Thus
\[ DB_\sigma(e^z - 1, \mathcal{P}) = z \frac{d}{dz} \log A_\sigma(y^{-1}z, \mathcal{P}). \]

As \( a_j \) has been chosen prime to \( (p) \), we define \( \beta(a_j) \) by
\[ (\psi(a_j)) = (1 + p)^{\beta(a_j)} \]
and \( a_\sigma(X; \chi, \mathcal{P}) \) by
\[ (16) \quad a_\sigma(X; \chi, \mathcal{P}) = \sum_{j \in J} n_j [N a_j - \chi(a_j) \omega(\psi(a_j)^\alpha)(1 + X)^{-\beta(a_j)}]. \]
It is clear that for all integers \( k \geq 0 \), with \( k \equiv \alpha \mod p - 1 \) we have
\[ a_\sigma((1 + p)^{-k} - 1; \chi, \mathcal{P}) = \sum_{j \in J} n_j [N a_j - \chi(a_j) \psi^k(a_j)]. \]
Using (15) and (16), we can prove the following Lemma.

**Lemma 12:** For all integers \( k \geq 1, k \equiv \alpha \mod p - 1 \) we have
\[ \sum_{\sigma \in \Omega(H(K))} \chi^{-1}(\sigma) \Gamma^\alpha(A_\sigma(t, \mathcal{P}))(k) a_\sigma((1 + p)^{-k} - 1; \chi, \mathcal{P}) = \gamma^{-k} \lambda_k (1 - \frac{\chi(p)\psi^k(p)}{Np}) L(\chi^{-1} \psi^k, k). \]
LEMMA 13: If either $\chi$ is non trivial or $\alpha$ different from 0, there exists a pair $P$ such that $a_{n}(X;\chi, P)$ is a unit in $\Lambda_{\chi}$.

PROOF: If $\chi$ is non trivial, there exists $\sigma$ such that $\chi(\sigma) \neq 1$. Let $e$ denote a generator of the ideal $12b \cap \mathbb{Z}$. Choose $a_{1} = (1 + ne\pi)$, $n_{2} = -(Na_{1} - 1)$; take $a_{2}$ to be an integral ideal of $K$, prime to $S$ and $p$, such that $\sigma a_{2} = \sigma$ and let $n_{1} = Na_{2} - 1$.

Now if $\chi$ is trivial and $\alpha \neq 0$, let $\eta$ be an element of $O$, whose image in $O/p$ is a generator of $(O/p)^{\times}$. Take $a_{1} = (1 + ne\pi)$. Choose $a_{2} = (a_{2})$ where $a_{2}$ is an algebraic integer in $K$, satisfying $a_{2} \equiv 1 \mod e\pi$ and $a_{2} \equiv \eta \mod \pi$. Let $n_{1} = Na_{2} - 1$ and $n_{2} = -(Na_{1} - 1)$. Then $n_{2}$ is prime to $p$ and because the conductor of $\psi$ divides $e$,

$$\omega(\psi(a_{1}))^{a} = \psi^{a}(a_{1}) = 1 \mod p$$

and

$$\omega(\psi(a_{2}))^{a} = \psi^{a}(a_{2}) \equiv \eta^{a} \mod p.$$ 

Such a choice is made in [1] Lemma 13.

$$\sum_{\sigma \in \text{Gal}(K)} \chi^{-1}(\sigma) \Gamma^{a}(A_{\sigma}(t, P))(s)$$

Now $\frac{\sum_{\sigma \in \text{Gal}(K)} \chi^{-1}(\sigma) \Gamma^{a}(A_{\sigma}(t, P))(s)}{a_{n}((1 + p)^{s} - 1; \chi, P)}$ is a continuous function on $\mathbb{Z}_{p}$, which is such that for all integers $k \geq 1$, $k \equiv \alpha \mod p - 1$

$$L_{p}(\chi^{\theta^{a}}, 1 - k) = \frac{\gamma}{T(\chi)w_{d}\sqrt{d}} \sum_{\sigma \in \text{Gal}(K)} \chi^{-1}(\sigma) \Gamma^{a}(A_{\sigma}(t, P))(k)$$

if either $\chi$ is non trivial or $\alpha$ different from zero.

LEMMA 14: If either $\chi$ is a non trivial character, or $\alpha$ a non zero residue class mod $(p - 1)$, for all $s \in \mathbb{Z}_{p}$,

$$L_{p}(\chi^{\theta^{a}}, 1 - s) = \frac{\gamma}{T(\chi)w_{d}\sqrt{d}} \sum_{\sigma \in \text{Gal}(K)} \chi^{-1}(\sigma) \Gamma^{a}(A_{\sigma}(t, P))(s)$$

(17) 

Remark: If $\chi$ is trivial and $\alpha$ is zero
But:

$$a_0(0; \chi_0, \mathcal{P}) = \sum_{j \in j} n_j (Na_i - 1) = 0.$$ 

$$I^0(A(t, \mathcal{P}))(0) = \hat{A}(0, \mathcal{P}) = B(0, \mathcal{P}) - \frac{1}{p} \sum_{\eta \in \mathcal{E}_x} B(\eta, \mathcal{P})$$

$$B(0, \mathcal{P}) = 0$$

and

$$\frac{1}{p} \sum_{\eta \in \mathcal{E}_x} B(\eta, \mathcal{P}) = \frac{1}{p} \sum_{\alpha} \log_p \frac{\Lambda(\alpha, \mathcal{P})}{\Lambda(0, \mathcal{P})}$$

where the sum on the right is taken over a set of representatives modulo $L$ of the $\pi$-division points of $L$. Then

$$\frac{1}{p} \sum_{\alpha} \log_p \frac{\Lambda(\alpha, \mathcal{P})}{\Lambda(0, \mathcal{P})} = \left(\frac{1}{p} - 1\right) \log_p \Lambda(0, \mathcal{P}).$$

But

$$\Lambda(0, \mathcal{P}) = N_{K(E_i) | K}(\rho, \mathcal{P})$$

where $\rho$ is a $\mathfrak{f}$-division point of $C$ mod $L$. This is a unit in $K$, then a root of unity and

$$\log_p N_{K(E_i) | K}(\rho, \mathcal{P}) = 0.$$

Even when $\chi$ is trivial and $\alpha$ is zero

$$\frac{I^0(A(t, \mathcal{P}))(s)}{a_0((1 + p)^s - 1; \chi_0, \mathcal{P})}$$

is a continuous function on $Z_p$ and we have

$$L_\psi(\theta^0, 1 - s) = \frac{\gamma}{w_d \sqrt{d}} \frac{I^0(A(t, \mathcal{P}))(s)}{a_0((1 + p)^s - 1; \chi_0, \mathcal{P})}.$$

But this formula is not useful for computing $L_\psi(\theta^0, 1)$.

Now we come back to the case where $\chi$ is non trivial, and $\alpha = 0$. From (17) we have
by lemma 4

\[ B_\sigma(0, \mathcal{P}) = 0 \]

and

\[ \frac{1}{p} \sum_{\eta \in E_\pi} B_\sigma(\eta, \mathcal{P}) = \frac{1}{p} \sum_{\alpha} \log \frac{A_\sigma(\alpha, \mathcal{P})}{A_\sigma(0, \mathcal{P})}. \]

where the sum on the right is taken over a set of representatives modulo \( L \) of the \( \pi \)-division points of \( L \). Now from Lemma 7, we have

\[ \Gamma^0(A_\sigma(t, \mathcal{P}))(0) = \frac{1}{p} \log \frac{A_{\sigma_0}(0, \mathcal{P})}{A_\sigma(0, \mathcal{P})^p}. \]

**Theorem 15:** If \( \chi \) is not trivial

\[
L_\pi(\chi, 1) = \gamma \frac{\sum_{\alpha \in \mathcal{O}(H/K)} \chi^{-1}(\sigma) I^0(A_\sigma(t, \mathcal{P}))(0)}{T(\bar{\chi}) w_\alpha \sqrt{d} a_0(0; \chi, \mathcal{P})}
\]

\[
\times \sum_{\sigma \in \mathcal{O}(H/K)} \chi^{-1}(\sigma) \log[N_{K(E_\delta)/H}\Theta(\rho, \mathcal{P})] a_0(0; \chi, \mathcal{P})
\]

\begin{equation}
\tag{18}
L_\pi(\chi, 1) = \gamma \frac{\chi(p)}{T(\bar{\chi}) w_\alpha \sqrt{d} \left( \frac{\chi(p)}{p} - 1 \right)}
\end{equation}

\[ \times \sum_{\sigma \in \mathcal{O}(H/K)} \chi^{-1}(\sigma) \log[N_{K(E_\delta)/H}\Theta(\rho, \mathcal{P})] a_0(0; \chi, \mathcal{P})
\]

We now proceed to find a similar formula for \( \alpha \neq 0 \). As before, define

\[ T(\tilde{\theta}) = \sum_{\lambda \pmod{\mathfrak{p}}} \tilde{\theta}(\lambda)e^{2\pi i \delta (\lambda \delta_\mathfrak{p})} \]

where \( \delta \) has been chosen once for all such that \( p^{-1}D^{-1} = (\delta) \) and \( \delta \sqrt{d} \) has exact denominator \( \mathfrak{p} \), and where \( \lambda \) runs through a full system of representatives of the residue classes mod \( \mathfrak{p} \). Let us denote by \( \zeta \) the \( p \)-th root of unity \( e^{2\pi i \delta} \). As \( p \) splits in \( K \), \( \mathcal{O}/\mathfrak{p} \) is isomorphic to \( \mathbb{Z}/p\mathbb{Z} \). Then, we will write

\[ T(\tilde{\theta}) = \sum_{\lambda \pmod{\mathfrak{p}}} \tilde{\theta}(\lambda) \zeta^\lambda. \]
LEMMA 16: For each $\alpha$, congruence class mod $(p-1)$ for each rational integer $n$, prime to $p$

$$\sum_{\lambda \mod p} \tilde{\theta}^\alpha(\lambda)\zeta^{\lambda n} = \omega^\alpha(n)T(\theta^\alpha).$$

PROOF: Let $m \in Z$, such that

$$m \equiv n \mod p$$

and

$$m \equiv 1 \mod f \ (where \ f = f \cap Z)$$

$$\sum_{\lambda \mod p} \tilde{\theta}^\alpha(\lambda)\zeta^{\lambda n} = \sum_{\lambda \mod p} \tilde{\theta}^\alpha(\lambda)\zeta^{\lambda m} = \theta^\alpha(m) \sum_{\lambda \mod p} \tilde{\theta}^\alpha(\lambda)\zeta^{\lambda}.$$

By definition

$$\theta(m) = \omega(\psi(m)) = \omega(m) = \omega(n).$$

Then Lemma 16 is proved.

Let $M$ be any complete subfield of $C_p$, and $A \in Q_M$. For each $\alpha$ congruence class mod$(p-1)$, let

$$A_\alpha(u) = \frac{1}{T(\theta^\alpha)} \sum_{\lambda \mod p} \tilde{\theta}^\alpha(\lambda)A(\zeta^\lambda(u+1) - 1).$$

$A_\alpha$ belongs to $Q_M$ and does not depend on $\zeta$.

LEMMA 17: For each $s \in Z_p$

$$\Gamma^{\alpha-\beta}(A)(s) = \Gamma^{\alpha-\beta}(A_\alpha)(s).$$

PROOF: Because of the linearity of $\Gamma^{\alpha-\beta}$ and $\Gamma^{\beta}$ we have just to prove the equality for $A(u) = (1+u)^n$. Then

$$A_\alpha(u) = \frac{1}{T(\theta^\alpha)} \sum_{\lambda \mod p} \tilde{\theta}^\alpha(\lambda)\zeta^{\lambda n}(1+u)^n$$

$$A_\alpha(u) = \omega^\alpha(n)(1+u)^n.$$

By definition [12]:
Now let us consider

\[ \Gamma^\alpha_{-\beta}(A)(s) = \omega^\alpha_{-\beta}(n)(n)^s \text{ if } p \nmid n = 0 \quad \text{if } p | n \]

and

\[ \Gamma^\beta_{-\gamma}(A_{\sigma})(s) = \omega^\beta(n)\omega^\gamma(n)(n)^s \text{ if } p \nmid n = 0 \quad \text{if } p | n. \]

Then

\[ \Gamma^\alpha_{-\beta}(A_{\sigma}(t, \mathcal{P}))(0) = \Gamma^\alpha_{0}(A_{\sigma,\alpha}(t, \mathcal{P}))(0). \]

Moreover

\[ \Gamma^\alpha_{0}(A_{\sigma,\alpha}(t, \mathcal{P}))(0) = A_{\sigma,\alpha}(0, \mathcal{P}) - \frac{1}{p} \sum_{\zeta} A_{\sigma,\alpha}(\zeta' - 1, \mathcal{P}) \]

where \( \zeta' \) runs over all \( p \)-th roots of unity in \( \mathbb{C}_p \). But:

\[ \sum_{\zeta} A_{\sigma,\alpha}(\zeta' - 1, \mathcal{P}) = \frac{1}{T(\theta^\alpha_\lambda)} \sum_{\zeta} \sum_{\lambda \mod p} \tilde{\theta}^\alpha(\lambda)A_{\sigma}(\zeta^\lambda \zeta' - 1, \mathcal{P}). \]

Then

\[ \sum_{\zeta} A_{\sigma,\alpha}(\zeta' - 1, \mathcal{P}) = 0 \]

and:

\[ \Gamma^\alpha_{0}(A_{\sigma,\alpha}(t, \mathcal{P}))(0) = \frac{1}{T(\theta^\alpha_\lambda)} \sum_{\lambda \mod p} \tilde{\theta}^\alpha(\lambda)A_{\sigma}(\zeta^\lambda - 1, \mathcal{P}). \]

Recall that by definition \( A_{\sigma}(t, \mathcal{P}) = B_{\sigma}(g(t), \mathcal{P}) \) where \( B_{\sigma}(t, \mathcal{P}) \) is given by the expansion \( \log \frac{A_{\sigma}(z, \mathcal{P})}{A_{\sigma}(0, \mathcal{P})} \).

Define:

\[ \Lambda^I(z, a_i) = \prod_{\beta \in B} \Theta(z + \psi(b)\beta + q, a_i) \]
and

\[ \Lambda^1(z, \mathcal{P}) = \prod_{j \in J} \Lambda^1(z, a_j)_{\eta_j} \]

where \( q \) is an element of \( C \) such that \( \xi(q) \) is the \( p \)-division point on \( E \) which corresponds to \( \zeta \). Then

\[ A_{\sigma}(\zeta - 1, \mathcal{P}) = \log \frac{A_{\sigma}(q, \mathcal{P})}{A_{\sigma}(0, \mathcal{P})} = \log \frac{\Lambda_{\sigma}^1(0, \mathcal{P})}{\Lambda_{\sigma}(0, \mathcal{P})}. \]

By Lubin Tate theory, we know that \( G(K(E_p)/K) \) is naturally isomorphic to the group of units of \( \mathcal{O}/p \); moreover \( G(H(E_p)/H) \) is isomorphic to \( G(K(E_p)/K) \) [14]. Then to each \( \lambda \mod p \) corresponds \( \Lambda^1 C \in G(H(E_p)/H) \) and

\[ A_{\sigma}(\zeta^\lambda - 1, \mathcal{P}) = \log \frac{A_{\sigma}^{\lambda}(0, \mathcal{P})}{A_{\sigma}(0, \mathcal{P})}. \]

**Theorem 18:** If \( \alpha \) is a non zero residue class mod \( p - 1 \)

(19)

\[ L_{\sigma}(\chi^\alpha, 1) = \frac{\gamma}{T(\chi)w_{\alpha}} \sum_{\sigma \in G(K(E_p))/K} \chi^{-1}(\sigma) \log p \theta(p + q, \mathcal{P})^\alpha \]

(2) **Analogy with complex formula**

Let \( H \) be an arbitrary finite abelian extension of \( K \) and let \( \mathfrak{b} \) be the least common multiple of the conductor of \( \psi \) and \( \mathfrak{b}_0 \), the conductor of \( H/K \). Let \( \chi' \) be a ray class character mod \( \mathfrak{b} \) such that \( \chi \), the proper ray-class character associated with \( \chi' \) has conductor \( \mathfrak{b}_0 \).

We will see that we have complex formula for \( L(\chi', 1) \) which is analogue of (17) and (18).

We take the notation of Robert [16]. Let us consider the set \( A(\mathfrak{b}) \) of pairs \( \{t, b\} \) where \( t \in \mathcal{O} \) and \( b \) is an ideal of \( K \), such that \( \mathfrak{b} = \{\alpha \in \mathcal{O} \mid at \in b\} \). One says that \( \{t, b\} \) is equivalent to \( \{t', b'\} \) if and only if, there exists \( \theta \in K^* \) such that \( t'/\theta t \) is congruent to 1 mod \( \mathfrak{b} \) and \( b' = \theta b \). Denote by \( \sim \) this equivalence. For each \( \{t, b\} \in A(\mathfrak{b}) \), \( t_\mathfrak{b}b^{-1} \) is an integral ideal, prime to \( \mathfrak{b} \). Denote by \( C_{\{t, b\}} \) the ideal class of \( t_\mathfrak{b}b^{-1} \). Robert has shown that the map \( \{t, b\} \mapsto C_{\{t, b\}} \) defines an isomorphism between \( A(\mathfrak{b}) \) and the ray class group mod \( \mathfrak{b} \), \( Cl(\mathfrak{b}) \). Let \( [w_1, w_2] \) be a basis of \( \mathfrak{b} \); we define

\[ \varphi^{12}(t, b) = \theta^{12}(t; w_1, w_2) \exp(-\mathcal{K}(t, t)/16) \]
where \( \mathcal{K}(t, t) = 12i\pi t/(w_2 \bar{w}_1 - w_1 \bar{w}_2) \). It can be shown that \( \varphi^{12h}(t, b) \) depends only on \( C_{(t, b)} \) and we set

\[
\varphi_b(C) = \varphi^{12h}(t, b).
\]

Now if we consider the pair \( \{\rho, \mathcal{C}\} \) where \( \rho = \frac{\Omega}{h} \). Then \( C_{(\rho, \mathcal{C})} = C_0 \) the identity in the ray class group mod \( \mathfrak{p} \). So

\[
\Theta^{12h}(\rho, a_j) = \varphi(C_0)^{N_a}/\varphi(C_0C_{a_j}).
\]

Then:

\[
\sum_{\sigma \in \mathcal{O}(K(E_0/K))} \chi'(\sigma) \log|\Theta(\rho, \mathcal{P})|^{\sigma} \frac{a_d(0; \chi, \mathcal{P})}{12h} = \sum_{C \in \mathcal{E}(b)} \chi'(C) \log|\varphi_b(C)| = \frac{X(\chi)}{w_h h_0} \sum_{C \in \mathcal{E}(b)} \chi(C) \log|\varphi_b(C)|
\]

Moreover it can be proved that [16]:

\[
\frac{1}{w_h h_0} \sum_{C \in \mathcal{E}(b)} \chi'(C) \log|\varphi_b(C)| = \frac{X(\chi)}{w_h h_0} \sum_{C \in \mathcal{E}(b)} \chi(C) \log|\varphi_b(C)|
\]

when

\[
X(\chi) = \prod_{q \mid b} (1 - \chi(q)).
\]

Now Siegel [18] has shown that

\[
L(\chi, 1) = \frac{2\pi}{T(\chi)\sqrt{d} w_h h_0} \sum_{C \in \mathcal{E}(b)} \chi(C) \log|\varphi_b(C)|.
\]

So, from (20), (21), (22) we have

\[
X(\chi)L(\chi, 1) = \frac{\pi}{T(\chi)w_0 \sqrt{d}} \sum_{\sigma \in \mathcal{O}(K(E_0/K))} \chi'(\sigma) \log|\Theta(\rho, \mathcal{P})|^{\sigma} a_d(0; \chi, \mathcal{P}).
\]

This formula is the complex analogue of (17) and (18). We will try to explain why this holds. We have

\[
L(\bar{\chi}', 0) = X(\chi)L(\bar{\chi}, 0)
\]
Moreover, from the functional equation [7], we have

\[ L(\tilde{\chi}, 0) = L(\psi^0 \tilde{\chi}, 0) = L(\psi^0, 0). \]

Then

\[ L(\tilde{\chi}, 0) = L(\chi, 1) \frac{\sqrt{dT(\tilde{\chi})}}{2\pi}. \]

and this is to compare with Lemma 12 and 13, if we could put \( k = 0 \).

**IV. \( p \)-adic residue formula**

Again, we suppose throughout this section that \( H \) is an arbitrary finite abelian extension of \( K \). As before, we write \( b \) for the least common multiple of the conductor of \( H \) over \( K \), and the conductor of the Grossencharacter \( \psi \) of \( E \) over \( K \). Finally, \( p \) is a rational prime, with \( p \neq 2, 3 \) and \( (p, b) = 1 \), which splits in \( K \), say \( (p) = pp \). For simplicity, we write

\[ F = H(E_p). \]

By analogy with Leopoldt’s work, in the cyclotomic case, our aim is to use the result of §.III to find the residue at \( s = 1 \) of a \( p \)-adic function that can be viewed almost as the \( p \)-adic zeta function of \( F \). Such a formula has been studied independently of us by Vishik [20] and Lichtenbaum. We begin by recalling the complex analogue of this formula. By class field theory, we have

\[ \zeta_F(s) = \zeta_k(s) \prod_{\chi \neq 1} L(\chi, s) \]

where the product on the right is taken over the non trivial characters \( \chi \) of the Galois group of \( F \) over \( K \), and \( L(\chi, s) \) is the primitive complex \( L \)-function attached to \( \chi \). Let \( \Delta \), \( W \), \( g \) denote respectively the discriminant of \( F \) over \( \mathbb{Q} \) the number of roots of unity in \( F \), and
the degree of $F$ over $K$. Let $d$, $w$ denote the discriminant of $K$ over $\mathbb{Q}$, and the number of roots of unity in $K$. Finally, let $R_\infty$ denote the regulator of $F$, and $h$ the class number of $F$. Multiplying by $s-1$ in the above formula and letting $s \to 1$ we obtain

$$
\frac{(2\pi)^h h R_\infty}{W \sqrt{|\Delta|}} = \frac{2\pi}{w \sqrt{|d|}} \prod_{\chi \neq 1} L(\chi, 1).
$$

Let $R_p$ be the $p$-adic regulator of $F$ over $K$, as defined on p. 13 of [4]. Also, we can view $\sqrt{\Delta}$ and $\sqrt{|d|}$ as lying inside $\mathbb{C}$, by taking their images under our fixed embedding $\tau: \overline{K} \to \mathbb{C}$ (for simplicity, we identify these elements with their images under $\tau$).

Let $\mathcal{P}$ be the pair defined in the previous section; $\rho = \frac{\Omega}{hp}$, where $(h) = b$. Let for $\sigma \in G(F/K)$

$$
E(\sigma) = \frac{\prod_{b \in B} \Theta(\psi(b) \rho, \mathcal{P})^\sigma}{\prod_{b \in B} \Theta(\psi(b) \rho, \mathcal{P})}.
$$

Let $\mathcal{E}_1$ be the group generated by the $E(\sigma)^{\sigma'}$, with $\sigma' \in G(F/K)$. It is a group of units in $F$.

Let us denote by

$$
A(\mathcal{P}) = \prod_{\chi \neq 1} a_0(0; \chi, \mathcal{P})
$$

by

$$
X = \prod_{\chi \neq 1} X(\chi)
$$

and

$$
w' = \prod_{\chi \neq 1} w_{\mathcal{E}_1, \chi}
$$

where $a_\chi$ is the least common multiple of the conductor of $\chi$ and $\psi$, where $\chi$ runs over all primitive character of $G(F/K)$.

**Lemma 19:** The index of $\mathcal{E}_1$ in the group of all units in $F$ is given by
\[ 2^{r-1} h \frac{w w'}{W} A(\mathcal{P}) X. \]

**Proof:** It is well known that the index of \( \mathcal{E}_1 \) in the group of all units in \( F \) is equal to \( \frac{U}{R_\infty} \) where \( U = \det(\log |E(\sigma)|) \) with \( \sigma, \sigma' \in G(F/K) \). From (24) we have

\[(29) \quad \prod_{\chi \neq 1} L(\chi, 1) = (2\pi)^{r-1} \frac{w}{W} \frac{R_\infty \sqrt{d}}{\sqrt{\Delta}} h.\]

Moreover from (23)

\[ \prod_{\chi \neq 1} L(\chi, 1) = \pi^{r-1} \frac{\sqrt{d}}{(\sqrt{d})^r} \frac{1}{\prod_{\chi \neq 1} T(\chi)} W' A(\mathcal{P}) X \times \prod_{\chi \neq 1} \sum_{\sigma \in G(F/K)} \chi(\sigma) \log \left| \prod_{b \in B} \Theta(\psi(b) \rho, \mathcal{P})^\sigma \right|. \]

But we know [18] that

\[(31) \quad U = \prod_{\chi \neq 1} \sum_{\sigma \in G(F/K)} \chi(\sigma) \log \left| \prod_{b \in B} \Theta(\psi(b) \rho, \mathcal{P})^\sigma \right|. \]

Combining (29) and (30), we have the lemma, recalling that \( (\sqrt{d})^r \prod_{\chi \neq 1} T(\chi) = \sqrt{\Delta} \).

Let us denote

\[ P = \left(1 - \frac{1}{p}\right)^{-1} \prod_{\mathfrak{p}} (1 - N(\mathfrak{p}))^{-1} \]

where the product is taken over all primes of \( F \) above \( p \).

**Theorem 20:**

\[ \prod_{\chi \neq 1} L_r(\chi, 1) = (2\gamma)^{r-1} h \frac{w}{W} \frac{R_\infty \sqrt{d}}{\sqrt{\Delta}} PX \text{ up to } \pm 1. \]

where the product on the left is taken over all non trivial character of \( G(F/K) \).

**Proof:** From (17) and (18), we know that
Then

\[ \sum_{\sigma \in G(F|K)} \chi(\sigma) \log_p \left( \prod_{\psi \in B} \Theta(\psi(b), \mathcal{P})^\sigma \right) \frac{1 - \chi(p)}{p}. \]

Then

\[ \prod_{\chi \neq 1} L_\chi(\chi, 1) = \frac{\gamma^{e-1} \sqrt{|d|}}{\sqrt{|\Delta|} w} \prod_{\chi \neq 1} \sum_{\sigma \in G(F|K)} \chi(\sigma) \log_p \left( \prod_{\psi \in B} \Theta(\psi(b), \mathcal{P})^\sigma \right). \]

Let

\[ U_\sigma = \prod_{\chi \neq 1} \sum_{\sigma \in G(F|K)} \chi(\sigma) \log_p \left( \prod_{\psi \in B} \Theta(\psi(b), \mathcal{P})^\sigma \right). \]

Then

\[ U_\sigma = \det(\log_p E(\sigma)'') \sigma', \sigma' \in G(F|K). \]

But \( U_\sigma / R_\sigma \) is equal to the index of \( \mathcal{E}_1 \) in the group of all units in \( F \), up to \( \pm 1 \). Then

\[ U_\sigma = R_\sigma 2^{e-1} h^{w w'} A(\mathcal{P}) X \text{ up to } \pm 1. \]

Then Theorem 20 is proved.

(2) **Kummer's criterion**

Let us recall what is known about Kummer's criterion in the elliptic case. Let \( L_\psi(\psi^k, s) \) be the *primitive* Hecke \( L \)-function of \( \psi^k \) for each \( k \geq 1 \). Let \( L_\psi^S(\psi^k, k) = w(k - 1)! L_\psi(\psi^k, k), k \equiv 0 \) mod \( w \). If \( p \) is a prime number not in the exceptional set \( S \), which splits in \( K \), it is shown in [4] that the numbers

\[(N) \quad L_\psi^S(\psi^k, k)(1 \leq k < p - 1 ; k \equiv 0 \mod w) \]

are all \( p \)-integral. Let \( (p) = p \mathfrak{p} \) and \( H_p \) the ray class field of \( K \) modulo \( \mathfrak{p} \). It is shown in [4] the Kummer's criterion i.e.

\( p \) divides at least one of the numbers \( (N) \) if and only if there exists a \( \mathbb{Z}/p \mathbb{Z} \)-extension of \( H_p \), which is unramified outside the prime of \( H \), above \( \mathfrak{p} \) and which is distinct from the ray class field mod \( \mathfrak{p}^2 \).
The proof of this theorem is divided in two parts. In the first part, the authors use class field theory to establish a Galois theoretic p-adic residue formula for $F$ an arbitrary finite extension of $K$. Denote by $K_\infty$ the unique $\mathbb{Z}_p$-extension of $K$, which is unramified outside $\mathfrak{p}$ and $F_\infty = K_\infty F$. The notations are those of the previous section.

**Lemma 21:** Let $M$ be the maximal abelian $p$-extension of $F$, which is unramified outside the primes of $F$ lying above $\mathfrak{p}$. Then $G(M/F_\infty)$ is finite if and only if $R_\mathfrak{p} \neq 0$. If $R_\mathfrak{p} \neq 0$, the order of $G(M/F_\infty)$ is equal to the inverse of the $p$-adic valuation of

$$\frac{p^eh R_\mathfrak{p} \sqrt{|d|}}{W \sqrt{|\Delta|}}$$

where the integer $e$ is defined by $F \cap K_\infty = K_e$.

Then they combine this with a function theoretic $p$-adic residue formula due to Katz and Lichtenbaum for the $p$-adic zeta function of $H_{\mathfrak{p}}$ over $K$.

Let now $H$ be an arbitrary finite abelian extension of $K$ and $F = H(E_{\mathfrak{p}})$. Let us consider the numbers

$$N' = \begin{cases} \lambda_k \left( 1 - \frac{\chi(p) \psi_k(p)}{Np} \right) L(\tilde{\chi} \tilde{\psi}^k, k)(1 \leq k < p - 1, k \equiv 0 \mod w) \\ \lambda_k \left( 1 - \frac{\chi(p) \psi_k(p)}{Np} \right) L(\tilde{\chi} \tilde{\psi}^k, k) \prod_{q \equiv k \equiv 0 \mod w} \left( 1 - \tilde{\chi}(q) \frac{\tilde{\psi}_k(q)}{Nq^k} \right)^{-1} \end{cases} \quad (1 \leq k < p - 1, k \equiv 0 \mod w)$$

for all primitive character $\chi$ associated to the characters of the Galois group $G(F/K)$.

Let $\mathfrak{p}$ denote any prime of $H$ above $\mathfrak{p}$.

**Theorem 22:** $\mathfrak{p}$ divides at least one of the numbers $(N')$ if and only if there exists a $\mathbb{Z}/p\mathbb{Z}$-extension of $F$, which is unramified outside the primes of $H(E_{\mathfrak{p}})$ above $\mathfrak{p}$ and which is distinct from $H(E_{\mathfrak{p}})$.

**Proof:** Theorem 20 shows that

$$\left| \prod_{\chi \neq 1} L_{\mathfrak{p}}(\chi \theta^a, 1) \right| = \left| \frac{h R_\mathfrak{p} \sqrt{|d|}}{W \sqrt{|\Delta|}} \chi P \right|.$$
For all $\chi\theta^\alpha$, $L_\phi(\chi\theta^\alpha, s)$ is an Iwasawa function. Then, for all integers $k \geq 0$

$$L_\phi(\chi\theta^\alpha, 1) \equiv L_\phi(\chi\theta^\alpha, 1 - k) \mod p.$$ 

But from theorem 10, if $k \equiv \alpha \mod (p - 1)$, $k \geq 1$

$$L_\phi(\chi\theta^\alpha, 1 - k) = \gamma^{1-k}\lambda_k \left(1 - \frac{\chi(p)\psi^k(p)}{N_p}\right)L(\chi^\alpha\psi^k, k).$$

This shows that if $k \equiv \alpha \mod p - 1$, $k \geq 1$

$$L_\phi(\chi\theta^\alpha, 1) \equiv \left(1 - \frac{\chi(p)\psi^k(p)}{N_p}\right)\gamma^{1-k}\lambda_k L(\chi^\alpha\psi^k, k) \mod p.$$ 

Moreover, if $k \equiv \alpha \mod p - 1$

$$\prod_{q \mid p} (1 - \chi\theta^\alpha(q)) = \prod_{q \mid f} (1 - \chi(q) \omega^{-k}(\psi(q))).$$

And if $k \equiv 0 \mod w$

$$X(\chi\theta^\alpha) \equiv \prod_{q \mid f} (1 - \chi(q)\psi^{-k}(q)) \mod p$$

Or

$$X(\chi\theta^\alpha) \equiv \prod_{q \mid f} \left(1 - \chi(q)\frac{\psi^k(q)}{Nq^k}\right) \mod p.$$ 

Thus, if $\alpha \equiv k \mod p - 1$

$$X(\chi\theta^\alpha)^{-1}L_\phi(\chi, 1) = \left(1 - \frac{\chi(p)\psi^k(p)}{N_p}\right)\lambda_k\gamma^{1-k}L(\chi^\alpha\psi^k, k) \mod p$$

if $k \equiv 0 \mod p - 1$

$$X(\chi\theta^\alpha)^{-1}L_\phi(\chi, 1) = \left(1 - \frac{\chi(p)\psi^k(p)}{N_p}\right)\lambda_k\gamma^{1-k}L(\chi^\alpha\psi^k, k)\prod_{q \mid f} \left(1 - \chi(q)\frac{\psi^k(q)}{Nq^k}\right) \mod p$$

if $k \equiv 0 \mod p - 1$.

Now we have just to use Lemma 20.
REFERENCES


(Oblatum 22-VII-1979)

Laboratoire associé au C.N.R.S. n°226
U.E.R. de Mathématiques et d'Informatique de l'Université de Bordeaux I
351, cours de la Libération
33405 TALENCE CEDEX