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VANISHING CRITERIA AND THE PICARD GROUP FOR PROJECTIVE VARIETIES OF LOW CODIMENSION

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Introduction

Let X and Y be closed subschemes of $\mathbb{P}^n$, where X is irreducible and smooth, and Y is a local complete intersection (characteristic 0) or Cohen-Macauley (characteristic $p > 0$). We shall be interested in two invariants: first, the cohomological dimension

$$\mathrm{cd}(X - Y) = \sup\left\{ i \,\middle|\, \begin{array}{c} H^i(X - Y, F) \neq 0 \\ \text{for some coherent} \\ \text{sheaf } F \text{ on } X \end{array} \right\}$$

and second, the Picard group $\mathrm{Pic}(X)$.

Let $s = \dim(X)$ and $t = \dim(Y)$. Our main result for $\mathrm{cd}(X - Y)$ is (2.1), which generalizes some earlier vanishing criteria: we have

$$s + t > n \Rightarrow \mathrm{cd}(X - Y) < s - 1.$$

In particular, we find

$$s + t > n \Rightarrow X \cap Y \text{ is connected},$$

a result recently proved [6] by Fulton and Hansen, under weaker assumptions. Their method is based on specialization of cycles.

For $\mathrm{Pic}(X)$, assume the characteristic is $p > 0$ and that $s = \dim(X) \geq \frac{1}{2}(n + 2)$. Then (3.1) the Picard group $\mathrm{Pic}(X)$ is a finitely generated abelian group of rank 1, with no torsion prime to p. (The assertion

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about the torsion is [5, Cor. 4.6], but the assertion about the rank, although expected, is new.) The analogue in characteristic 0, (compare, for example, Ogus [8]), has been known for some time.

Our proofs are mainly based on ideas used, for example, in [2], [9], [8], [5] and [11]. The approach to (2.1) is though a study of the cohomology of coherent sheaves on a formal completion, with the needed preliminary results in §1. For (3.1) we use Hartshorne's version of the Barth-Lefschetz theorem for l-adic cohomology.

Notation will be standard, except that all schemes will tactily be assumed separated and of finite type over the spectrum of an algebraically closed field k, of arbitrary characteristic.

§1. Formal Neighborhoods in $\mathbb{P}^n$

Fix a closed subscheme $X \subset \mathbb{P}^n$, and denote by $\hat{\mathbb{P}}^n$ the formal completion of $\mathbb{P}^n$ along X. For a coherent sheaf F on $\mathbb{P}^n$, write $\hat{F}$ for the formal completion of F. In particular, the standard invertible sheaf $\mathcal{O}(\nu)$ on $\mathbb{P}^n$ has completion $\hat{\mathcal{O}}(\nu)$.

For a coherent sheaf F on $\mathbb{P}^n$, the homological dimension hd(F) is the maximum projective dimension dim proj(F_x) of a stalk of F, where x ranges over all scheme points $x \in \mathbb{P}^n$, and the projective dimension of F_x is taken over the local ring of $\mathbb{P}^n$ at x.

For each integer i, define a $\underline{k}$-vector space V^i as follows. In characteristic zero, let X_0 be $\mathbb{V}(\mathcal{O}_X(-1))$ minus the zero-section, where of course $\mathcal{O}_X(-1) = \mathcal{O}(-1)|X$; then set

$$V^i = H^i_{DR}(X_0),$$

the algebraic de Rham cohomology group [13]. In characteristic $p > 0$, set

$$V^i = H^i(X, \mathcal{O}_X)_s,$$

the stable part of $H^i(X, \mathcal{O}_X)$ under the action of the p^{th}-power endomorphism of $\mathcal{O}_X$.

THEOREM (1.1): *Let $X \subset \mathbb{P}^n$ be a closed subscheme. If the characteristic is zero, suppose X is a local complete intersection, or, if the characteristic is >0, that X is Cohen-Macaulay. Given any coherent sheaf F on $\mathbb{P}^n$ there are, for all integers k, natural maps*

$$\beta^k : \bigoplus_{i+j=k} H^i(\mathbb{P}^n, F) \underset{\underline{k}}{\otimes} V^j \to H^k(\hat{\mathbb{P}}^n, \hat{F})$$

which are bijective for $k < \dim(X) - \mathrm{hd}(F)$, *and injective for* $k = \dim(X) - \mathrm{hd}(F)$.

Let $S = \underline{k}[X_0, \ldots, X_n]$ be the homogeneous co-ordinate ring of $\mathbb{P}^n$. Then

$$M^i = \sum_{\nu \in Z} H^i(\hat{\mathbb{P}}^n, \hat{\mathcal{O}}(\nu))$$

is a graded S-module under the cup product. With $F = \mathcal{O}(\nu)$ in (1.1), then, summing over ν and taking into account the standard results on the cohomology of invertible sheaves on $\mathbb{P}^n$, we obtain the following:

COROLLARY (1.2): *Let* $X \subset \mathbb{P}^n$ *be as in* (1.1). *We have a natural map of graded* S*-modules*

$$\beta^i : S \underset{\underline{k}}{\otimes} V^i \to M^i$$

which is bijective for $i < \dim(X)$ *and injective for* $i = \dim(X)$.

PROOF OF (1.1): In characteristic zero, (1.1) is an immediate special case of Ogus' result [8, Th. 2.1, p. 1091]. In characteristic $p > 0$, (1.2) is [2, Cor. 6.6, p. 140], plus the Lemma of Enriques and Severi [2, Ex. 6.13, p. 143]. To deduce (1.1) from (1.2), one can repeat Ogus' argument, once one has the maps β^k, to which we now proceed.

Construction of the β^k

Here we work in the category of graded (S, F)-modules; [2, p. 127–143] contains the definitions and the foundational results we shall need. Since S is regular, the p^{th}-power endomorphism $F : S \to S$ is flat [loc. cit. Cor. 6.4, p. 138], hence the functor G [loc. cit., p. 132] is left exact.

LEMMA (1.3): *Let* M *be a graded* (S, F)*-module such that* M_s *is finite dimensional over* $\underline{k}$. *Then*:

(a) *there is a natural injection of* (S, F)*-modules*

$$S \underset{\underline{k}}{\otimes} (M_0)_s \to G(M);$$

(b) *if* $M_\nu = 0$ *for* $\nu \ll 0$, *this map is a bijection.*

PROOF: If $M_\nu = 0$ for $\nu \ll 0$, this is precisely [2, Theorem 6.1, p. 133]. If not, consider the functor acting on (S, F)-modules via

$$M \mapsto M^+ = \sum_{\nu \geq 0} M_\nu,$$

where the image M^+ is an (S, F)-module by restriction. We have

$$M_s = (M_0)_s = (M^+)_s,$$

and, since G is left exact, we have a natural inclusion $G(M^+) \subset G(M)$ induced by the inclusion of M^+ in M. Since (b) holds for M^+, we obtain a composite injective morphism of functors of M:

$$S \otimes_{\underline{k}} (M_0)_s = S \otimes_{\underline{k}} (M_0^+)_s \cong G(M^+) \to G(M).$$

This proves (1.3).

We can now construct the β^k. By [2, Theorem 6.3, p. 135], we have a natural isomorphism

$$M^i \cong G\Big(\sum_{\nu \in Z} H^i(X, \mathcal{O}_X(\nu))\Big),$$

hence an injection

$$S \otimes_{\underline{k}} V^i = S \otimes_{\underline{k}} H^i(X, \mathcal{O}_X)_s \to M^i$$

by (1.3)(a), since, plainly, $(\Sigma H^i(X, \mathcal{O}_X(\nu)))_s = H^i(X, \mathcal{O}_X)_s$. This injection restricts to the subspace $1 \otimes V^i$; hence in degree 0 we have an injection

$$V^i \xrightarrow{\alpha} H^i(\hat{P}^n, \hat{\mathcal{O}}_{\mathbb{P}^n}) \subset M^i.$$

Finally, composing α with the cup product

$$H^i(\mathbb{P}^n, F) \otimes_{\underline{k}} H^j(\hat{P}^n, \hat{\mathcal{O}}_{\mathbb{P}^n}) \to H^{i+j}(\hat{\mathbb{P}}^n, \hat{F}),$$

we obtain β^k, and this establishes (1.1).

REMARK: Unfortunately for us, [2] only treats the case $F = \mathcal{O}_{\mathbb{P}^n}(\nu)$; we shall need general coherent F, however, in order to prove (2.1) below.

§2. Vanishing Criteria

We shall be concerned for the rest of this section with the following situation: X and Y will be *connected* closed subschemes of $\mathbb{P}^n$, with $s = \dim(X)$ and $t = \dim(Y)$. We shall assume X is smooth and that Y is a local complete intersection if the characteristic is zero, or, if the characteristic is >0, that Y is Cohen-Macaulay. We want bounds on the cohomological dimension $\mathrm{cd}(X\text{-}Y)$.

By Lichtenbaum's Theorem (Cf. [7] or [2, Cor. (3.3), p. 98]), $\mathrm{cd}(X - Y) < \dim(X) = s$ if and only if $Y \cap X = \emptyset$; for lower cohomological dimensions the situation is more complicated.

Here is our main result:

THEOREM (2.1): *With closed subschemes X and Y of $\mathbb{P}^n$ as above, suppose $s + t > n$. Then we have*

$$\mathrm{cd}(X - Y) < s - 1;$$

in particular, $X \cap Y$ is connected.

COROLLARY (2.2) (A weak form of Hartshorne's Second Vanishing Theorem [3, Theorem 7.5, p. 444]): *Let $Y \subset \mathbb{P}^n$ be a positive dimensional closed subscheme satisfying the hypotheses of* (2.3). *Then*

$$\mathrm{cd}(\mathbb{P}^n - Y) < n - 1.$$

COROLLARY (2.3) (Compare [10, Theorem D, p. 179]): *Let $X \subset \mathbb{P}^n$ be a smooth hypersurface, $Y \subset X$ a closed subscheme as in* (2.3), *of dimension $t > 1$. Then*

$$\mathrm{cd}(X - Y) < n - 2.$$

The corollaries are immediate consequences of (2.1). To prove (2.1), we shall need the following result.

LEMMA (2.4) (Hartshorne [2, Theorem 3.4, p. 96]): *Let X be a smooth projective variety, $Y \subset X$ a closed subset. The following are equivalent:*

(a) $\mathrm{cd}(X - Y) \leq a$
(b) *the natural map*

$$\alpha^i: H^i(X, F) \to H^i(\hat{H}, \hat{F})$$

is bijective for $i < \dim(X)\text{-}a\text{-}1$ *and injective for* $i = \dim(X)\text{-}a\text{-}1$, *for all locally free sheaves* F *on* X.

PROOF OF (2.1): Let F be a locally free sheaf on X. Hence $\mathrm{hd}(F) = \mathrm{hd}(\mathcal{O}_X) = n - s$. Denote by "^" the operation of formal completion along Y. Applying (1.1) to the t-dimensional subscheme $Y \subset \mathbb{P}^n$, we find that the natural maps

$$\beta^k: \bigoplus_{i+j=k} H^i(\mathbb{P}^n, F) \bigotimes_{\underline{k}} V^j \to H^k(\hat{\mathbb{P}}^n, \hat{F})$$

are bijective for $k = 0$ and injective for $k = 1$. Hence, since Y is connected, $V^0 = \underline{k}$. Thus the bijection β^0 reduces to the natural map

$$\alpha^0: H^0(X, F) \to H^0(\hat{X}, \hat{F});$$

indeed, X is the support of F, so we can replace $\hat{\mathbb{P}}^n$ with $\hat{X}$. Similarly the injection β^1 induces

$$\alpha^1: H^1(X, F) \to H^1(\hat{X}, \hat{F})$$

on the summand corresponding to $i = 0$, $j = 1$. Since α^0 is bijective and α^1 is injective, our assertion about $\mathrm{cd}(X - Y)$ follows from (2.4). Finally, to see that $X \cap Y$ is connected, one applies [2, Cor. 3.9, p. 101].

EXAMPLE: Let $X = \mathbb{P}^m \times \mathbb{P}^1$, $Y = \mathbb{P}^m \times \{P\}$ for a closed point $P \in \mathbb{P}^1$. By [12, (1.3)], $\mathrm{cd}(\mathbb{P}^m \times \mathbb{P}^1 - Y) = m$. (Since $Y \neq \emptyset$, we can't have $\mathrm{cd} = m + 1$, by Lichtenbaum's Theorem.) We therefore obtain the bound

$$n \geq 2m + 1$$

for any $\mathbb{P}^n$ containing $\mathbb{P}^m \times \mathbb{P}^1$. Now any smooth projective variety of dimension $m + 1$ can be projected isomorphically into $\mathbb{P}^{2(m+1)+1} = \mathbb{P}^{2m+3}$; Hartshorne, however, shows [4, pp. 1025–1026] that $\mathbb{P}^m \times \mathbb{P}^1$ can be projected isomorphically into $\mathbb{P}^{2m+1}$, two dimensions less. In other words, the inequality of (2.1) actually gives the embedding dimension of $\mathbb{P}^m \times \mathbb{P}^1$, hence is best possible.

REMARK: Related techniques (compare [5], [8], [9]) give other, perhaps better known, results. For example, with $F = \mathcal{O}_X(1)$ in (1.1), we find

$$V^j = \begin{cases} \underline{k} \text{ if } j = 0 \\ 0 \text{ if } 0 < j \leq 2 \dim(X) - n. \end{cases}$$

Then a straightforward inspection of the β^k, with F locally free on X, gives the inequality

$$\text{(2.5)} \qquad \operatorname{cd}(X - Y) < n + s - t - \inf(s, t).$$

With $s \leq t$, we obtain $\operatorname{cd}(X - Y) < n - t$. On the other hand, with $t \leq s$ (e.g., if $Y \subset X$), we find $\operatorname{cd}(X - Y) < n + s - 2t$.

Taking $X = \mathbb{P}^n$, the last inequality gives

$$\operatorname{cd}(\mathbb{P}^n - Y) < 2n - 2t,$$

a result due originally to Barth [1, §7, Cor. of Th. III] in the complex case.

§3. The Main Result for Pic(X)

THEOREM (3.1): *Let $X \subset \mathbb{P}^n$ be a smooth closed subscheme of dimension s, over the spectrum of an algebraically closed field of characteristic $p > 0$. If $s \geq \frac{1}{2}(n + 2)$, then* Pic($X$) *is a finitely generated abelian group of rank* 1, *with no torsion prime to p.*

PROOF: $\operatorname{Pic}(\mathbb{P}^n) \to \operatorname{Pic}(X)$ *is injective*; if not, $\mathcal{O}_{\mathbb{P}^n}(d)|X$ would be trivial for some $d > 0$, but $\mathcal{O}_{\mathbb{P}^n}(d)|X$ is very ample. By [5, Cor. 4.6, p. 74], Pic(X) has no torsion prime to p, so $\underline{\operatorname{Pic}}^0(X)_{\text{red}}$, an abelian variety, is trivial. Hence

$$\operatorname{Pic}(X) = \underline{\operatorname{Pic}}(X)/\underline{\operatorname{Pic}}^0(X) = NS(X),$$

a finitely generated abelian group, by the Theorem of the Base.

For the rest of the proof, let l be a prime different from p, and consider the l-adic *étale* cohomology of X. Then, by [4, Remark 2, p.

1021], the natural maps

$$H^i(\mathbb{P}^n_{\underline{\text{ét}}}, \mathbb{Q}_l) \xrightarrow{\gamma^i} H^i(X_{\underline{\text{ét}}}, \mathbb{Q}_l)$$

are bijective for $i \leq 2s - n$. In particular, γ^2 is bijective. Recall that, as functors, we have

$$H^i(*_{\underline{\text{ét}}}, \mathbb{Q}_l) = H^i(*_{\underline{\text{ét}}}, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l,$$

where

$$H^i(*_{\underline{\text{ét}}}, \mathbb{Z}_l) = \varinjlim_r H^i(*_{\underline{\text{ét}}}, \mathbb{Z}/l^r\mathbb{Z}).$$

Hence, bijectivity of γ^2 implies that $H^2(X_{\underline{\text{ét}}}, \mathbb{Z}_l)$ has rank 1.

Since the base field is algebraically closed, we can make a non-canonical identification $\mu_{l^r} = \mathbb{Z}/l^r\mathbb{Z}$. Then the Kummer sequence reads

$$0 \to \mathbb{Z}/l\mathbb{Z} \to \mathbb{G}_m \xrightarrow{l^r} \mathbb{G}_m \to 1.$$

Passing to cohomology, we obtain the exact sequence

$$H^1(X_{\underline{\text{ét}}}, \mathbb{G}_m) \xrightarrow{l^r} H^1(X_{\underline{\text{ét}}}, \mathbb{G}_m) \to H^2(X_{\underline{\text{ét}}}, \mathbb{Z}/l^r\mathbb{Z}).$$

Since $H^1(X_{\underline{\text{ét}}}, \mathbb{G}_m) = \text{Pic}(X)$ we have a natural inclusion

$$\text{Pic}(X)/l^r\,\text{Pic}(X) \hookrightarrow H^2(X_{\underline{\text{ét}}}, \mathbb{Z}/l^r\mathbb{Z})$$

compatible with the reduction maps on both sides are r varies. Letting $r \to \infty$, we find

$$\text{rank}(\text{Pix}(X)) \leq \text{rank}(H^2(X_{\underline{\text{ét}}}, \mathbb{Z}_l)) = 1,$$

and this completes the proof.

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