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ON A CHARACTERIZATION OF AN ABELIAN VARIETY IN THE CLASSIFICATION THEORY OF ALGEBRAIC VARIETIES

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In this paper we shall prove the following theorem which was conjectured by S. Iitaka (see p. 131 in [1]) and proven by K. Ueno for \( n = 3 \) [2]. In this paper everything is defined over the complex number field \( \mathbb{C} \).

**Main Theorem:** Let \( X \) be an algebraic variety and let \( f : X \to A \) be a dominant generically finite morphism to an abelian variety. If the Kodaira dimension \( \kappa(X) = 0 \), then \( f \) is birationally equivalent to an étale morphism and \( X \) is birationally equivalent to an abelian variety.

To prove the main theorem we shall reduce it to the following theorem 1.

**Theorem 1:** Let \( A \) be abelian variety of dimension \( n \), let \( X \) be a reduced irreducible divisor on \( A \) and let \( \tilde{X} \) be a resolution of \( X \). If \( X \) is an algebraic variety of general type, then \( q_k(\tilde{X}) \equiv \dim H^0(\tilde{X}, \Omega^k_{\tilde{X}}) \geq \binom{n}{k} \), for \( k = 1, \ldots, n - 1 \). Moreover, if \( p_g(\tilde{X}) = q_{n-1}(\tilde{X}) = n \), then \( q_k(\tilde{X}) = \binom{n}{k} \), for \( k = 1, \ldots, n - 2 \), and in particular \( |X(O_{\tilde{X}})| = 1 \).

The following lemma is just a special case of a theorem of Ueno (3.3 of [2]).
Lemma 2: Let the notations and assumptions be as in the Main Theorem. Then \( \dim(H^0(X, \Omega_X^{n-1})) \leq n \).

Proof: We want to show that \( f^*(dz_1 \wedge \cdots \wedge dz_n) = 0 \) are generators of \( H^0(X, \Omega_X^{n-1}) \), for a global coordinate system \((z_1, \ldots, z_n)\) of \( A \). Take \( \omega \in H^0(X, \Omega_X^{n-1}) \) and \( a_i \in \mathbb{C} \), such that \( \omega \wedge f^*(dz_i) = a_i \cdot f^*(dz_1 \wedge \cdots \wedge dz_n) \). This is always possible, since \( H^0(X, \Omega_X^{n-1}) \) is generated by \( f^*(dz_1 \wedge \cdots \wedge dz_n) \). Replacing \( \omega \) by \( \omega - \sum_{i=1}^{n} (-1)^{n-1} a_i \omega_i \) we may assume that \( a_i = 0 \) for \( i = 1, \ldots, n \). Choose a small open subset \( U \subseteq X \), such that \( f|_U \) is an embedding. \((z_1, \ldots, z_n)\) is a local coordinate system of \( U \). Since \( \omega \wedge dz_i = 0 \) for \( i = 1, \ldots, n \), \( \omega \) must be 0 on \( U \) and hence on \( X \).

Lemma 3: Let the notations and assumptions be as in the main theorem. Let \( f_0: X_0 \to A \) be the normalisation of \( A \) in \( C(X) \). Let \( D_1, \ldots, D_m \) be the irreducible components of the discriminant \( \Delta(X_0/A) \) and let \( \tilde{D}_1, \ldots, \tilde{D}_m \) be their desingularisations. Then

\[
\sum_{i=1}^{m} p_j(\tilde{D}_i) \leq \dim(A).
\]

Proof: Choose \( \Delta_i \) to be one irreducible component of \( f^{-1}(D_i) \), such that \( \Delta_i \) is ramified over \( A \). We may assume, that \( X \) is projective and that \( \Delta_1 \cup \cdots \cup \Delta_m \) is a regular subvariety of \( X \). Let \( \omega_X = \Omega_X^X \). Then \( \omega_X \otimes O_X \left( \bigoplus_{i=1}^{m} \Delta_i \right) \subseteq \omega_X^X \) and, since \( H^0(X, \omega_X) = H^0(X, \omega_X^X) = \mathbb{C} \), we know that \( \bigoplus_{i=1}^{m} H^0(\Delta_i, \omega_{\Delta_i}) \) is a subspace of \( H^1(X, \omega_X) = H^{n-1}(X, O_X) = H^0(X, \Omega_X^{n-1}) \). However, \( H^0(\tilde{D}_i, \omega_{\tilde{D}_i}) \subseteq H^0(\Delta_i, \omega_{\Delta_i}) \).

Now we recall the following Theorem of Ueno (p. 120 in [1]):

Theorem 4: Let \( B \) be a subvariety of an abelian variety \( A \). Then there exist an abelian subvariety \( A_1 \) of \( A \) and an algebraic variety \( W \) which is a subvariety of an abelian variety such that
(1) \( B \) is an analytic fibre bundle over \( W \) whose fibre is \( A_1 \),
(2) \( \kappa(W) = \dim W = \kappa(B) \).

\( A_1 \) is characterized as the maximal connected subgroup of \( A \) such that \( A_1 + B \subseteq B \).

Proof of "Theorem 1 \( \Rightarrow \) Main Theorem": Let \( \eta: A' \to A \) be any
étale covering and $X_\eta = X \times_A A'$. Then $X_\eta \to A'$ also satisfies the conditions of the main theorem. Let $X_{\eta,0}$ be the normalisation of $A'$ in $C(X_\eta)$. Then $\Delta(X_{\eta,0}/A')$ is the pullback of $\Delta(X_0/A)$ by $\eta$. Suppose $\Delta(X_0/A)$ is not empty. Any abelian variety has étale coverings of arbitrary high degree (for example "multiplication with $r \gg 0$"). Every subvariety of an abelian variety has $p_g > 0$. Hence, replacing $A$ by some étale covering we may assume, that for every étale covering $\eta : A' \to A$ the number of irreducible components of $\Delta(X_0/A)$ and $\Delta(X_{\eta,0}/A')$ is the same (Lemma 3).

Put $B = \{ x \in A ; x + \Delta(X_0/A) \subseteq \Delta(X_0/A) \}^0$. Again replacing $A$ by an étale covering, we may assume that $A = B' \times B$. Let $Y_0$ be the Stein factorisation of $X_0 \to A \to B'$ and $Y$ any desingularisation of $Y_0$. Since $X_0$ is a finite covering of $Y_0 \times B$ we have $\kappa(X) = 0 \geq \kappa(Y) + \kappa(B) \geq 0$ and $\kappa(Y) = 0$.

Assume that $\Delta(Y_0/B') = \emptyset$. Since $\Delta(X_0/A) = \Delta' \times B$ for some positive divisor $\Delta' \subseteq B'$, the ramification divisor of $X_0 \to Y_0 \times B$ must be a (rational) multiple of the pullback of some divisor $\Delta$ of $Y_0$. Then $\kappa(X) \geq \kappa(Y_0, O(\Delta)) > 0$, in contradiction to our assumptions. Therefore $\Delta(Y_0/B') \neq \emptyset$ and, repeating this step if necessary, we may assume $B = 0$.

Let $B_i = \{ x \in A ; x + D_i \subseteq D_i \}^0$ for $i = 1, \ldots, m$. We have $\cap B_i = 0$. By Theorem 4 each $D_i$ is a fibre bundle over a certain $E_i$ with fibre $B_i$, for $i = 1, \ldots, m$. We have $p_g(\bar{D}_i) \geq p_g(\bar{E}_i)$ for a desingularisation $\bar{E}_i$ of $E_i$ and by Theorem 1 $p_g(\bar{E}_i) \geq \text{codim}_A B_i$. Since the equalities must be true by Lemma 3, we have $|X(O_{\bar{E}_i})| = 1$ by theorem 1, for $i = 1, \ldots, m$.

Let $r$ be a natural number, with $r \geq 2$, and let $r : A \to A$ be the multiplication with $r$. Using the notation introduced above, $\Delta(X_{\eta,0}/A)$ must have components $D_{ir}, i = 1, \ldots, m$ such that the corresponding base space $E_{r,i}$ satisfies $|X(O_{\bar{E}_{r,i}})| = \text{degree}(r) \cdot |X(O_{\bar{E}_i})| \geq 2$. This is a contradiction.

PROOF OF THEOREM 1: Let $\{x_1, \ldots, x_n\}$ be a global coordinate system on $A$ such that the set $\{dx_1, \ldots, dx_n\}$ gives a basis of 1-forms on $A$. Let $\alpha : \bar{X} \to A$ be the canonical map and let $\omega_i = \alpha^*(dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n)$ for $i = 1, \ldots, n$. We shall prove first that these are linearly independent $(n - 1)$-forms on $\bar{X}$. Suppose the contrary: $\sum_{i=1}^n a_i \omega_i = 0$ for $a_i \in \mathbb{C}$. Pick a smooth point $p$ on $X$. 2
Suppose $X$ is defined in $A$ near $p$ by an equation $x_n = F(x_1, \ldots, x_{n-1})$, where $F$ is a certain holomorphic function. Then $\omega_i = (-1)^{n-i-1} \frac{\partial F}{\partial x_i} \omega_n$ for $i = 1, \ldots, n - 1$. Therefore, $\sum_{i=1}^{n-1} (-1)^{n-i-1} a_i \frac{\partial F}{\partial x_i} + a_n = 0$, which means that there is a non-zero subgroup $B$ of $A$ such that $B + X \subseteq X$, which is a contradiction.

Put $\omega_I = \alpha_*(dx_{i_1} \wedge \cdots \wedge dx_{i_k})$ for each set $I$ of $k$-distinct integers $1 \leq i_1 < \cdots < i_k \leq n$. Since $\omega_I$ are linearly independent, $\{\omega_I\}_I$ gives a linearly independent system of $k$-forms on $\tilde{X}$. Thus, $q_k(\tilde{X}) \geq \binom{n}{k}$.

Before we prove the second part of theorem 1, we shall prove the following theorem, due to the first author.

**Theorem 5:** Let $A$ and $X$ be as in theorem 1. Let $f : X \to \mathbb{P}^{n-1}$ be the rational map defined by the system $\{\omega_1, \ldots, \omega_n\}$. If $X$ is an algebraic variety of general type, then $f$ is dominant.

**Proof:** Assume the contrary. Let $Y$ be the image variety of $f$ and let $q$ be a smooth point of $Y$ such that $f^{-1}(q)$ is also smooth near some smooth point $p \in f^{-1}(q)$ of $X$. Our assumption means that $\dim f^{-1}(q) \geq 1$. Consider everything in the universal cover $C^n$ of $A$. Let $H$ be the tangent plane of $X$ at $p$, which we assume is defined by an equation $x_n = 0$. Then, $X$ is defined near $p$ by an equation $x_n = F(x_1, \ldots, x_{n-1})$, where $\{x_1, \ldots, x_n\}$ is a global coordinate system centered at $p$ and $\frac{\partial F}{\partial x_i}(0) = 0$ for $i = 1, \ldots, n - 1$. $f^{-1}(q)$ is defined near $p$ by the equations $\frac{\partial F}{\partial x_i} = 0$ for $i = 1, \ldots, n - 1$. $Y$ is contained near $q$ in a smooth divisor $D$ of $\mathbb{P}^{n-1}$ (near $q$). After a suitable linear transformation of $x_1, \ldots, x_{n-1}$, the equation of $D$ can be written as $\frac{\partial F}{\partial x_i} = G(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_{n-1}})$, where $G$ is a holomorphic function of degree $\geq 2$.

By the rule of derivation of products, we have on $f^{-1}(q) \frac{\partial}{\partial x_1} \left( \frac{\partial F}{\partial x_i} \right) = \frac{\partial G}{\partial x_i} = 0$ for $i = 1, \ldots, n - 1$. Thus, $f^{-1}(q)$ is invariant under translations in the direction of $x_1$ and hence contains a translation of an abelian subvariety of $A$ generated by the line $x_2 = \cdots = x_n = 0$. Let $B$ be the maximal abelian subvariety of $A$ such that $p + B$ is contained in $X$. We have proved that $B \neq 0$. Since there are only countably many
abelian subvarieties, \( B \) does not depend on \( p \). Thus, \( B + X \subseteq X \), a contradiction. 

**Proof of Theorem 1 continued:** Suppose \( p, (\tilde{X}) = n \). Let \( p \) be a smooth point of \( X \) and let \( x_1, \ldots, x_n \) be as in the proof of theorem 5. Let \( \omega \) be an arbitrary \( k \)-form on \( \tilde{X} \). Write near \( p \) \( \omega = \sum_{\alpha \in I} g_\alpha(x_1, \ldots, x_{n-1}) \omega_\alpha \). Put \( I^c = \{1, \ldots, n-1\} - I \). Then \( \omega \wedge \omega_{I^c} = \epsilon(I, I^c) g_\alpha \omega_\alpha \), where \( \epsilon \) is the sign of permutations. Therefore, we have

\[
g_\alpha = g_\alpha(0) + \sum_{i=1}^{n-1} a_{i\alpha} \frac{\partial F}{\partial x_i}
\]

for some \( a_{i\alpha} \in \mathbb{C} \). Let \( J \) be a subset of \( \{1, \ldots, n-1\} \) such that \( \text{Card } J = n - k - 2 \). Since

\[
\omega \wedge \omega_f \wedge dx_n = \sum_{\{i\} \subseteq J^c} \epsilon(I, J, i) g_i \frac{\partial F}{\partial x_i} \omega_n,
\]

we have

\[
\sum_{\{i\} \subseteq J^c} \epsilon(I, J, i) \left( g_i(0) + \sum_{j=1}^{n-1} a_{j\alpha} \frac{\partial F}{\partial x_j} \right) \frac{\partial F}{\partial x_i} = \sum_{i=1}^{n-1} b_{i\alpha} \frac{\partial F}{\partial x_i}
\]

for some \( b_{i\alpha} \in \mathbb{C} \). Since \( \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_{n-1}} \) are algebraically independent,

we can compare the coefficients and we get (1) \( a_{i\alpha} = 0 \) for \( j \in J \), (2) \( a_{i\alpha} = 0 \) for \( I \cup \{i\} = J^c \), and (3) for each \( K = \{i_1, \ldots, i_{k-1}\} \) such that \( i_1 < \cdots < i_{k-1} \) and \( K \cup \{i\} \cup \{j\} = J^c \),

\[
e(K \cup \{i\}, J, j) a_{K \cup \{i\}, j} + \epsilon(K \cup \{j\}, J, i) a_{K \cup \{j\}, i} = 0,
\]

that is,

\[
e(K, i) a_{K \cup \{i\}, i} = \epsilon(K, j) a_{K \cup \{j\}, j}.
\]

Put \( a_K = \epsilon(K, i) a_{K \cup \{i\}, i} \). Then, \( \omega = \sum_{\alpha \in I} q_\alpha(0) \omega_\alpha + \sum_K a_K \omega_K \wedge dx_n \). 

Q.E.D.

**References**


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