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# $p$-LOCAL UNCONDITIONAL STRUCTURE OF BANACH SPACES*

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## Abstract

There are Banach spaces which fail to have $p$-local unconditional structure ($p$-l.u.st.) for any $p$, $\infty > p > 0$. In particular, there exist $n$-dimensional Banach spaces $E_n$, $n = 1, 2, \ldots$, whose $p$-l.u.st. constants are "almost" the largest possible theoretical value $\min\{n^{1/2}, n^p\}$. The $p$-l.u.st. constant is smaller and not equivalent to the usual l.u.st. constant.

## 1. Introduction

Given any $\infty > p \geq 0$, let $\eta_p$ be the ideal norm defined in the following manner: If $T \in L(E, F)$ is a bounded operator from a Banach space $E$ to a Banach space $F$ which can be written the form $Tx = \Sigma_{i\geq 1} A_i x$ ($x \in E$), where $A_i$ ($i = 1, 2, \ldots$) are in the class $\mathscr{F}(E, F)$ of the finite-rank operators from $E$ to $F$, then

$$\eta_p(T) = \inf \sup \left\| \sum_{i\leq N} \pm (r(A_i))^p A_i \right\|$$

where $r(A)$ denotes the rank of an operator $A$, the supremum ranges over all choices of $\pm$ signs and integers $N$, and the infimum is taken over all the possible representations of the operator $T$.

$\eta_p(T)$ is a non-decreasing function of $p$, and $\eta_p$ is a Banach ideal norm, that is has the following properties:

(1) $\eta_p$ is a norm and $\eta_p(E, F) = \{T \in L(E, F); \eta_p(T) < \infty\}$ is a Banach space under the norm $\eta_p$.

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(2) $\eta_p(T) = \|T\|$ whenever $r(T) = 1$.

(3) If $u \in L(G, E)$, $T \in \eta_p(E, F)$, $v \in L(F, H)$, then $vTu \in \eta_p(G, H)$ and $\eta_p(vTu) \leq \|v\|\|u\|\eta_p(T)$.

We recall some well known facts about a general Banach ideal norm $\alpha$ which may be found in [9]. If $T \in L(E, F)$, the adjoint ideal norm $\alpha^*(T)$ is defined as the least $C$ such that the inequality

$$\operatorname{trace}(SvTu) \leq C\|v\|\|u\|\alpha(S)$$

holds for any finite-dimensional normed spaces $X$ and $Y$, $u \in L(X, E)$, $v \in L(F, Y)$ and $S \in L(Y, X)$. If $X$ and $Y$ are finite-dimensional normed spaces, the dual space $(\alpha(X, Y))'$ can be naturally identified with $\alpha^*(Y, X) = (L(Y, X), \alpha^*)$ via the identity $\langle T, S\rangle = \operatorname{trace}(ST)$ for $T \in \alpha(X, Y)$, $S \in \alpha^*(Y, X)$. Hence, if $T \in L(E, F)$, $\alpha^{**}(T) = \sup \alpha(vTu)$, where the supremum ranges over all finite-dimensional normed spaces $X$ and $Y$, $u \in L(X, E)$ and $v \in L(F, Y)$ with $\|u\| = \|v\| = 1$. From this we get immediately that $\eta_p^{**}(T) = \eta_p^{**}(T')$ for every operator $T \in L(E, F)$.

If in the definition of $\eta_p(T)$, $T$ is further restricted only to representations for which $r(A_i) = 1$ for all $i$, then the corresponding resulting norm which is independent of $p$ was called in [8] the weakly nuclear norm of $T$ and denoted by $\eta(T)$. It follows that $\|\cdot\| \leq \eta_p \leq \eta_q \leq \eta$ for $0 \leq p < q < \infty$, and since $\eta_0 = \|\cdot\|$ on finite dimensional spaces, taking double adjoints we obtain $\eta_0^{**} = \|\cdot\|^{**} = \|\cdot\| \leq \eta_p^{**} \leq \eta_q^{**} \leq \eta^{**}$.

Using ultraproducts it can be shown (see for example [16]) that $T \in \eta^{**}(E, F)$ if and only if $j_F T$ factors through some Banach lattice, more precisely, $\eta^{**}(T) = \inf\|v\|\|u\|$, where the infimum ranges over all Banach lattices $L$ and $u \in L(E, L)$, $v \in L(L, F'')$, satisfying $j_F T = vu$, where $j_F : F \to F''$ is the canonical inclusion. Thus, if $T$ is a map on, or, into, a norm one complemented subspace of a Banach lattice, then $\|T\| = \eta_p^{**}(T) = \eta^{**}(T)$. If $T = I_E$ the identity operator on a Banach space $E$, $\eta^{**}(I_E)$ is generally better known as the local unconditional structure (l.u.st.) constant of $E$ which is usually denoted by $x_u(E)$ [7]. If $\alpha$ is an ideal norm, $\alpha(E)$ denotes $\alpha(I_E)$. For $p > 0$, $\eta_p^{**}(E)$ will be called the $p$-l.u.st. constant of $E$. $x(E)$ will denote the unconditional basis constant of $E$.

If $\dim(E) = n$ and $0 \leq p < q < \infty$, then trivially we get from the definitions

$$1 \leq \eta_p(E) \leq \eta_q(E) \leq \eta(E) = x_u(E) \leq x(E) \leq d(E, \ell_2^n) \leq \sqrt{n},$$

and also, since $r(A_i) \leq n$, $\eta_q(E) \leq n^{q-p}\eta_p(E)$. Moreover, the representation $I_E = I_E$ shows that $\eta_p(E) \leq n^p$, thus $\eta_p(E) \leq \min\{n^{1/2}, n^p\}$ for all $p \geq 0$.

The main result here shows that the last inequality is asymptotically "almost" the best possible. There exists a sequence $E_n$, $n = 1, 2, \ldots$, of $n$-dimensional spaces for which $\eta_p(E_n) \geq \min\{an^{1/2}, an^p\} \exp(-\sqrt{\log n})$ where $a$ is an absolute positive constant. Since the exponential factor tends to zero more slowly than any negative power of $n$, this implies that if $p \neq q$ and $0 < p < \frac{1}{2}$, then $\eta_p$ and $\eta_q$ are not equivalent ideal norms, and in particular $\eta_p$ and $\eta$ are not equivalent ideal norms. Since $\eta_p(E_n) \to \infty$ as $n \to \infty$, this also implies there exists a reflexive separable Banach space which fails to have $p$-l.u.st. for all $p > 0$.

Regarding $\chi_u$ it was proved in [3] that there is an absolute constant $c > 0$ and a sequence of spaces $F_n$, $\dim(F_n) = n$, such that $\chi_u(F_n) \geq c\sqrt{n}$. Our result therefore is of interest for the smaller $p$-l.u.st. constants $\eta_p$. We do not know if $\eta_p(E)$ can be asymptotically equivalent to $\min\{n^{1/2}, n^p\}$ for a sequence of spaces $E_n$, $\dim(E_n) = n$. It is also an open question whether for $q > p \geq \frac{1}{2}$ $n_p(E)$ and $\eta_q(E)$ are always equivalent when $\dim(E) < \infty$; the same question is also open for the constants $\chi_u(E)$ and $\chi(E)$. It was proved recently by Johnson, Lindenstrauss and Schechtman, that there exists a Banach spaces $E$ with $\chi_u(E) = \infty$, that is $E$ does not have local unconditional structure, yet $E$ has an unconditional Schauder decomposition into 2-dimensional spaces. This fact implies that $\chi_u(E)$ and $\eta_p(E)$ are not equivalent since $\eta_p(E)$ is finite for such spaces. Also unknown is whether many of the spaces which fail l.u.st. also fail $p$-l.u.st. for some $p > 0$. Does $L_q(\infty > q \geq 1)$ have a subspace without $p$-l.u.st.? G. Pisier proved that if $p > 2$, $L_q$ has a subspace without l.u.st. (See [15] for $q > 4$; for $2 < q$ we know of an unpublished proof).

To obtain the lower estimates for $\eta_p(E_n)$ we use the characterization of the adjoint norm $\eta_p^*$ proved in the next section and an inequality due to S. Chevet which was communicated to us by G. Pisier who has used the inequality to prove that l.u.st. constant $\chi_u$ of

$$\underbrace{\ell_1^n \hat{\otimes}_\epsilon \ell_1^n \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon \ell_1^n}_{2^k}$$ is bigger than $C_\alpha N^\alpha$, where $N = n^{2^k}$ and $n, k$ are chosen in some appropriate relation to $N$, and where $\alpha$ is any scalar $< \frac{1}{2}$, and $c_\alpha > 0$ is a constant depending only on $\alpha$.

## 2. $p$-local unconditional structure

A characterization of $\eta_p^*$ is given by the following proposition.

PROPOSITION 1: *If $p, C$ are non-negative constants and $T \in L(E, F)$, then the following statements are equivalent*:

(1) $\eta_p^*(T) \leq C$.

(2) $\sum_{i=1}^n \operatorname{trace}(TA_i) \leq C \max_{\pm} \|\sum_{i=1}^n \pm (r(A_i))^p A_i\|$ *for any choice of* $\{A_i\}_{i=1}^n \subset \mathcal{F}(F, E)$.

(3) *If $K_{E'}$ denotes the $w^*$-closure of the extreme points of the unit ball of $E'$ equipped with the $w^*$ topology, there exists a probability measure on the compact topological product space $K = K_{E'} \times K_{F''}$ such that for every $A \in \mathcal{F}(F, E)$ holds the inequality*

$$\operatorname{trace}(TA) \leq C(r(A))^p \int_K |\langle A'(x'), y''\rangle| \,\mathrm{d}\mu(x', y'').$$

PROOF: Let $C_i$ $(i = 1, 2, 3)$ denote a constant $C$ which appears in the inequality of statement (i). Let $X$, $Y$ be finite-dimensional spaces and $\epsilon > 0$, and let $S = \sum_{i=1}^n A_i$ where $A_i \in L(Y, X)$ are chosen to satisfy $(1+\epsilon)\eta_p(S) \geq \max_{\pm}\|\sum \pm (r(A_i))^p A_i\|$. Then, for any $u \in L(X, E)$, $v \in L(F, Y)$, we get

$$\begin{aligned}\operatorname{trace}(SvTu) &= \sum_{i=1}^n \operatorname{trace}(A_i vTu) = \sum_{i=1}^n \operatorname{trace}(uA_i vT)\\ &\leq C_2 \max_{\pm}\left\|\sum_{i=1}^n \pm (r(uA_iv))^p uA_iv\right\|\\ &\leq C_2\|u\|\|v\| \max_{\pm}\left\|\sum_{i=1}^n \pm (r(A_i))^p A_i\right\|\\ &\leq C_2\|u\|\|v\|(1+\epsilon)\eta_p(S),\end{aligned}$$

this implies $\inf C_1 \leq C_2(1+\epsilon)$, therefore $\inf C_1 \leq \inf C_2$.

Given arbitrary $B_i \in \mathcal{F}(F, E)$, $i = 1, 2, \ldots, n$,

$$\begin{aligned}\sum_{i=1}^n \operatorname{trace}(TB_i) &\leq C_3 \int_K \sum_{i=1}^n (r(B_i))^p |\langle B_i'(x'), y''\rangle| \,\mathrm{d}\mu\\ &\leq C_3 \sup\left\{\sum_{i=1}^n (r(B_i))^p |\langle B_i'(x'), y''\rangle|; \|x'\| = \|y''\| = 1\right\}\\ &= C_3 \max_{\pm}\left\|\sum_{i=1}^n \pm (r(B_i))^p B_i\right\|,\end{aligned}$$

hence $\inf C_2 \leq C_3$.

If $A \in \mathcal{F}(F, E)$, let $\tilde{A} \in C(K)$ be the function defined by: $\tilde{A}(x', y'') = \langle A'(x'), y''\rangle (r(A))^p$, and denote by $M$ the convex hull of the set $\{C_2\tilde{A};\ A \in \mathcal{F}(F, E), \text{trace}(TA) = 1\}$. Statement (2) implies that $M$ is disjoint from the set $N = \{f \in C(K), f < 1\}$ which is also convex and contains the open unit ball of $C(K)$, therefore there exists a probability measure $\mu \in M(K) = (C(K))'$ such that $\mu(g) \geq 1$ for all $g \in M$, this shows that $\inf C_3 \leq C_2$.

Let now $\{A_i\}_{i=1}^n \subset \mathcal{F}(F, E)$, and consider the space $X = \text{span}\{A_i(y);\ y \in F,\ i = 1, 2, \ldots, n\}$. Let $u: X \to E$ be the inclusion map, and $S = \sum_{i=1}^n A_i$ be the map of $F$ into $X$, $S'$ maps $X'$ to $F'$ and $(S')_a$ will denote the map $S'$ of $X'$ onto $S'(X')$. Let $j$ be the inclusion of $S'(X')$ in $F'$, then $v = j'j_F$ maps $F$ to $Y = (S'(X'))'$. Both $X$ and $Y$ are now finite-dimensional spaces, and

$$\begin{aligned}\sum_{i=1}^n \text{trace}(TA_i) &= \text{trace}(vTu(S')_a')\\ &\leq C_1\|u\|\|v\|\eta_p((S')_a')\\ &\leq C_1 \max_{\pm}\left\|\sum_{i=1}^n \pm A_i(r(A_i))^p\right\|\end{aligned}$$

the last inequality is because $\|u\| = \|v\| = 1$ and the fact that if we denote by $\tilde{A}_i$ the operator $A_i''$ considered as a map of $Y$ to $X$, then

$(S')_a' = \sum_{i=1}^n \tilde{A}_i$ and so

$$\eta_p((S')_a') \leq \max_{\pm}\left\|\sum_{i=1}^n \pm (r(\tilde{A}_i))^p \tilde{A}_i\right\| = \max_{\pm}\left\|\sum_{i=1}^n \pm (r(A_i))^p A_i\right\|.$$

Therefore, $\inf C_2 \leq C_1$, and the proof is complete. □

We need a preliminary lemma which was used in [6].

Lemma 2: *If $x_i, y_i$, $i = 1, 2, \ldots, m$, are arbitrary vectors in $\ell_2^n$, then $n \sum_{i=1}^m \sum_{j=1}^m \langle x_i, y_j\rangle^2 \geq (\sum_{i=1}^m \langle x_i, y_i\rangle)^2$.*

Proof: Without loss of generality we can assume $\{y_i\}_{i=1}^m$ are fixed such that the operator $T = \sum_{i=1}^m y_i \otimes y_i$ has rank $n$. We shall maximize the function $f(x_1, x_2, \ldots, x_m) = \sum_{i=1}^m \langle x_i, y_i\rangle$ subject to the constraint $\sum_{i=1}^m \sum_{j=1}^m \langle x_i, y_j\rangle^2 = 1$. At the maximum point, the function

$$\varphi = \sum_{i=1}^m \langle x_i, y_i\rangle - \lambda\left(\sum_{i=1}^m \sum_{j=1}^m \langle x_i, y_j\rangle^2 - 1\right)$$

satisfies $\partial\varphi/\partial x_{ik}=0$, where $x_i=(x_{ik})_{k=1}^n$. This yields $y_i = 2\lambda\sum_{j=1}^m \langle x_i, y_j\rangle y_j = 2\lambda T(x_i)$, hence $f=\sum_{i=1}^m\langle x_i,y_i\rangle = 2\lambda\sum_{i=1}^m\sum_{j=1}^m\langle x_i,y_j\rangle^2 = 2\lambda$. $T=\sum_{i=1}^m y_i\otimes y_i = 2\lambda\sum_{i=1}^m y_i\otimes Tx_i$, therefore $I_{\ell_2^n}=2\lambda\sum_{i=1}^m x_i\otimes y_i$ and taking trace, $n=2\lambda\sum_{i=1}^m\langle x_i,y_i\rangle = 4\lambda^2$, so that $2\lambda=\sqrt{n}$. □

Given Banach spaces $E$ and $F$ let $E\hat{\otimes}_\epsilon F$ denote the completion of the tensor product space $E\otimes F$ under the $\epsilon$-norm, that is the ordinary norm induced on it as a subspace of $L(E',F)$. $E\otimes_\pi F$ denotes the completion of $E\otimes F$ under the $\pi$-norm, that is, on $E\otimes F$ the norm $|\cdot|_\pi$ is defined as

$$\left|\sum_{i=1}^n x_i\otimes y_i\right|_\pi = \sup\left\{\sum\langle y_i, Bx_i\rangle;\ \|B\|\le 1, B\in L(E,F')\right\}.$$

If $k$ is a positive integer, $E_\epsilon^k$ will denote the space $\underbrace{E\hat{\otimes}_\epsilon E\hat{\otimes}_\epsilon\cdots\hat{\otimes}_\epsilon E}_{k}$, and for $\{x_i\}_{i=1}^k\subset E$, $\vec{x}=\otimes_{i=1}^k x_i = x_1\otimes x_2\otimes\cdots\otimes x_k$ will be a $k$ tensor in $E_\epsilon^k$. If $u_i\in L(E,E)$ are isometries on $E$ $(i=1,2,\dots,k)$, let $\vec{u}=\otimes_{i=1}^k u_i = u_1\otimes u_2\otimes\cdots\otimes u_k$ be the isometry of $E_\epsilon^k$ defined by: $\vec{u}(\vec{x}) = \otimes_{i=1}^k u_i(x_i)$. This definition makes $\vec{u}$ also an isometry on $E_\pi^k = \underbrace{E\hat{\otimes}_\pi\otimes\cdots\hat{\otimes}_\pi E}_{k}$. We shall denote by $\pi_p$ $(1\le p<\infty)$ the $p$-absolutely summing ideal norm [14].

Lemma 3: *Let $E$ have a normalized symmetric basis $\{e_i\}_{i=1}^n$,* and let *$T:E\to\ell_2^n$ be the basis to basis map, $T(e_i)=(\underbrace{0,\dots,0,1}_{i},0,\dots,0)$, $i=1,2,\dots,n$. Let $A=\sum_{i_1=1}^n\dots\sum_{i_k=1}^n a_{i_1,i_2,\dots,i_k}e_{i_1}\otimes e_{i_2}\otimes\cdots\otimes e_{i_k}$ be any norm-one element in $E_\epsilon^k$, then*

$$\eta_{1/2}^*(E_\epsilon^k)\sum|a_{i_1,i_2,\dots,i_k}|\le(n\sqrt{2}\pi_1(T))^k.$$

Proof: By Pietsch [14] there exists a probability measure $\mu$ on $K_{E'}$ such that for every $x=\sum_{i=1}^n\xi_i e_i\in E$

$$\|Tx\|_2=\left(\sum_{i=1}^n\xi_i^2\right)^{1/2}\le(\pi_1(T)\int_{K_{E'}}|\langle x,x'\rangle|d\mu(x').$$

Let $\vec{d\mu}=\underbrace{d\mu\times d\mu\times\cdots\times d\mu}_{k}$ be the product measure on the set of extreme points of the unit ball of $(E_\epsilon^k)'=(E')_\pi^k$,

$$K_{(E')^k_\pi} = \left\{ \vec{x}' = \bigotimes_{i=1}^{k} x'_i;\ x'_i \in K_{E'} \right\}.$$

Let $u = \Sigma_{i=1}^m A_i \otimes B_i$ be any rank-$m$ operator in $L(E^k_\epsilon, E^k_\epsilon)$, where $A_i \in (E^k_\epsilon)'$ and $B_i \in E^k_\epsilon$. Suppose $A_i$ and $B_i$ have the representations

$$A_i = \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} a^{(i)}_{i_1,\ldots,i_k} e'_{i_1} \otimes \cdots \otimes e'_{i_k}, \quad \text{and}$$

$$B_i = \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} b^{(i)}_{i_1}, \ldots, {}_{i_k} e_{i_1} \otimes \cdots \otimes e_{i_k}.$$

Let $\epsilon^{(i)} = (\epsilon^{(i)}_j)^n_{j=1}$, $\epsilon^{(i)}_j = \pm 1$, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, n$, and $g_{\epsilon^{(i)}}$ be the isometry of $E$ defined by $g_{\epsilon^{(i)}}(e_j) = \epsilon^{(i)}_j e_j$. Let $\pi^{(i)}$ be any permutation of the integers $\{1, 2, \ldots, n\}$, and $g_{\pi^{(i)}}$ be the isometry of $E$ defined by $g_{\pi^{(i)}}(e_j) = e_{\pi^{(i)}(j)}$. Set $g_i = g_{\epsilon^{(i)}} g_{\pi^{(i)}}$, and let $\vec{g}_{\epsilon,\pi} = \otimes^k_{i=1} g_i$ be the isometry of $E^k_\epsilon$.

Denote by $Av_\epsilon$ and $Av_\pi$ the averages with respect to signs and permutations, that is, if $f(\epsilon^{(1)}, \ldots, \epsilon^{(k)})$ is a real function then

$$Av_\epsilon(f) = 2^{-nk} \sum_{\epsilon} f(\epsilon^{(1)}, \ldots, \epsilon^{(k)})$$

where the sum is taken over all possible distinct elements $(\epsilon^{(1)}, \epsilon^{(2)}, \ldots, \epsilon^{(k)})$; and similarly for a function $h(\pi^{(1)}, \ldots, \pi^{(k)})$

$$Av_\pi(h) = (n!)^{-k} \sum_{\pi} h(\pi^{(1)}, \ldots, \pi^{(k)})$$

where the sum ranges over all $(n!)^k$ possible choices of $(\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(k)})$.

We shall find a lower bound for the integral

$$I = Av_\epsilon Av_\pi \int_{K_{(E')^k_\pi}} |\langle u'(\vec{x}'), \vec{g}_{\epsilon,\pi}(A)\rangle| d\vec{\mu}(\vec{x}')$$

which will give the claim of the lemma. First observe that

$$|\langle u'(\vec{x}'), \vec{g}_{\epsilon,\pi}(A)\rangle| =$$
$$= \Bigg| \sum_{\ell_1,\ldots,\ell_k=1}^{n} \langle e_{\ell_1}, x'_1\rangle \ldots \langle e_{\ell_k}, x'_k\rangle \sum_{i_1,\ldots,i_k=1}^{n} \sum_{j_1,\ldots,j_k=1}^{n} \langle g_1(e_{i_1}), e'_{j_1}\rangle$$
$$\ldots \langle g_k(e_{i_k}), e'_{j_k}\rangle a_{i_1,\ldots,i_k} \sum_{i=1}^{m} a^{(i)}_{j_1}, \ldots, {}_{j_k} b^{(i)}_{\ell_1}, \ldots, {}_{\ell_k} \Bigg|.$$

Integrating with respect to $d\mu(x_1')$ first we get

$$\int_{K_{E'}} |\langle u'(\vec{x}'), \vec{g}_{\epsilon,\pi}(A)\rangle| d\mu(x_1') \geq (\pi_1(T)^{-1}) \left[\sum_{\ell_1=1}^{n} \left(\sum_1\right)^2\right]^{1/2}$$

where

$$\sum_1 = \sum_{\ell_2,\ldots,\ell_k=1}^{n} \langle e_{\ell_2}, x_2'\rangle \ldots \langle e_{\ell_k}, x_k'\rangle \sum_{i_1,\ldots,i_k=1}^{n} \sum_{j_1,\ldots,j_k=1}^{n} \langle g_1(e_{i_1}), e_{j_1}'\rangle$$
$$\ldots \langle g_k(e_{i_k}), e_{j_k}'\rangle a_{i_1,\ldots,i_k} \sum_{i=1}^{m} a_{j_1,\ldots,j_k}^{(i)} b_{\ell_1,\ldots,\ell_k}^{(i)}.$$

Next, integrating with respect to $d\mu(x_2')$ and using the fact, which we shall also use throughout, that

$$\int \left[\sum_{\ell_1=1}^{n} \left(\sum_1\right)^2\right]^{1/2} d\mu(x_2') \geq \left[\sum_{\ell_1=1}^{n} \left(\int \left|\sum_1\right| d\mu(x_2')\right)^2\right]^{1/2},$$

we obtain

$$\int_{K_{E'}}\int_{K_{E'}} |\langle u'(\vec{x}'), \vec{g}_{\epsilon,\pi}(A)\rangle| d\mu(x_1') d\mu(x_2') \geq (\pi_1(T))^{-2} \left[\sum_{\ell_1,\ell_2=1}^{n} \left(\sum_2\right)^2\right]^{1/2}$$

where $\Sigma_2 = \Sigma_{\ell_3,\ldots,\ell_k=1}^{n} \langle e_{\ell_3}, x_3'\rangle \ldots$ (the $\ldots$ represent the same terms which appear in $\Sigma_1$).

If we continue to integrate with respect to $d\mu(x_3')$ and so on, finishing with $d\mu(x_k')$ we obtain

$$\int_{K_{(E')_\pi^k}} |\langle u'(\vec{x}'), \vec{g}_{\epsilon,\pi}(A)\rangle| d\vec{\mu}(\vec{x}') \geq (\pi_1(T))^{-k} \left[\sum_{\ell_1,\ldots,\ell_k=1}^{n} \left(\sum_k\right)^2\right]^{1/2}$$

where

$$\sum_k = \sum_{i_1,\ldots,i_k=1}^{n} \sum_{j_1,\ldots,j_k=1}^{n} \langle g_1(e_{i_1}), e_{j_1}'\rangle \ldots \langle g_k(e_{i_k}), e_{j_k}'\rangle a_{i_1,\ldots,i_k}$$
$$\cdot \sum_{i=1}^{m} a_{j_1,\ldots,j_k}^{(i)} b_{\ell_1,\ldots,\ell_k}^{(i)}.$$

Let $\sigma^{(i)} = \pi^{(i)^{-1}}$, i.e. $\sigma^{(i)}(r) = j$ iff $\pi^{(i)}(j) = r$, then

$$\langle g_1(e_{i_1}), e_{j_1}'\rangle = \langle \epsilon_{j_1}^{(1)} e_{\pi^{(1)}(i_1)}, e_{j_1}'\rangle = \begin{cases} 0; & i \neq \sigma^{(1)}(j_1) \\ \epsilon_{j_1}^{(1)}; & i_1 = \sigma^{(1)}(j_1) \end{cases}.$$

Using Khintchine's inequality [17]

$$Av_{\epsilon^{(1)}}\left|\sum_{i_1j_1} c_{i_1,j_1}\epsilon^{(1)}_{j_1}\right| \geq 2^{-1/2}\left(\sum_{j_1=1}^{n}\left(\sum_{i_1=1}^{n} c_{i_1,j_1}\right)^2\right)^{1/2}$$

and averaging over all $\epsilon^{(1)}, \epsilon^{(2)}, \ldots, \epsilon^{(k)}$, we get

$$2^{k/2}Av_\epsilon\left|\sum_k\right| \geq \left[\sum_{j_1,\ldots,j_k=1}^{n}(a_{\sigma^{(1)}(j_1),\ldots,\sigma^{(k)}(j_k)})^2\left(\sum_{i=1}^{m} a^{(i)}_{j_1,\ldots,j_k}b^{(i)}_{\ell_1,\ldots,\ell_k}\right)^2\right]^{1/2}.$$

Hence,

$$Av_\epsilon\int_{K_{(E')^k_\pi}}|\langle u'(\vec{x}'), \vec{g}_{\epsilon,\pi}(A)\rangle|d\vec{\mu}(\vec{x}') \geq$$
$$\geq(\sqrt{2}\pi_1(T))^{-k}\left[\sum_{\ell_1,\ldots,\ell_k=1}^{n}\sum_{j_1,\ldots,j_k=1}^{n}(a_{\sigma^{(1)}(j_1),\ldots,\sigma^{(k)}(j_k)})^2\right.$$
$$\left.\times\left(\sum_{i=1}^{m} a^{(i)}_{j_1,\ldots,j_k}b^{(i)}_{\ell_1,\ldots,\ell_k}\right)^2\right]^{1/2}.$$

Now we shall average over all permutations, and use the fact that

$$Av_\pi|a_{\sigma^{(1)}(j_1),\ldots,\sigma^{(k)}(j_k)}| = n^{-k}\sum_{i_1,\ldots,i_k}^{n}|a_{i_1,\ldots,i_k}|,$$

this gives the following estimate for I

$$(n\sqrt{2}\pi_1(T))^k I \geq$$
$$\geq\sum_{i_1,\ldots,i_k=1}^{n}|a_{i_1,\ldots,i_k}|\left[\sum_{\ell_1,\ldots,\ell_k=1}^{n}\sum_{j_1,\ldots,j_k=1}^{n}\left(\sum_{i=1}^{m} a^{(i)}_{j_1,\ldots,j_k}\right.\right.$$
$$\left.\left.\times\, b^{(i)}_{\ell_1,\ldots,\ell_k}\right)^2\right]^{1/2} \geq \sum|a_{i_1,\ldots,i_k}|m^{-1/2}|\mathrm{trace}(u)|$$

the last inequality follows from Lemma 2. The proof is completed by applying Proposition 1 for $p=\frac{1}{2}$ while noting that $\vec{g}_{\epsilon,\pi}(A)$ are norm-one elements of $E^k_\epsilon$ and the $\vec{x}'$ which appear in I are norm-one elements of the dual space $(E')^k_\pi$. □

LEMMA 4: *With the notation of Lemma* 3,

$$(\sqrt{2}\pi_1(T))^k\eta_{1/2}(E^k_\epsilon) \geq \sum|a_{i_1,i_2,\ldots,i_k}|.$$

PROOF: This follows immediately from Lemma 3 and the obvious inequality $\alpha(F)\alpha^*(F) \geq \dim(F)$ for any ideal norm $\alpha$ and finite-dimensional space $F$. □

We shall next use the following inequality due to S. Chevet [1], for the sake of completeness we include the proof. If $\{x_i\}_{i=1}^n \subset E$, we shall denote by $\epsilon_2(\{x_i\}) = \sup\{(\Sigma_{i=1}^n |\langle x_i, x'\rangle|^2)^{1/2};\ x' \in E', \|x'\| = 1\}$.

LEMMA 5: *If $\{x_i\}_{i=1}^n$ and $\{y_j\}_{j=1}^n$ are elements in Banach spaces E and F respectively, and $g_{i,j}$ $(i, j = 1, 2, \ldots, n)$ is a sequence of equidistributed, independent, orthonormal random Gaussian variables, then*

(*)
$$\frac{\Lambda}{2} \leq \mathsf{E}\left(\left\|\sum_{i,j} g_{i,j} x_i \otimes y_j\right\|_{E\hat{\otimes}_\epsilon F}\right) \leq \sqrt{2}\Lambda$$

*where* $\Lambda = \epsilon_2(\{y_i\})\mathsf{E}(\|\Sigma_i g_{i,1} x_i\|) + \epsilon_2(\{x_i\})\mathsf{E}(\|\Sigma_j g_{j,1} y_j\|)$.

PROOF: Let $T = \{(\xi, \eta);\ \xi \in E', \eta \in F', \|\xi\| = \|\eta\| = 1\}$. For each $t = (\xi, \eta) \in T$, define the random variables

$$X_t = \sum_{i,j} g_{i,j}\langle x_i, \xi\rangle\langle y_j, \eta\rangle \quad \text{and}$$

$$Y_t = \alpha \sum_j g_{j,1}\langle y_j, \eta\rangle + \beta \sum_i g_{i,2}\langle x_i, \xi\rangle$$

where $\alpha = \epsilon_2(\{x_i\})$, $\beta = \epsilon_2(\{y_j\})$. It is easy to see that if $s = (\xi_1, \eta_1) \in T$, then

$$\begin{aligned}
\mathsf{E}(|X_t - X_s|^2) &= \sum_{i,j} (\langle x_i, \xi\rangle\langle y_j, \eta\rangle - \langle x_i, \xi_1\rangle\langle y_j, \eta_1\rangle)^2 \\
&= \sum_{i,j} (\langle x_i, \xi - \xi_1\rangle\langle y_j, \eta\rangle + \langle x_i, \xi_1\rangle\langle y_j, \eta - \eta_1\rangle)^2 \\
&\leq 2\sum_{i,j} (\langle x_i, \xi - \xi_1\rangle\langle y_j, \eta\rangle)^2 + (\langle x_i, \xi_1\rangle\langle y_j, \eta - \eta_1\rangle)^2 \\
&= 2\mathsf{E}(|Y_t - Y_s|^2)
\end{aligned}$$

hence $\mathsf{E}(|X_t - X_s|^2) \leq 2\mathsf{E}(|Y_t - Y_s|^2)$. By a result due to Sudakov ([2], Corollaire 2.1.3) this implies $\mathsf{E}(\vee_T X_t) \leq \sqrt{2}\mathsf{E}(\vee_T Y_t)$, from which the right hand side of (*) follows.

For the other side, pick $\xi_0 \in E'$, $\|\xi_0\| = 1$, such that $\alpha = \epsilon_2(\{x_i\})$, and define the random variables $Z_\eta = \Sigma_{i,j} g_{i,j}\langle x_i, \xi_0\rangle\langle y_j, \eta\rangle$ and $W_\eta = \alpha\ \Sigma_{j=1}^n g_{j,1}\langle y_j, \eta\rangle$. Then

$$\mathsf{E}(|Z_\eta - Z_{\eta_1}|^2) = \alpha^2 \sum_j \langle y_j, \eta - \eta_1\rangle^2 = \mathsf{E}(|W_\eta - W_{\eta_1}|^2),$$

so again $\mathsf{E}(\sup_{\|\eta\|=1} Z_\eta) = \mathsf{E}(\sup_{\|\eta\|=1} W_\eta)$, but

$$\begin{aligned}\mathsf{E}\left(\left\|\sum_{i,j} g_{i,j} x_i \otimes y_j\right\|_{E\hat{\otimes}_\epsilon F}\right) &\geq \mathsf{E}\left(\left\|\sum_{i,j} g_{i,j}\langle x_i, \xi_0\rangle y_j\right\|\right)\\ &= \mathsf{E}\left(\sup_{\|\eta\|=1} Z_\eta\right) = \mathsf{E}\left(\sup_{\|\eta\|=1} W_\eta\right)\\ &= \alpha \mathsf{E}\left(\left\|\sum_i g_{i,1} y_i\right\|\right), \quad \text{and similarly, } \geqq \beta \mathsf{E}\left(\left\|\sum_i g_{i,1} x_i\right\|\right),\end{aligned}$$

hence the left side of (*). □

THEOREM 6: *Let $\{e_i\}_{i=1}^n$ be a symmetric basis for a Banach space $E$, and let $T: E \to \ell_2^n$ be the natural basis to basis map. Then, if* $E_\epsilon^{2^k} = \underbrace{E\hat{\otimes}_\epsilon E\hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon E}_{2^k}$

$$\eta_{1/2}(E_\epsilon^{2^k})(\|T^{-1}\|\pi_1(T))^{2^k} \geq \sqrt{\frac{2}{\pi n}}\, n^{2^k} 2^{-3k/2-2^{k-1}}.$$

PROOF: For each integer $k = 1, 2, \ldots$, let $I_k$ denote a set consisting of $n^{2^k}$ elements, and let $\{b_\nu\}_{\nu\in I_k}$ denote the natural basis of $E_\epsilon^{2^k}$ (each $b_\nu$ has the form $e_{i_1}\otimes e_{i_2}\otimes\cdots\otimes e_{i_k}$, where $1 \leq i_j \leq n$). Consider the random vectors of $E_\epsilon^{2^k} = E_\epsilon^{2^{k-1}}\hat{\otimes}_\epsilon E_\epsilon^{2^{k-1}}$ of the form $A^k = \sum_{\alpha,\beta\in I_{k-1}} g_{\alpha,\beta} b_\alpha \otimes b_\beta$. By Lemma 5

$$\mathsf{E}(\|A^k\|_{E_\epsilon^{2^k}}) \leq 2\sqrt{2}\epsilon_2(\{b_\alpha\}_{\alpha\in I_{k-1}})\mathsf{E}(\|A^{k-1}\|_{E_\epsilon^{2^{k-1}}}).$$

Since $\{\otimes_{i=1}^{2^{k-1}} x_i';\ x_i' \in K_{E'}\}$ is the set of extreme points of the unit ball of $(E_\epsilon^{2^{k-1}})' = (E')_\pi^{2^{k-1}}$, we have

$$\epsilon_2(\{b_\alpha\}_{\alpha\in I_{k-1}}) = \sup\left\{\left(\sum_{i_1,\ldots,i_{2^{k-1}}=1}^{n} (\langle e_{i_1}, x_1'\rangle \ldots \langle e_{i_{2^{k-1}}}, x_{2^{k-1}}'\rangle\right)^2;\right.$$

$$\left. x_i' \in K_{E'}\right\} = \left[\sup\left\{\left(\sum_{i=1}^n \langle e_i, x'\rangle^2\right)^{1/2}; x' \in K_{E'}\right\}\right]^{2^{k-1}} = \|T^{-1}\|^{2^{k-1}},$$

hence we get the reduction formula

$$\mathsf{E}(\|A^k\|_{E_\epsilon^{2^k}}) \leq 2\sqrt{2}\|T^{-1}\|^{2^{k-1}}\mathsf{E}(\|A^{k-1}\|_{E_\epsilon^{2^{k-1}}}),$$

and so

$$\mathsf{E}(\|A^k\|_{E_\epsilon^{2^k}}) \leq (2\sqrt{2})^k\|T^{-1}\|^{2^{k-1}+2^{k-2}+\cdots+2^0}\mathsf{E}\left(\left\|\sum_{i=1}^n g_i e_i\right\|_E\right)$$

$$= (2\sqrt{2})^k \|T^{-1}\|^{2^k-1} \mathsf{E}\left(\left\|\sum_{i=1}^n g_i e_i\right\|\right)$$
$$\leq (2\sqrt{2})^k \|T^{-1}\|^{2^k} \mathsf{E}\left(\left(\sum g_i^2\right)^{1/2}\right)$$
$$\leq (2\sqrt{2})^k \|T^{-1}\|^{2^k} \sqrt{n}.$$

On the other hand, $\mathsf{E}(\sum_{\alpha,\beta\in I_{k-1}} |g_{\alpha,\beta}|) = \sqrt{2/\pi}\, n^{2^k}$, hence

$$\sup\left(\sum |a_{i_1,\ldots,i_{2^k}}| \Big/ \left\|\sum a_{i_1,\ldots,i_{2^k}} e_{i_1} \otimes \cdots \otimes e_{i_{2^k}}\right\|_{E_\epsilon^{2^k}}\right)$$
$$\geq \mathsf{E}\left(\sum_{\alpha,\beta\in I_{k-1}} |g_{\alpha,\beta}|\right) \Big/ \mathsf{E}(\|A^k\|)$$
$$\geq \sqrt{\frac{2}{\pi}}\, n^{2^k} / (2/\sqrt{2})^k \sqrt{n} \|T^{-1}\|^{2^k},$$

and the inequality of Lemma 4 establishes the Theorem. □

REMARK: More generally, the same proof can show that if $E_i$ ($i = 1, 2, \ldots, 2^k$) are $n_i$-dimensional spaces with symmetric bases, and if $T_i : E_i \to \ell_2^{n_i}$ are the natural basis to basis maps, then for $F = E_1 \hat{\otimes}_\epsilon E_2 \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon E_{2^k}$,

$$\eta_{1/2}(F) \prod_{j=1}^{2^k} \|T_j^{-1}\| \pi_1(T_j) \geq \sqrt{2/\pi} \prod_{j=1}^{2^k} n_j / 2^{k/2+2^{k-1}} \left(\sum_{j=1}^{2^k} \sqrt{n_j}\right).$$

It is not essential that $2^k$ spaces appear in $F$, however, if the number is not $2^k$ then the bottom line on the right hand side of the inequality will be different.

EXAMPLE: If $E = \ell_2^n$, then $\|T^{-1}\| = 1$, $\pi_1(T) \leq \sqrt{\pi n/2}$ [5], hence if $E_N = \underbrace{\ell_2^n \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon \ell_2^n}_{2^k}$ where $N = n^{2^k}$,

$$\eta_{1/2}(E_N) \geq \sqrt{2/\pi} \sqrt{n^{2^k} / (\sqrt{\pi})^{2^k} 2^{3k/2} \sqrt{n}} \geq a\sqrt{N} e^{-\sqrt{\log N}}$$

where $a > 0$ is constant, and $k, n$ are chosen in an appropriate relation to $N$. If $0 < p < \frac{1}{2}$, then $\eta_p(E_N) \geq N^{p-1/2} \eta_{1/2}(E_N) \geq a N^p e^{-\sqrt{\log N}} \to \infty$ as $N \to \infty$.

COROLLARY 7: *If $E$ is an $n$-dimensional normed space such that $d(E, \ell_2^n) < (n/\pi)^{1/2}$, then $\eta_{1/2}(E_\epsilon^k) \to \infty$ as $k \to \infty$.*

PROOF: First observe that if $k > \ell$ then $E_\epsilon^\ell$ is isometric to a norm one-complemented subspace of $E_\epsilon^k$, therefore the ideal property of the norm $\eta_{1/2}$ implies that $\eta_{1/2}(E_\epsilon^k)$, $k = 1, 2, \ldots$, is a nondecreasing sequence. Clearly $d(E_\epsilon^k, (\ell_2^n)_\epsilon^k) \leq (d(E, \ell_2^n))^k$, hence using the estimates for $\eta_{1/2}((\ell_2^n)_\epsilon^{2k})$ we obtain

$$\eta_{1/2}(E_\epsilon^{2k}) \geq \eta_{1/2}((\ell_2^n)_\epsilon^{2k})/(d(E, \ell_2^n))^{2k} \geq \sqrt{2/\pi}\left(\frac{\sqrt{n/\pi}}{d(E, \ell_2^n)}\right)^{2k} / 2^{3k/2}\sqrt{n}$$

which tends to $\infty$ with $k$.

REMARKS: (1) It may be true that $\eta_{1/2}(E_\epsilon^k) \to \infty$ as $k \to \infty$ whenever $d(E, \ell_2^n) < \sqrt{n}$. It is obviously false if $d(E, \ell_2^n) = \sqrt{n}$, as in the case $E = \ell_\infty^n$.

(2) If $1 < q < \infty$, and $c > |\frac{1}{2} - 1/q|$, there exists $N$ such that if $n \geq N$ and $E$ is any $m$-dimensional subspace of $L_q(\mu)$, then $\eta_c(E_\epsilon^k) \to \infty$ as $k \to \infty$. The reason for this is that $d(E, \ell_2^n) \leq n^{|1/2 - 1/q|}$ by [13], so if $c \leq \frac{1}{2}$

$$\eta_c(E_\epsilon^{2k}) \geq {}^{(c-1/2)2k}\eta_{1/2}(E_\epsilon^{2k})$$

which tends to $\infty$ by applying the inequality in the proof of Corollary 7.

THEOREM 8: *There exists a reflexive separable Banach space E with both $\eta_p(E)$ and $\eta_p^{**}(E)$ infinite for all values of $p > 0$.*

PROOF: Let $E_N$ be the space in the Example, and let $E = (\Sigma_{i=1}^\infty \oplus E_{N_i})_{\ell_2}$ where $N_i \to \infty$, $N_i = (n_i)^{2k_i}$, which the proper relation maintained between $n_i$ and $k_i$ with $N_i$. Since $E_{N_i}$ is norm one complemented in $E$, it follows that

$$\eta_p(E) \geq \eta_p^{**}(E) \geq \eta_p^{**}(E_{N_i}) = \eta_p(E_{N_i}) \geq aN_i^p \exp(-\sqrt{\log N_i}) \xrightarrow[i\to\infty]{} \infty.$$

□

In order to estimate $\pi_1(T)$ for general spaces it may be useful to apply the following proposition.

PROPOSITION 9: *For any $0 < p$, $q$, $r < \infty$ there exist constants $a_{r,q}$, $b_{p,q} > 0$ such that for any Banach space E and any operator $T: E \to \ell_2^n$*

*(1) $b_{p,q}^{-1}\pi_p(T) \leq \sqrt{n}(\int_{S_n} \|T'x\|^q dm(x))^{1/q} \leq a_{r,q}^{-1}\pi_r(T')$, where $S_n = \{x \in \ell_2^n;\ \|x\|_2 = 1\}$ and $dm(x)$ is the rotation invariant normalized measure on $S_n$.*

(2) *If $E'$ is a subspace of an $L_s$-space, $1 \le s < \infty$, and if $0 < p \le s \le r < \infty$ and $0 < q < \infty$, the inequalities of* (1) *becomes equivalence relations and the constants of equivalence are independent of n, T and E.*

(3) *If* $\dim(E) = n$ *and E has a symmetric basis and $T : E \to \ell_2^n$ is the basis to basis map, then all the values $I_q = (\int_{S_n} \|T'x\|^q dm(x))^{1/q}$ $(0 < q < \infty)$ are equivalent and the constants of equivalence are independent of n and E.*

Proof: (1) By [14] there exists a probability measure $\mu$ on $S_n$ such that for all $x \in \ell_2^n$

$$\|T'x\| \le \pi_q(T') \left( \int_{S_n} |\langle x, x' \rangle|^q d\mu(x') \right)^{1/q}$$

hence by integrating with respect to $dm$

$$I_q \le \pi_q(T') \left( \int_{S_n} \int_{S_n} |\langle x, x' \rangle|^q d\mu(x') dm(x) \right)^{1/q} = \pi_q(T')/\pi_q(\ell_2^n)$$

since $(\int_{S_n} |\langle x, x' \rangle|^q dm(x))^{1/q} = \|x'\|_2 (\pi_q(\ell_2^n))^{-1}$ [5]. Since $\pi_q(\ell_2^n) \sim \sqrt{n}$, and $\pi_q(T')$ is a non-increasing function of $q$, and $I_q$ is a non-decreasing function of $q$, the right hand side of (1) readily follows.

Without loss of generality we may assume that $T'$ is a $1-1$ map, and define the probability measure $\nu$ on the unit ball $B_{E'}$ of $E'$ by

$$\int_{B_{E'}} f d\nu = \int_{S_n} f(T'x/\|T'x\|) \|T'x\|^q dm(x) \Big/ \int_{S_n} \|T'x\|^q dm(x)$$

for $f \in C(B_{E'})$. Taking $f = |\langle \xi, \cdot \rangle|^q$ where $\xi \in E$, we obtain

$$\int_{B_{E'}} |\langle \xi, x' \rangle|^q d\nu(x') = \int_{S_n} |\langle T\xi, x \rangle|^q dm(x) \Big/ \int_{S_n} \|T'x\|^q dm(x)$$
$$= \|T\xi\|^q \Big/ (\pi_q(\ell_2^n))^q \int_{S_n} \|T'x\|^q dm(x),$$

therefore, $\pi_q(T) \le \pi_q(\ell_2^n)(\int_{S_n} \|T'x\|^q dm(x))^{1/q}$, and as above this proves the left hand side of (1).

(2) Let $j : E' \to L_s$ be an isometric embedding, then by [12] $\pi_s(jT') \le \pi_s((jT')')$, that is

$$\pi_s(T') = \pi_s(jT') \le \pi_s(T''j') \le \pi_s(T'') = \pi_s(T),$$

and (2) follows from the inequalities

$$\pi_s(T') \leq \pi_s(T) \leq \pi_p(T) \leq b_{p,q}\sqrt{n} I_q \leq b_{p,q} a_{r,q}^{-1} \pi_r(T') \leq b_{p,q} a_{r,q}^{-1} \pi_s(T').$$

(3) Set $|x|_{E'} = \|T'x\|$ for vectors $x \in \ell_2^n = (R^n, \|\cdot\|_2)$. Since $E$ is symmetric, by [10] $\|T\|\|T^{-1}\| = d(E, \ell_2^n)$, and $d(E, \ell_2^n) \leq \sqrt{n}$, so $a\|x\|_2 \leq |x|_{E'} \leq b\|x\|_2$ for all $x \in \ell_2^n$, where $b/a \leq \sqrt{n}$. From the remark following Lemma 2.7 in [4], the values $I_q = (\int_{S_n} |x|_{E'}^q dm(x))^{1/q}$ $(0 < q < \infty)$ are all equivalent to the Levy mean $M^*$, which is by definition the unique number such that $m(\{x \in S_n;\ |x|_{E'} \leq M^*\}) \leq \frac{1}{2}$ and $m(\{x \in S_n;\ |x|_{E'} \geq M^*\}) \leq \frac{1}{2}$, that is, there exist absolute positive constants $a_q, b_q$ such that $a_q M^* \leq I_q \leq b_q M^*$. □

COROLLARY 10: *If* $\dim(E) = n$, *there are absolute constants* $a, b > 0$ *such that for any* $T: E \to \ell_2^n$

$$a\pi_1(T) \leq \sqrt{n} \int_{S_n} \|T'x\| dm(x) \leq b\pi_1(T)\sqrt{\ln n}\, x_u(E).$$

PROOF: Let $\gamma_p$ denote the best factorization through an $L_p$-space norm [9]. Interpolation technique as in Theorem 7 [11] shows that $\pi_p(T') \leq n^{1/p}\gamma_\infty(T')$. Since $\gamma_\infty(T') = \gamma_1(T) \leq x_u(E)\pi_1(T)$ [7], and $\pi_p(\ell_2^n) \geq c\sqrt{n/p}$ [5], we obtain

$$\int_{S_n} \|T'x\| dm(x) \leq \left(\int_{S_n} \|T'x\|^p dm(x)\right)^{1/p} \leq \pi_p(T')/\pi_p(\ell_2^n) \leq \sqrt{p/n}\, c^{-1} n^{1/p} \pi_1(T) x_u(E)$$

and the estimate follows by taking $p = \ln n$, and from Proposition 9. □

COROLLARY 11: *Let* $E_N = \underbrace{\ell_p^n \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon \ell_p^n}_{2^k}$, $N = n^{2^k}$, *where* $1 \leq p \leq 2$. *There exists* $b > 0$, *such that* $\eta_{1/2}(E_N) \geq b\sqrt{N} e^{-\sqrt{\log N}}$ *for the proper relation between* $n, k$ *with* $N$.

PROOF: Factor $T: \ell_p^n \xrightarrow{A} \ell_1^n \xrightarrow{B} \ell_2^n$, where $A, B$ are the inclusions. Then $\|T^{-1}\| = n^{1/p - 1/2}$, and $\pi_1(T) \leq \|A\|\pi_1(B) = \sqrt{2}n^{1-1/p}$, since $\pi_1(B) = \sqrt{2}$ is the Khintchine constant. Therefore, $\|T^{-1}\|\pi_1(T) \leq \sqrt{2n}$. The proof is concluded as in the Example following Theorem 6. □

REMARK: Since the distance between $E^k_\epsilon$ and $(\ell^n_p)^k_\epsilon$ is $\leq (d(E, \ell^n_p))^k$, it follows from the estimates of Corollary 10, Theorem 6 and the inequality

$$\eta_{1/2}(E^k_\epsilon) \geq \eta_{1/2}((\ell^n_p)^k_\epsilon)(d(E, \ell^n_p)^{-k},$$

that if $E$ is any $n$-dimensional Banach space such that $\inf_{1\leq p\leq 2} d(E, \ell^n_p) < \sqrt{n}/2$, then $\eta_{1/2}(E^{2k}_\epsilon) \to \infty$ as $k \to \infty$. □

Denote by $r_i(t)$, the $i$-th Rademacher function on [0, 1].

COROLLARY 12: *Let $1 < p \leq 2 \leq q < \infty$, $1/p < 1/q + \frac{1}{2}$. Assume $F$ is a Banach space of type $p$ and cotype $q$. Then, for any $c$ satisfying $c > 1/p - 1/q$, there exists an integer $N$ such that if $n > N$ and if $E$ is any $n$-dimensional symmetric subspace of $F$, then $\eta_c(E^k_\epsilon) \to \infty$ as $k \to \infty$.*

PROOF: Let $\alpha$ be the type-$p$ constant and $\beta$ be the cotype-$q$ constant of $F$ respectively. Let $\{e_i\}_{i=1}^n$ denote the symmetric basis of a subspace $E \subset F$, and $\{e_i'\}_1^n$ be the biorthogonal functionals. The inequality

$$\left(\sum_i^n |\xi_i|^q\right)^{1/q} \leq \beta \int_0^1 \left\|\sum_{i=1}^n \xi_i r_i(t) e_i\right\| dt = \beta \left\|\sum_{i=1}^n \xi_i e_i\right\|$$

implies that $\|\sum \xi_i e_i'\| \leq \beta \|\xi\|_{q'}$. By Proposition 9

$$b_{1,q'}^{-1} n^{-1/2} \pi_1(T) \leq \left(\int_{S_n} \|T'x\|^{q'} dm(x)\right)^{1/q'} \leq \beta \left(\int_{S_n} \|\xi\|_{q'}^{q'} dm(\xi)\right)^{1/q'}$$
$$= \beta n^{1/q'} (\pi_{q'}(\ell^n_2))^{-1},$$

hence $\pi_1(T) \leq \beta c_q n^{1/q'}$, where $c_q > 0$ is constant. Select scalars $\{x_i\}_{i=1}^n$ so that $\sum x_i^2 = 1$ and $\|T^{-1}\| = \|\sum x_i e_i\|$. Then,

$$\|T^{-1}\| = \int_0^1 \left\|\sum x_i r_i(t) e_i\right\| dt \leq \alpha \|x\|_p \leq \alpha n^{1/p - 1/2},$$

therefore $\|T^{-1}\| \pi_1(T) \leq \alpha\beta c_q n^{1/p - 1/q + 1/2}$. Combining this with the inequality $\eta_c(E^{2k}_\epsilon) \geq \eta_{1/2}(E^{2k}_\epsilon)(n^{2k})^{c-1/2}$, and Theorem 6, establishes that for $c > 1/p - 1/q$ and $n$ sufficiently large, $\eta_{1/2}(E^{2k}_\epsilon)(n^{2k})^{c-1/2} \to \infty$ as $k \to \infty$. □