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TOTAL CURVATURE AND THE TOPOLOGY OF COMPLETE SURFACES

Victor Bangert

1. Introduction

The Gauß-Bonnet theorem for compact Riemannian manifolds with boundary can be used to study the relations between the total curvature and the topology of complete, non-compact Riemannian manifolds. The first results in this direction were obtained by Cohn-Vossen in his fundamental article [2]. His major theorem is the following

THEOREM 1: *If M is a finitely connected, complete Riemannian manifold of dimension 2 whose total curvature $\int_M K dA$ exists as an extended real number, then $\int_M K dA \leq 2\pi\chi(M)$.*

For a modern version of the proof see [1].
The purpose of this paper is to give a geometrical proof of the following theorem originally due to Huber [4].

THEOREM 2: *Let M be a complete, connected Riemannian manifold of dimension 2 whose total curvature exists as an extended real number. If M is not finitely connected then $\int_M K dA = -\infty$.*

This means in particular that finite total curvature implies finite connectivity. Furthermore, using Theorem 1, we obtain: If M is non-compact, connected and $\int_M K dA$ exists in $[-\infty, \infty]$ then $\int_M K dA \leq 2\pi$.

We define $\chi(M) := -\infty$ if M is connected and not finitely connected. Then Theorem 2 complements Theorem 1 in the following sense.

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COROLLARY: *In Theorem* 1 *the hypothesis "finitely connected" can be replaced by "connected".*

Huber's proof of Theorem 2 depends primarily on function theoretic methods. Here we will give an entirely different proof based on Cohn-Vossen's geometrical ideas and techniques. However, to overcome the difficulties arising from the more general topological situation we will be using some additional methods developed in the theory of closed geodesics.

Finally we made two remarks concerning the existence of closed geodesics and of almost-geodesic loops on complete Riemannian manifolds of arbitrary dimension.

2. Notation and definitions

Let (M, g) be a complete, connected, non-compact Riemannian manifold of class C^∞ and dimension $m = 2$. The metric induced by g is denoted by $d: M \times M \to \boldsymbol{R}$. The space of closed curves $\gamma: \frac{[0,1]}{\{0,1\}} \to M$ carries the topology defined by the metric

$$d_\infty(\gamma_0, \gamma_1) := \max_{t \in [0,1]} d(\gamma_0(t), \gamma_1(t)).$$

Here curves are at least piecewise C^1. The length of a curve γ is denoted by $L(\gamma)$. Since Theorems 1 and 2 are true if they are true for a finite covering of M we always assume M to be oriented. For a compact, 2-dimensional submanifold N with boundary ∂N the Gauß-Bonnet theorem takes the form

$$\int_N K dA + \int_{\partial N} k_g ds = 2\pi\chi(M). \tag{2.1}$$

Here dA, ds denote the volume elements of $M, \partial N$, and K, k_g denote the Gaussian curvature of M and the geodesic curvature of ∂N. If $\gamma: \frac{[0,1]}{\{0,1\}} \to M$ is a regular, differentiably closed C^2-curve we define the total geodesic curvature $G(\gamma)$ of γ by

$$G(\gamma) := \int_0^1 k_g(t) \mid \dot{\gamma}(t) \mid dt$$

where k_g denotes the geodesic curvature of γ. If we parametrize ∂N

by regular C^2-curves γ_i then

$$\int_{\partial N} k_g ds = \sum G(\gamma_i) \tag{2.2}$$

if the parametrization of each γ_i is such that N lies to the left of γ_i. This means that a vector field v points into the interior of N along γ_i if and only if $(\dot{\gamma}_i, v \circ \gamma_i)$ is positively oriented.

Finally we want to generalize the concept of total geodesic curvature to regular piecewise C^2-curves $\gamma : \frac{[0, 1]}{\{0, 1\}} \to M$. By definition such a curve γ has the following two properties

(i) There exists a subdivision $0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0, 1]$ such that $\gamma \mid [t_i, t_{i+1}]$ is regular and of class C^2.
(ii) $\dot{\gamma}_-(t_i) = \alpha \dot{\gamma}_+(t_i)$ implies $\alpha > 0$, $(0 \le i \le n-1)$.

Here $\dot{\gamma}_-(t_i)$, $\dot{\gamma}_+(t_i)$ denote the left resp. right hand limit of $\dot{\gamma}(t)$ at t_i, $(\dot{\gamma}_-(0) := \dot{\gamma}_-(1))$.

The oriented exterior angle $\alpha_i \epsilon (-\pi, \pi)$ of γ at t_i is defined by

$$| \dot{\gamma}_-(t_i) | \, | \dot{\gamma}_+(t_i) | \cos \alpha_i = g(\dot{\gamma}_-(t_i), \dot{\gamma}_+(t_i))$$

where, for $\alpha_i \neq 0$, the sign of α_i is determined by $\alpha_i > 0$ if and only if $(\dot{\gamma}_-(t_i), \dot{\gamma}_+(t_i))$ is positively oriented.

Now we define

$$G(\gamma) := \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} k_g^i(t) \, | \dot{\gamma}(t) | \, dt + \sum_{i=0}^{n-1} \alpha_i,$$

where k_g^i denotes the geodesic curvature of $\gamma \mid [t_i, t_{i+1}]$.

Note that equations (2.1) and (2.2) hold for a compact, 2-dimensional submanifold N whose boundary is parametrized by regular piecewise C^2-curves γ_i.

3. Outline of the proof

In order to apply the Gauß-Bonnet theorem we construct a sequence M_j of compact, connected submanifolds with boundary having the following properties:

$$M_j \subseteq \mathring{M}_{j+1} \text{ and } \bigcup_{j=1}^{\infty} M_j = M. \tag{3.1}$$

(3.2) Each component of $M - \mathring{M}_j$ is non-compact and has connected boundary.

Because of (3.2) one obtains M_{j+1} from M_j by gluing surfaces with nonpositive Euler characteristic to M_j, one for each component of ∂M_j. Hence $\chi(M_{j+1}) \leq \chi(M_j)$ and $\lim \chi(M_j) = \chi(M)$. Taking the limit in the Gauß-Bonnet formula (2.1) applied to M'_j, Theorems 1 and 2 follow from:

THEOREM 3: *For every $\epsilon > 0$ and every $j \in N$ there exists a submanifold $M'_j \supseteq M_j$, diffeomorphic to M_j, such that*

$$\int_{\partial M'_j} k_g ds \geq -\epsilon.$$

From now on we will consider one of the finitely many components P of $M - \mathring{M}_j$. Let $\beta : \frac{[0,1]}{\{0,1\}} \to \partial P$ parametrize ∂P such that P lies to the right of β. The set of curves which are freely homotopic to β within P will be denoted by $[\beta]_P$. It suffices to show that there exists a simple closed curve $\gamma \epsilon [\beta]_P$ such that $G(\gamma) \geq -\epsilon$. Then γ and β bound a compact cylinder $C_P \subseteq P$ and we obtain M'_j by adding, for every component P of $M - \mathring{M}_j$, the cylinder C_P to M_j.

In the finitely connected case, i.e. when M is homeomorphic to a compact surface with finitely many points deleted, we can assume that every P is homeomorphic to a punctured disk. For such P the construction of an appropriate curve γ is due to Cohn-Vossen. Because of the following lemma his result will be useful in the more general case as well.

LEMMA 1: *If P is not homeomorphic to a punctured disk then there exists a compact subset K of P such that every curve $\alpha \in [\beta]_P$ which is not longer than β lies in K.*

This lemma is related to Thorbergsson's results [6] on the existence of closed geodesics on complete surfaces.

The construction of an appropriate curve γ now proceeds as follows: We assume that P has a broken geodesic boundary. We shorten the boundary curve β by replacing parts of β by geodesic segments in P. When we iterate this process the resulting curves may leave any compact subset of P. By Lemma 1 this can only happen if P is a punctured disk and then Cohn-Vossen's result applies. Otherwise the procedure will eventually lead to a limiting curve. If such a limiting curve exists in Cohn-Vossen's case, i.e. when P is a punctured disk, then it is a simple closed broken geodesic having non-negative total geodesic curvature. In the general case, however, the limiting curve may have self-intersections and one cannot easily decide if its total geodesic curvature is non-negative. Now Klingenberg [5] devised a specific shortening process which provides a limiting curve γ_0 which at least can be approximated by simple closed curves in P. Investigating the self-intersections of γ_0 closely we will prove

LEMMA 3: *The total geodesic curvature of γ_0 is non-negative.*

Smoothing the corners of γ_0 we easily obtain a regular C^∞-curve $\gamma_i \in [\beta]_P$ such that $G(\gamma_1) > -\epsilon$ and γ_1 can be approximated by simple closed curves in P. We can assume that these approximating curves γ_i are smooth and contained in $\mathring{P}$. Finally we prove a lemma to the effect that $\lim G(\gamma_i) = G(\gamma_1)$. Now the curve γ we are looking for can be chosen from the γ_i since $\gamma_i \in [\beta]_P$ holds for almost all $i \in N$. This concludes the proof of Theorem 3.

4. Proofs for the lemmata

An appropriate exhaustion $(M_j)_{j\in N}$ can be constructed as follows: Let $(N_j)_{j\in N}$ be an exhaustion of M by compact, connected submanifolds with boundary. The N_j can be obtained from the sublevels of a proper function $f: M \to \boldsymbol{R}$ with minimum. It is sufficient to find for every N_j a compact, connected submanifold $M_j \supseteq N_j$ such that property (3.2) holds. We first add all compact components of $M - \mathring{N}_j$ to N_j. If two boundary components of the resulting submanifold N_j' belong to the same component P of $M - \mathring{N}_j'$ they can be joined in P by a regular curve without self-intersections. We attach an appropriate neighborhood of this curve to N_j' thus reducing the number of boundary components of N_j' by one. Iterating this process we obtain a submanifold M_j with property (3.2).

Lemma 1 is not contained in Thorbergsson's results [6] but it could be proved along the lines of his Lemma 3.1. Instead we give a proof based on a different concept, namely the homotopy invariance of intersection numbers.

LEMMA 1: *Suppose the component P of $M - \mathring{M}_j$ is not homeomorphic to a punctured disk and ∂P is parametrized by β. Then there exists a compact subset K of P such that every curve $\alpha \in [\beta]_P$ which is not longer than β lies in K.*

PROOF: We will construct certain curves which intersect every curve in $[\beta]_P$. The following two cases are treated separately:

(a) There exists $i > j$ such that $M_i \cap P$ has at least three boundary components, i.e. there exist at least two components P_1, P_2 in $P - \mathring{M}_i$.

(b) $M_i \cap P$ has two boundary components for all $i > j$.

In case (a) there exist two arc-length-parametrized curves $\alpha_1 : [0, \infty) \to M - P_2$, $\alpha_2 : [0, \infty) \to M - P_1$ such that for $i = 1, 2$

(i) α_i intersects ∂P exactly once and transversely, and
(ii) there exists $A > 0$ such that $\alpha_i(t) \in P_i$ for $t \geq A$, and
(iii) $\alpha_i([0, \infty))$ is a closed submanifold with boundary.

The intersection numbers $\#(\alpha_i, \beta)$ equal 1. This implies that every curve $\alpha \in [\beta]_P$ intersects both α_1 and α_2, see [3], p. 132 and note that $\alpha_i(0) \notin P$. Hence any $\alpha \in [\beta]_P$ with $L(\alpha) \leq L(\beta)$ is contained in the compact set $K := \{p \in M \mid d(p, \partial P) \leq A + L(\beta)\}$.

In case (b) there exists $i > j$ such that $M_i \cap P$ is not homeomorphic to the cylinder $S^1 \times [0, 1]$ since otherwise $P = \cup_{i>j}(M_i \cap P)$ would be homeomorphic to a punctured disk $S^1 \times [0, 1)$. Attaching a disk D to ∂P we obtain a manifold $(M_i \cap P) \cup D$ which is homeomorphic to a torus of genus $g \geq 1$ with an open disk removed. Hence there exists a regular curve $\alpha_0 : [0, 1] \to P$ such that

(i) α_0 meets ∂P transversely and only in $\alpha_0(0) = \alpha_0(1)$, and
(ii) $\alpha_0([0, 1])$ does not separate $P \cup D$.

It suffices to prove that every $\alpha \in [\beta]_P$ intersects α_0. Let $F : N \to P \cup D$ be the universal Riemannian covering. By (ii) the curve α_0 is not contractible within $P \cup D$. Hence a lift α_0' of α_0 joins two different lifts β' and β'' of β. Extending α_0' a bit inside the corresponding copies of D we see that $\#(\alpha_0', \beta') = 1$. Hence every curve which is freely homotopic to β' within $N - F^{-1}(\mathring{D})$ intersects α_0'. Now an application of the homotopy lifting property of F completes the proof.

From now on we will assume that P has a broken geodesic boundary ∂P. The new boundary curve β can be obtained by approximating a given smooth one. Obviously Lemma 1 remains true in this context. Again we assume that P is situated on the right hand side of β. We are going to use Klingenberg's deformation $\mathfrak{D}$ described in [5], A.2. Since we apply it to the single curve β only we do not need the continuity of $\mathfrak{D}$. However we have to modify $\mathfrak{D}$ so as to obtain curves in P. This modification consists in replacing the minimal geodesic segments used in the definition of $\mathfrak{D}$ by minimal segments with respect to the inner metric $d_P: P \times P \to \boldsymbol{R}$ of P,

$$d_P(q, q') := \inf\{L(\alpha) \mid \alpha \text{ joins } q \text{ and } q' \text{ within } P\}.$$

Subsequently we summarize some properties of d_P which should make clear that the modified deformation has the same properties as $\mathfrak{D}$ itself.

DEFINITION: A curve $\alpha:[a,b]\to P$ is P-minimal if $d_P(\alpha(a), \alpha(b)) = L(\alpha)$. A curve $\alpha:[a,b]\to P$ is P-geodesic if α is locally P-minimal and parametrized proportionally to arc-length.

Cohn-Vossen investigates these concepts in [2], §§8, 9, even for more general P. His results are:

A P-geodesic is a geodesic as long as it does not hit ∂P. It can be broken in concave corners of ∂P only and then it bends into the same direction as ∂P. Convex balls with respect to d_P exist in the same way they do exist for d. P-geodesic segments converge in the same way as usual geodesic segments. Hence, locally, the situation is in complete analogy to the case of a polygonal domain in the Euclidean plane.

Using Lemma 1 and replacing "geodesic" by "P-geodesic" everywhere in [5], A.2, the following Lemma 2 is a consequence of [5], Lemma A.2.2. Note that it is also possible to prove Lemma 2 directly without any reference to $\mathfrak{D}$.

LEMMA 2: *If P is not homeomorphic to a punctured disk there exists a P-geodesic $\gamma_0 \in [\beta]_P$ which can be approximated by simple closed curves in P.*

However, contrary to Klingenberg's case, γ_0 need not be simple, since different P-geodesics can intersect without intersecting transversely. One can actually construct examples showing that γ_0 may have self-intersections. This accounts for the difficulties in the proof of

LEMMA 3: *The total curvature of γ_0 is non-negative.*

PROOF: Let $\gamma_i: \dfrac{[0,1]}{\{0,1\}} \to P$ be a sequence of simple closed curves converging to γ_0. Using standard methods from differential topology we can assume that the γ_i are regular C^2-curves in $\mathring{P}$. Because of $\gamma_0 \in [\beta]_P$ almost all γ_i belong to $[\beta]_P$. Hence β and each of these γ_i bound a topological cylinder in P. Since P is situated to the right of β the set M-P lies to the left of each γ_i. Let $p \in \partial P$ be a vertex of γ_0

and $\{t_1, \dots, t_n\} = \gamma_0^{-1}(p)$. We will prove that we can order the $t_1, \dots, t_n$ such that the following properties hold for the corresponding exterior angles α_i:

(i) $|\alpha_1| \geq |\alpha_2| \geq \cdots \geq |\alpha_n|$

(ii) $\alpha_k \geq 0$ if k is odd and $\alpha_k \leq 0$ otherwise.

Then $\Sigma_{k=1}^n \alpha_k \geq 0$ and since this is true for every vertex of γ_0 we obtain $G(\gamma_o) \geq 0$.

Let B be a closed convex d-ball about p such that p is the only vertex of γ_0 within B. We assume that $\gamma_i(t_k) \in B$ holds for all $k \in \{1, \dots, n\}$ and all $i \in N$. If $[t_k - \epsilon_k^i, t_k + \delta_k^i]$ denotes the component of t_k in $\gamma_i^{-1}(B)$ then $\lim_i \epsilon_k^i = \epsilon_k^0$ and $\lim_i \delta_k^i = \delta_k^0$. This follows from the convergence of the γ_i to γ_0 and the fact that γ_0 intersects ∂B transversely in $t_k - \epsilon_k^0$ and in $t_k + \delta_k^0$. Here $t_k^i - \epsilon_k^i, t_k^i + \delta_k^i$ have to be considered mod 1 if necessary. Let q be a point in $\partial B \cap (M-P)$ and let $\epsilon := \min\{d(x,y) \mid x \neq y; x,y \epsilon (\gamma_0([0,1]) \cup \{q\} \cap \partial B\}$. Choose $\gamma_i \in [\beta]_P$ such that for all $k \in \{1, \dots, n\}$

(a) the distance of any pair of corresponding points $\gamma_i(t_k - \epsilon_k^i)$, $\gamma_0(t_k - \epsilon_k^0)$ resp. $\gamma_i(t_k + \delta_k^i)$, $\gamma_0(t_k + \delta_k^0)$ is smaller than $\epsilon/2$,

(b) there exists a ball $B' \subseteq B$ about p such that $\gamma_i(t_k) \in B'$ and $\gamma_i^{-1}(B') \subseteq \cup_{k=1}^n [t_k - \epsilon_k^i, t_k + \delta_k^i]$.

Let T_k denote the components of $B - \gamma_i([t_k - \epsilon_k^i, t_k + \delta_k^i])$ containing $B \cap (M-P)$. We order the t_k so that $T_1 \subseteq T_2 \subseteq \dots \subseteq T_n$ holds. Because of (a) this implies for the corresponding components S_k of $B - \gamma_0([t_k - \epsilon_k^0, t_k + \delta_k^0])$:

$$S_1 \subseteq S_2 \subseteq \dots \subseteq S_n.$$

Hence

$$|\alpha_1| \geq |\alpha_2| \geq \dots \geq |\alpha_n|.$$

By assumption T_1 is situated to the left of $\gamma_i \mid [t_i - \epsilon_1^i, t_1 + \delta_1^i]$. Since γ_i separates M condition (b) implies that T_k is situated to the left of $\gamma_i \mid [t_k - \epsilon_k^i, t_k + \delta_k^i]$ if k is odd and to the right otherwise. Because of (a) the same is true for S_k and $\gamma_0 \mid [t_k - \epsilon_k^0, t_k + \delta_k^0]$. This implies $\alpha_k \geq 0$ for odd k and $\alpha_k \leq 0$ for even k. Thus Lemma 3 is proved.

Now we want to construct a regular C^∞-curve $\gamma_1 \in [\beta]_P$ such that $G(\gamma_1) > -\epsilon$ and γ_1 can be approximated by simple closed curves in P. For every vertex p of γ_0 let B denote a ball about p as in the proof of Lemma 3. We can assume that these balls are disjoint for different vertices. We obtain γ_1 from γ_0 by smoothing, for every vertex p of γ_0, the curves $\gamma_0^k := \gamma_0 \mid [t_k - \epsilon_k, t_k + \delta_k]$ where $t_k, \epsilon_k, \delta_k$ are defined as in the proof of Lemma 3. Choosing the regular C^∞-curves $\gamma_1^k := \gamma_1 \mid [t_k -$

$\epsilon_k, t_k + \delta_k]$ within $B \cap P$ we have $\gamma_1 \in [\beta]_P$. Furthermore the curves γ_1^k should neither have self-intersections nor should they pass through each other. Then γ_1 can as well be approximated by simple closed curves in P. Finally, provided the area of the sets bounded by γ_0^k and γ_1^k is small enough we conclude $G(\gamma_1) > -\epsilon$ from $G(\gamma_0) \geq 0$. This follows from the Gauß-Bonnet theorem applied to the sets bounded by γ_0^k and γ_1^k.

Let $\gamma_i : \frac{[0,1]}{\{0,1\}} \to P$ be a sequence of simple closed curves converging to γ_1. As in the proof of Lemma 3 we can assume that the γ_i are regular curves in $\mathring{P}$ and as smooth as we want them to be. We conclude the proof of Theorem 3 by noting that the curve γ we are looking for can be chosen from the sequence γ_i. This is a consequence of the following lemma which is stated in a more general setting since it may be considered of independent interest.

For $t, s \in \frac{[0,1]}{\{0,1\}}$ we define $d_1(t,s) := \min\{|t-s|, 1-t+s, 1-s+t\}$.

LEMMA 4: *Let $\alpha_i : \frac{[0,1]}{\{0,1\}} \to M$ be a sequence of regular piecewise C^2-curves converging to a regular piecewise C^2-curve α. Suppose the α_i are uniformly locally injective, i.e. there exists $\eta > 0$ such that $0 < d_1(t,s) < \eta$ implies $\alpha_i(t) \neq \alpha_i(s)$ for all $i \in N$. Then the total geodesic curvatures $G(\alpha_i)$ converge to $G(\alpha)$.*

PROOF: In order to keep technicalities down to a minimum we assume that α is a regular C^∞-curve. This suffices to prove Theorem 3. Let $N \simeq S^1 \times \mathbf{R}$ be the normal bundle of α and $\exp_N : N \to M$ the exponential map of N. Let α' be a parametrization of the 0-section such that $\exp_N \circ \alpha' = \alpha$. Let U be a neighborhood of the 0-section such that $\exp_N | U$ is an immersion. There exists $\delta > 0$ such that for every $t \in \frac{[0,1]}{\{0,1\}}$ there is a neighborhood $U_t^\delta \subseteq U$ of $\alpha'(t)$ which is mapped diffeomorphicly onto $B(\alpha(t), \delta)$ by $\exp_N$. Hence, if $d_\infty(\alpha_i, \alpha) < \delta$ we can lift α_i via $\exp_N$ to a piecewise C^2-curve α_i' in U. Obviously the α_i' are freely homotopic to α' and converge to α'. If we choose $\delta > 0$ so small that $d_1(t,s) \geq \eta$ implies $U_t^\delta \cap U_s^\delta = \emptyset$ then the a_i' are simple since by assumption $\alpha_i'(t) = \alpha_i'(s)$ can only hold for $t = s$ or for $d_1(t,s) \geq \eta$.

Let β' be a simple closed curve in U which is freely homotopic to α' and which is situated to the left of α' and of all α_i'. Then β' and α' resp. α_i' bound topological cylinders C resp. C_i. We consider the metric $g' = \exp_N^* g$ on U. The metric objects G, K, dA with respect to g' are denoted by G', K', dA'.

By (2.1) and (2.2) we have

$$G'(\alpha_i') - G'(\alpha') = \int_{C_i} K' dA' - \int_C K' dA'.$$

Because of $G'(\alpha_i') = G(\alpha_i)$ and $G'(\alpha') = G(\alpha)$ we obtain

$$\left| G(\alpha_i) - G(\alpha) \right| \leq \int_{\Delta(C_i, C)} \left| K' \right| dA'.$$

This proves the lemma since the area of the symmetric difference $\Delta(C_i, C) = (C_i - C) \cup (C - C_i)$ converges to zero.

REMARKS: (1) The method Thorbergsson applies in [6] to construct closed geodesics on complete surfaces is interchangeable with the technique used in the proof of Lemma 1. The latter method may have the advantage to generalize to higher dimensions. As an example we note the following

THEOREM: *Let M be a differentiable manifold such that there exists a compact hypersurface $N \subseteq M$ which does not separate M (i.e. such that $M - N$ is connected). Then, for every complete Riemannian metric on M, there exists a non-trivial closed geodesic.*

For a proof note that there exists a loop in M which intersects N exactly once and transversely. Hence the arguments used in Lemma 1 and 2 apply.

(2) Bleecker [1] investigates the integral over the absolute value of the geodesic curvature of a closed curve γ. This quantity is equally defined in the higher dimensional case and will be denoted by $|G|(\gamma)$. A closed geodesic c is characterized by $|G|(c) = 0$. Let M be a complete Riemannian manifold and let $[\gamma_0]$ denote a non-trivial free homotopy class of loops in M.

Provided $\dim M = 2$ Bleecker proves that $\inf\{|G|(\gamma) \mid \gamma \in [\gamma_0]\} = 0$. Studying [2] closely one remarks that Cohn-Vossen's methods can be used to simplify Bleecker's proof and, at the same time, to extend his result to arbitrary dimensions. The idea is as follows:

For given $[\gamma_0]$ let $F: M \to \mathbf{R}$ be defined by

$$F(p) := \inf\{L(\gamma) \mid \gamma \in [\gamma_0] \text{ and } \gamma(0) = \gamma(1) = p\}.$$

Since M is complete there exists a geodesic loop γ at p such that

$L(\gamma) = F(p)$. For this loop $|G|(\gamma) \in [0, \pi)$ is the non-oriented exterior angle that γ makes at p. Now assume $|G|(\gamma) \geq a > 0$ for all $\gamma \in [\gamma_0]$. Then, for every $p \in M$, there exists a ball B_p about p of radius $r(p) > 0$ such that

$$(*) \qquad \inf(F \mid B_p) < F(p) - \cos\left(\frac{a}{2}\right) r(p).$$

This follows from the first variation formula. Cohn-Vossen's proof [2], p. 91 generalizes to higher dimensions. Now (*) contradicts the fact that F is bounded below, see [2], p. 129. Hence $\inf\{|G|(\gamma) \mid \gamma \in [\gamma_0]\} = 0$.

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Added in proof

A.L. Verner has kindly brought to my attention the following paper of his: Tapering saddle surfaces. Sibirsk. Matem. Zh. 11 (1970) 750–769. In §1 he treats Theorem 2 with similar methods. His proof, however, appears to be incomplete.