COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 40, nº 3 (1980), p. 417-423

<http://www.numdam.org/item?id=CM_1980__40_3_417_0>

COMPOSITIO MATHEMATICA, Vol. 40, Fasc. 3 1980, pag. 417–423

Printed in the Netherlands

TWO THEOREMS ON DE RHAM COHOMOLOGY

William E. Lang*

In this paper, we prove two theorems on the de Rham cohomology of a nonsingular variety over a perfect field of characteristic $p > 0$. The first theorem relates the differential in the conjugate spectral sequence of de Rham cohomology to the Bockstein operation of Serre's Witt vector cohomology. Applications of this theorem to quasi-elliptic surfaces will be published later. The second theorem gives the de Rham cohomology of an Enriques surface, and, as a corollary, gives the number of vector fields on a non-classical Enriques surface.

The two theorems are logically independent, but Theorem 1 was discovered in the course of my work on Theorem 2, and both theorems are closely related to fundamental work of Oda [6]. Therefore, it seems best to publish them together. After the first version of this paper was written, I learned that L. Illusie has obtained the results of Part B independently, using the deep theory of the de Rham–Witt complex [4]. The proof given here is more elementary, but Illusie's gives additional information on the crystalline cohomology of an Enriques surface. Nevertheless, I believe that this proof is still of some interest.

These results originally appeared in my 1978 Harvard doctoral dissertation. I wish to thank my thesis advisor, David Mumford, for suggesting the area of research and for innumerable helpful comments. I also wish to thank the National Science Foundation for supporting me throughout my graduate work and during the preparation of this paper, and the faculty and staff of the Institute for Advanced Study.

* Supported in part by an NSF grant.

0010–437X/80/03/0417–07$00.20/0

A. De Rham cohomology and the first Bockstein operation

The algebraic de Rham cohomology was introduced by Grothendieck in [2]. The facts which we need concerning this theory are proved in Katz [5] and summarized in [3, III, 7–8]. We give a brief summary below to fix notation.

Recall that if X is a non-singular variety of dimension n over a perfect field k, the de Rham cohomology $H^*_{DR}(X)$ is defined as the hypercohomology of the de Rham complex

$$\Omega^{\cdot}_X = 0 \to 0_X \to \Omega^1_X \to \Omega^2_X \to \cdots \to \Omega^n_X \to 0.$$

The first spectral sequence of hypercohomology

$$E_1^{pq} = H^q(X, \Omega^p_X) \Rightarrow H^n_{DR}(X)$$

is called the Hodge–de Rham spectral sequence. It degenerates in characteristic zero, but does not always degenerate in characteristic p.

There is another spectral sequence which is useful in computing de Rham cohomology in characteristic p. We note that the Frobenius morphism F induces a p-linear isomorphism between the hypercohomology of $\Omega^{\cdot}_X$ and $F_*\Omega^{\cdot}_X$. Let Z^i denote the kernel of $F_*d\colon F_*\Omega^i_X \to F_*\Omega^{i+1}_X$ and let B^i denote the image of $F_*d\colon F_*\Omega^{i-1}_X \to F_*\Omega^i_X$. The Cartier operator C induces an isomorphism $Z^i/B^i \cong \Omega^i$. Then the second spectral sequence of hypercohomology gives the conjugate spectral sequence

$$E_2^{pq} = H^p(X, \Omega^q) \Rightarrow H^*_{DR}(X)$$

of de Rham cohomology.

THEOREM 1: *Let X be a non-singular variety over a perfect field k of characteristic $p > 0$. Then the following diagram is commutative:*

$$\begin{array}{ccc} H^0(X, B^1) & \longrightarrow & H^0(X, \Omega^1_X) \\ \downarrow \delta & & \downarrow d_2 \\ H^1(X, 0_X) & \xrightarrow[-\beta_1]{} & H^2(X, 0_X) \end{array}$$

The horizontal arrow is the obvious inclusion, β_1 is the first Bockstein operation of Witt vector cohomology, the left vertical arrow comes from the exact sequence $0 \longrightarrow 0_X \xrightarrow{F} F_*0_X \xrightarrow{d} B^1 \longrightarrow 0$, and the right vertical arrow is the differential in the conjugate spectral sequence of de Rham cohomology.

PROOF: This will be a straightforward computation in Čech cohomology. Pick a cover of X by open affines $\{U_i\}$. Then on U_i, an element of $H^0(X, B^1)$ can be written as dg_i. Let $h = \delta(dg_i)$. Then h can be represented by an alternating 1-cochain such that $h_{ij}^p = g_j - g_i$, $h_{jk}^p = g_k - g_j$, $h_{ik}^p = g_k - g_i$.

Recall that β_1 is the connecting homomorphism in the cohomology sequence of the following exact sequence:

$$0 \to 0_X \to W_2 \to 0_X \to 0$$

where W_2 is the sheaf of Witt vectors of length 2. Recall also that Witt vectors of length 2 add as follows: If $a = (a_0, a_1)$ and $b = (b_0, b_1)$ then

$$a + b = \left(a_0 + b_0, a_1 + b_1 - \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} a_0{}^m b_0^{p-m}\right),$$

where $p^{-1}\binom{p}{m}$ is defined as an element of k by the formula $((p-1)!)/(m!(p-m)!)$. (See Serre [7].)

Assume $p \neq 2$. To find a representative cocycle for $d = \beta_1(h)$, we compute

$$\begin{aligned}
&(h_{jk}, 0) - (h_{ik}, 0) + (h_{ij}, 0)\\
&= \left(h_{jk} + h_{ij}, - \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} h_{jk}^m h_{ij}^{p-m}\right) + (-h_{ik}, 0)\\
&= \left(h_{jk} + h_{ij} - h_{ik}, - \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} h_{jk}^m h_{ij}^{p-m}\right.\\
&\qquad \left. - \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} (-1)^m h_{ik}^m (h_{jk} + h_{ij})^{p-m}\right).
\end{aligned}$$

Since $h_{jk} + h_{ij} = h_{ik}$, this reduces to

$$\left(0, - \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} h_{jk}^m h_{ij}^{p-m} - \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} (-1)^m h_{ik}^p\right).$$

Since $p \neq 2$, $\sum_{m=1}^{p-1} p^{-1}\binom{p}{m}(-1)^m = 0$. So $\beta_1(h)$ is represented by the 2-cocycle

$$d_{ijk} = - \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} h_{jk}^m h_{ij}^{p-m}. \tag{1}$$

Now we go around the other way. Since the conjugate spectral sequence is a second spectral sequence of hypercohomology, d_2 is the composition of connecting homomorphisms obtained from the cohomology of the following exact sequences.

$$0 \longrightarrow B^1 \longrightarrow Z^1 \xrightarrow{C} \Omega^1 \longrightarrow 0 \qquad H^0(\Omega^1) \xrightarrow{\delta_1} H^1(B^1)$$

$$0 \longrightarrow 0_X \xrightarrow{F} F_*0_X \xrightarrow{d} B^1 \longrightarrow 0 \qquad H^1(B^1) \xrightarrow{\delta_2} H^2(0_X).$$

Let $e = \delta_1(dg_i)$. To find a representative cocycle for e, observe that $dg_i = C(g_i^{p-1}dg_i)$. Then

$$\begin{aligned} e_{ij} &= g_j^{p-1}dg_j - g_i^{p-1}dg_i = (g_j^{p-1} - g_i^{p-1})dg_i = ((g_i + h_{ij}^p)^{p-1} - g_i^{p-1})dg_i \\ &= \sum_{m=1}^{p-1} \binom{p-1}{m} g_i^{p-1-m} h_{ij}^{pm} dg_i = d\left(\sum_{m=1}^{p-1} p^{-1} \binom{p}{m} g_i^{p-m} h_{ij}^{pm}\right). \end{aligned}$$

Similarly,

$$e_{ik} = d\left(\sum_{m=1}^{p-1} p^{-1} \binom{p}{m} g_i^{p-m} h_{ik}^{pm}\right), \quad e_{jk} = d\left(\sum_{m=1}^{p-1} p^{-1} \binom{p}{m} g_j^{p-m} h_{jk}^{pm}\right).$$

Let $f = \delta_2(e)$, then f can be represented by a 2-cocycle f_{ijk} such that

$$\begin{aligned} (2) \qquad f_{ijk}^p &= \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} g_j^{p-m} h_{jk}^{pm} - \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} g_i^{p-m} h_{ik}^{pm} \\ &\quad + \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} g_i^{p-m} h_{ij}^{pm} \\ &= \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} (g_i + h_{ij}^p)^{p-m} h_{jk}^{pm} \\ &\quad - \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} (h_{ij}^p + h_{jk}^p)^m g_i^{p-m} + \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} g_i^{p-m} h_{ij}^{pm} \\ &= \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} \sum_{q=0}^{p-m} \binom{p-m}{q} g_i^q h_{ij}^{p(p-m-q)} h_{jk}^{pm} \\ &\quad - \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} \sum_{q=0}^{m} \binom{m}{q} h_{ij}^{pq} h_{jk}^{p(m-q)} g_i^{p-m} \\ &\quad + \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} g_i^{p-m} h_{ij}^{pm} \\ &= \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} h_{ij}^{p(p-m)} h_{jk}^{pm} + (\text{terms involving } g_i). \end{aligned}$$

Comparing (2) with (1), we see that it is enough to show that the sum of the terms involving g_i is 0. To see this, choose ℓ, $1 \le \ell \le p-1$. Then the contribution of g_i^ℓ is

$$
\begin{aligned}
&g_i^{\ell}\Bigg(\sum_{m=1}^{p-\ell} p^{-1}\binom{p}{m}\binom{p-m}{\ell} h_{ij}^{p(p-m-\ell)} h_{jk}^{pm} - p^{-1}\binom{p}{p-\ell} \\
&\quad \times \sum_{q=0}^{p-j}\binom{p-\ell}{q} h_{ij}^{pq} h_{jk}^{p(p-\ell-q)} + p^{-1}\binom{p}{p-\ell} h_{ij}^{p(p-\ell)}\Bigg) \\
&= g_i^{\ell}\Bigg(\sum_{m=1}^{p-\ell} p^{-1}\binom{p}{m}\binom{p-m}{\ell} h_{ij}^{p(p-m-\ell)} h_{jk}^{pm} - p^{-1}\binom{p}{p-\ell} \\
&\quad \times \sum_{q=0}^{p-\ell-1}\binom{p-\ell}{q} h_{ij}^{pq} h_{jk}^{p(p-\ell-q)}\Bigg).
\end{aligned}
$$

Now match up terms by putting $m = p - \ell - q$. We need only show that

$$\binom{p}{p-\ell-q}\binom{\ell+q}{\ell} = \binom{p}{p-\ell}\binom{p-\ell}{q}.$$

This is left to the reader.

We must also check the case $p = 2$. We know that $\beta_1(x) = x \cup x$ for all $x \in H^1(0_X)$ [7]. Therefore $d = \beta_1(h)$ is represented by $d_{ijk} = h_{ij}h_{jk}$. However, the computation going around the other way does not use $p \neq 2$, hence the result is

$$\left(\sum_{m=1}^{1} \frac{1}{2}\binom{2}{m} h_{ij}^{2} h_{jk}^{2}\right)^{1/2} = h_{ij}h_{jk}.$$

B. De Rham cohomology and Enriques surfaces

For the results on Enriques surfaces that we use in this section, see [1].

THEOREM 2: *The first de Rham cohomology group of an Enriques surface is* 0, *if* char $k \neq 2$. *If* char $k = 2$, *it is* 1-*dimensional. Furthermore, if* char $k = 2$,

$$h^0(\Omega_X^1) = \begin{cases} 1 \text{ if } X \text{ is classical} \\ 0 \text{ if } X \text{ is singular} \\ 1 \text{ if } X \text{ is supersingular.} \end{cases}$$

In the supersingular case, the injection $H^0(\Omega_X^1) \to H_{DR}^1(X)$ induced by the Hodge–de Rham spectral sequence is an isomorphism.

PROOF: We use a result of Oda [6] relating the first de Rham cohomology group to the Picard scheme. The conjugate spectral sequence gives an exact sequence

$$0 \longrightarrow H^1(0_X) \longrightarrow H^1_{DR}(X) \xrightarrow{V} H^0(\Omega^1_X).$$

Oda defines a map V^2 on $\ker(d \cdot V)$ by $V^2(x) = C \cdot Vx$, where C is the Cartier operator. He defines V^3 on $\ker(d \cdot V^2)$ similarly, and so on. He shows that $\bigcap \ker(d \cdot V^n)$ is isomorphic to the dual of the Dieudonné module of the finite group scheme ${}_p\mathrm{Pic}^\tau(X)$, where ${}_p\mathrm{Pic}^\tau(X)$ is the kernel of multiplication by p on $\mathrm{Pic}^\tau(X)$. If all 1-forms on X are closed, then it is clear that Oda's subspace is all of $H^1_{DR}(X)$. In the case of the Enriques surface, the explicit computation of $\mathrm{Pic}^\tau(X)$ is Bombieri–Mumford III shows that Oda's subspace is 0 if $p \neq 2$ and is 1-dimensional if $p = 2$. Therefore, we need only prove the following lemma.

LEMMA: *If X is an Enriques surface, all 1-forms on X are closed.*

PROOF: This is obvious if $p_g = 0$, therefore we need only check the singular and supersingular cases in characteristic 2. In these cases, K_X is trivial and therefore $h^0(\Omega^2_X) = 1$. Suppose this lemma is not true. Then $d: H^0(\Omega^1_X) \to H^0(\Omega^2_X)$ is surjective. This implies that d: $H^2(0_X) \to H^2(\Omega^1_X)$ is injective. (This implication is a special case of a well-known duality theorem in de Rham cohomology which is proved as follows. If X is a variety of dimension n, the conjugate spectral sequence shows that $H^{2n}_{DR}(X) \simeq k$. Therefore, the differential in the Hodge–de Rham spectral sequence d_1: $H^n(\Omega^{n-1}) \to H^n(\Omega^n)$ is zero. Now let $a \in H^p(\Omega^q)$ and let $b \in H^{n-p}(\Omega^{n-q-1})$. Then $0 = d_1(a \cup b) = da \cup b \pm a \cup db$. This shows that d: $H^p(\Omega^q) \to H^p(\Omega^{q+1})$ is (up to sign) the transpose of d: $H^{n-p}(\Omega^{n-q-1}) \to H^{n-p}(\Omega^q)$.) Let a be a non-zero class in $H^1(0_X)$. Then $a \cup a \neq 0$ in $H^2(0_X)$, since the Bockstein operation is injective ([1], Lemma 3.1). Now (using characteristic 2) we see that $d(a \cup a) = a \cup da + da \cup a = 2(a \cup da) = 0$, contradiction. Q.E.D.

This proves our assertions about $H^1_{DR}(X)$. To finish the proof, we need only remark that a class in $H^1(0_X)$ fixed by Frobenius lives forever in the Hodge–de Rham spectral sequence, so $H^0(\Omega^1)$ must be zero in the singular case; in the supersingular case, it is known that $h^0(\Omega^1) \geq 1$.

We may use Theorem 2 to compute $h^i(\theta_X)$ in the singular and supersingular cases in characteristic 2, since K_X is trivial, and therefore θ_X is isomorphic to Ω^1_X. We get

	$h^0(\theta_X)$	$h^1(\theta_X)$	$h^2(\theta_X)$
Singular	0	10	0
Supersingular	1	12	1

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(Oblatum 8-I-1979 & 18-VII-1979)

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