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ON ISOSPECTRAL DEFORMATIONS OF RIEMANNIAN METRICS

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1. Introduction

Let M be an n-dimensional compact orientable C^∞ manifold, and g be a C^∞ Riemannian metric on M. It is known that the Laplace-Beltrami operator $\Delta_g = -g^{ij}\nabla_i\nabla_j$ acting on C^∞ functions on M has an infinite sequence of eigenvalues (denoted by Spec(M, g))

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \uparrow +\infty$$

each eigenvalue being repeated as many as its multiplicity.

Consider the following problem [1, p. 233]: Let $g(t)$ $(-\epsilon < t < \epsilon, \epsilon > 0)$ be a 1-parameter C^∞ deformation of a Riemannian metric on M. Then, is there a deformation $g(t)$ such that Spec$(M, g(t)) =$ Spec$(M, g(0))$ for every t? Such a deformation is called an *isospectral deformation.*

First, we give some definitions to state the results of this article. The deformation $g(t)$ is called *trivial* if for each t there is a diffeomorphism $\eta(t)$ such that $g(t) = \eta(t)^*g(0)$ (the pull-back of $g(0)$ by $\eta(t)$). For a deformation $g(t)$ the symmetric covariant 2-tensor $h \equiv g'(0)$ is called the *infinitesimal deformation* (*i-deformation,* for short) [2]. By Berger-Ebin [3] h is decomposed as

$$h = \tilde{h} + L_X g(0),$$

where $\nabla^i \tilde{h}_{ij} = 0$ (∇ being the connection induced by $g(0)$) and L_X is the Lie derivative with respect to X. The i-deformation h is called *trivial* if $\tilde{h} = 0$.

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The main result of this article is the following.

THEOREM A: *There is no non-trivial isospectral i-deformation of a metric of flat torus.*

REMARK: Concerning the isospectral deformation of a metric of constant curvature, we can easily get the following: *There is no non-trivial isospectral deformation of a metric of constant curvature* K *if* $2 \leq \dim M \leq 5$, *or* $\dim M = 6$ *and* $K > 0$. This result is obtained by combining the results of spectral geometry [1], [4] and those concerning the non-deformability of a metric of constant curvature, the latter being directly derived from the results of Berger-Ebin [3], Mostow [5] and Koiso [2]. (See Tanaka [6], for the case $\dim M = 2$ and $K < 0$.) In the case of flat metrics Sunada [7] showed that there are only finitely many isometry classes of flat manifolds with a given spectrum.

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2. Proof of Theorem A

The main part of the proof is to study the variation of the coefficients of Minakshisundaram's expansion:

$$\sum_{k=0}^{\infty} \exp(-\lambda_k s) \underset{s \to +0}{\sim} \left(\frac{1}{4\pi s}\right)^{n/2} \sum_{j=0}^{\infty} a_j s^j.$$

If $g(t)$ is an isospectral deformation, the coefficients $a_j(t)$ must be constants.

For a Riemannian metric g on M, $\mathrm{d}V(g)$, R^i_{jkm}, R_{ij} and τ denote the volume element, the curvature tensor, the Ricci tensor and the scalar curvature, respectively. The coefficients a_j $(j = 0, 1, 2)$ are given by

$$a_0 = \mathrm{vol}(M, g) = \int_M \mathrm{d}V(g), \quad a_1 = \tfrac{1}{6} \int_M \tau \, \mathrm{d}V(g),$$

$$a_2 = \tfrac{1}{360} \int_M (2|R|^2 - 2|\rho|^2 + 5\tau^2) \, \mathrm{d}V(g),$$

where $|R|^2 = R_{ijkm} R^{ijkm}$ and $|\rho|^2 = R_{ij} R^{ij}$.

Let $g(t)$ be a deformation and $h(t) \equiv g'(t)$ the i-deformation of $g(t)$

at t. Let $a_j(t)$ be the Minakshisundaram's coefficients for $(M, g(t))$. Then the following formulas are obtained by straightforward calculation.

$$a_0'(t) = \tfrac{1}{2}\int_M h_s^s \, \mathrm{d}V(g(t)), \tag{2.1}$$

$$a_1'(t) = \tfrac{1}{6}\int_M (\tfrac{1}{2}\tau h_s^s - R_{ij}h^{ij}) \, \mathrm{d}V(g(t)), \tag{2.2}$$

$$\begin{aligned} a_2'(t) = \tfrac{1}{360}\int_M [&12(\nabla_j\nabla_i\tau)h^{ji} - 6(\nabla_k\nabla^k R_{ji})h^{ji} + 8R_{jk}R_i^k h^{ji} \\ &- 4R_{kjmi}R^{km}h^{ji} - 4R_{jkms}R_i^{kms}h^{ji} + 9(\Delta\tau)h_s^s \\ &- 10\tau R_{ji}h^{ji} + |R|^2 h_s^s - |\rho|^2 h_s^s + \tfrac{5}{2}\tau^2 h_s^s] \, \mathrm{d}V(g(t)). \end{aligned} \tag{2.3}$$

Let (M, g_0) be a flat manifold. Then we have

LEMMA 2.1: *If $g(t)$ is an isospectral deformation with $g(0) = g_0$, and $\nabla^i h_{ij} = 0$ holds at $t = 0$, then*

$$h_s^s = 0, \quad \nabla_k h_{ji} = 0 \tag{2.4a, b}$$

hold at $t = 0$.

PROOF: Starting from (2.3), we have by tedious calculation,

$$a_2''(0) = \tfrac{1}{120}\int_M [(\nabla_k\nabla^k h^{ji})(\nabla_m\nabla^m h_{ji}) + 3(\Delta h_s^s)^2] \, \mathrm{d}V(g_0). \tag{2.5}$$

(Note that $\nabla^i h_{ij} = 0$ and $R^i_{jkm} = 0$ at $t = 0$.) From (2.1) and (2.5), $a_0'(0) = a_2''(0) = 0$ holds if and only if (2.4a, b) hold good. Q.E.D.

LEMMA 2.2: *Let $g(t)$ be as in Lemma 2.1. Then*

$$\int_M h_{ij}\phi(\nabla^i\nabla^j\phi) \, \mathrm{d}V(g_0) = 0 \tag{2.6}$$

holds for each eigenfunction ϕ of $\Delta_{g(0)}$.

PROOF: For the eigenvalue $\lambda_k(t)$ of $\Delta_{g(t)}$, the following was obtained by Berger [8]:

$$\lambda_k'(0) = \int_M [h_{ij}\phi(\nabla^i\nabla^j\phi) + (\nabla^i h_{ij} - \tfrac{1}{2}\nabla_j h_s^s)\phi(\nabla^j\phi)] \, \mathrm{d}V(g_0), \tag{2.7}$$

where ϕ is the eigenfunction for $\lambda_k(0)$. Therefore, (2.6) follows from (2.4). Q.E.D.

Now, let us prove Theorem A. Let (M, g_0) be a flat torus given by R^n/L, where L is a lattice, i.e., a discrete abelian subgroup of the group of Euclidean motions in R^n. Let L^* denote the dual lattice, consisting of all $x \in R^n$ such that $(x, y) = \Sigma_{i=1}^n x^i y^i$ is an integer for all $y \in L$. Then the sets of eigenfunctions and eigenvalues are given by $\{\phi_x(y) = \cos 2\pi(x, y),\ \psi_x(y) = \sin 2\pi(x, y);\ x \in L^*\}$ and $\{4\pi^2(x, x);\ x \in L^*\}$, respectively. Recalling (2.4b), we see that h_{ij} are constants in the coordinates induced from R^n. Therefore, from (2.6) we have

$$4\pi^2 h_{ij} x^i x^j \int_{R^n/L} \{\cos 2\pi(x, y)\}^2 \, \mathrm{d}y = 0, \quad x \in L^*,$$

hence $h_{ij}x^i x^j = (hx, x) = 0$ for $x \in L^*$. This leads to $h = 0$, because the set $\{x/\|x\|;\ x \in L^*\}$ is obviously dense in $\{x \in R^n;\ \|x\| = 1\}$. Q.E.D.

3. Conformal deformations

In this section we restrict our study to the conformal deformation,

$$g(t) = \mathrm{e}^{2\rho(t)} g_0, \quad \text{with } \rho(0) \equiv 0. \tag{3.1}$$

Set $\sigma(t) = \rho'(t)$, and straightforward calculation gives

$$a_1''(t) = \tfrac{1}{3}(n-1)(n-2) \int_M \sigma \Delta \sigma \, \mathrm{d}V(g(t)) + \tfrac{1}{6}(n-2)^2 \int_M \tau \sigma^2 \, \mathrm{d}V(g(t)) + \tfrac{1}{6}(n-2) \int_M \tau \frac{\partial \sigma}{\partial t} \, \mathrm{d}V(g(t)). \tag{3.2}$$

LEMMA 3.1: *Assume that (M, g_0) has a constant scalar curvature. If $g(t)$ is a volume-preserving deformation, we have*

$$a_1''(0) = \tfrac{1}{3}(n-1)(n-2) \int_M \sigma \Delta \sigma \, \mathrm{d}V(g_0) - \tfrac{1}{3}(n-2)\tau \int_M \sigma^2 \, \mathrm{d}V(g_0). \tag{3.3}$$

PROOF: It is easy to see that $a_0''(0) = 0$ is written as

$$\int_M \frac{\partial \sigma}{\partial t} \, \mathrm{d}V(g_0) + n \int_M \sigma^2 \, \mathrm{d}V(g_0) = 0.$$

By this equation and $\tau = \text{const.}$, (3.2) is led to (3.3). Q.E.D.

From now on, we assume $n = \dim M > 2$.
By virtue of the above lemma we have the following theorem.

THEOREM B: *Let (M, g_0) have a constant scalar curvature and λ_1 be the non-zero first eigenvalue of $\Delta_{g(0)}$. If*

$$\lambda_1 > \frac{\tau}{n-1} \tag{3.4}$$

holds, there is no conformal isospectral deformation of g_0.

PROOF: Let $g(t)$ be a conformal isospectral deformation of g_0. Since $a_0'(0) = 0$, we have $\int_M \sigma \, \mathrm{d}V(g_0) = 0$. Therefore,

$$\int_M \sigma \Delta \sigma \, \mathrm{d}V(g_0) \geq \lambda_1 \int_M \sigma^2 \, \mathrm{d}V(g_0)$$

holds (see [1, p. 186], for example). Accordingly, if (3.4) holds, we have $a_1''(0) > 0$ unless $\sigma = 0$, i.e., $h = 0$. Q.E.D.

The condition (3.4) is obviously satisfied if $\tau \leq 0$. Further, as shown by Obata [9], it is also satisfied for an Einstein space not isometric with a sphere. In the case of a sphere, $\lambda_1 = \tau/(n-1)$, hence $a_1''(0) \geq 0$ holds. The equality holds only when σ is the eigenfunction for λ_1, which is equivalent to $\nabla_i \nabla_j \sigma + \{\tau/n(n-1)\} \, \sigma g_{ij} = 0$ (see [9]). Therefore,

$$h_{ij} = 2\sigma g_{ij} = -\frac{2n(n-1)}{\tau} \nabla_i \nabla_j \sigma.$$

Thus h is trivial.

As a consequence we have the following theorem.

THEOREM C: *Suppose (M, g_0) is an Einstein space, or a space of non-positive constant scalar curvature. Then there is no non-trivial conformal isospectral i-deformation of g_0.*

REMARK: In the case of $\dim M = 2$, it follows from the Gauss-Bonnet formula that the coefficient $a_1(t)$ is invariant.

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