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HYPERSPACES OF NONCOMPACT METRIC SPACES

D.W. Curtis

0. Introduction

For a metric space X , the hyperspace 2^X of nonempty compact subsets and the hyperspace $C(X)$ of nonempty compact connected subsets are topologized by the Hausdorff metric, defined by $\rho(A, B) = \inf\{\epsilon : A \subset N_\epsilon(B) \text{ and } B \subset N_\epsilon(A)\}$. It is easily seen that the hyperspace topologies induced by ρ are invariants of the topology on X . It is known that $2^X \approx Q$, the Hilbert cube, if and only if X is a nondegenerate Peano continuum, and $C(X) \approx Q$ if and only if X is a nondegenerate Peano continuum with no free arcs [6]. In this paper we obtain various characterization theorems for hyperspaces of noncompact connected locally connected metric spaces.

THEOREM 1.6: 2^X is an ANR (AR) if and only if X is locally continuum-connected (connected and locally continuum-connected).

THEOREM 3.3: $2^X \approx Q \setminus \text{point}$ if and only if X is noncompact, connected, locally connected, and locally compact.

THEOREM 4.2: X admits a Peano compactification $\tilde{X}$ such that $(2^{\tilde{X}}, 2^X) \approx (Q, S)$ if and only if X is topologically complete, separable, connected, locally connected, nowhere locally compact, and admits a metric with Property S.

Analogous results are obtained for $C(X)$. Additionally, we discuss two examples relating to local continuum-connectedness, and an example relating to Property S.

1. Hyperspaces which are ANR's

A *growth hyperspace* $\mathcal{G}$ of a metric space X is any closed subspace of 2^X satisfying the following condition: if $A \in \mathcal{G}$ and $B \in 2^X$ such that $B \supset A$ and each component of B meets A , then $B \in \mathcal{G}$. Both 2^X and $C(X)$ are growth hyperspaces of X . Another growth hyperspace of particular interest is $\mathcal{G}_A(X)$, the smallest growth hyperspace containing $A \in 2^X$. Thus $\mathcal{G}_A(X) = \{B \in 2^X : B \supset A \text{ and each component of } B \text{ meets } A\}$. Growth hyperspaces of Peano continua were studied in [4].

LEMMA 1.1: (Kelley [11]). *Let $A, B \in 2^X$ such that $B \in \mathcal{G}_A(X)$ and B has finitely many components. Then there exists a path $\sigma : I \rightarrow \mathcal{G}_A(B)$ such that $\sigma(0) = A$ and $\sigma(1) = B$.*

DEFINITION: A metric space X is *continuum-connected* if each pair of points in X is contained in a subcontinuum. X is *locally continuum-connected* if it has an open base of continuum-connected subsets.

Note that in verifying the local property it is sufficient to produce, for each neighborhood U of a point x , a neighborhood $V \subset U$ of x such that each $y \in V$ is connected to x by a subcontinuum in U . For topologically complete metric spaces, the properties of local connectedness, local continuum-connectedness, and local path-connectedness are equivalent, since every complete connected locally connected metric space is path-connected. Examples given later show that in general these properties are not equivalent.

LEMMA 1.2: *Let $A \in 2^X$, with X a locally continuum-connected metric space. Then for arbitrary $\epsilon > 0$ there exists $\tilde{A} \in \mathcal{G}_A(X)$ such that $\rho(A, \tilde{A}) < \epsilon$ and $\tilde{A}$ has finitely many components.*

PROOF: For each $n \geq 1$ choose $\epsilon_n > 0$ such that, whenever $x \in A$ and $y \in X$ with $d(x, y) < \epsilon_n$, there exists a continuum in X connecting x and y with diameter less than $\min\{1/n, \epsilon\}$. For each n let $A_n \subset A$ be a finite ϵ_n -net for A . Then for each $p \in A_{n+1}$ there exists a continuum L_p in X with diameter less than $\min\{1/n, \epsilon\}$, connecting p and some point of A_n . Set $\tilde{A}_1 = A_1$ and $\tilde{A}_{n+1} = \bigcup \{L_p : p \in A_{n+1}\}$ for each $n \geq 1$. Then $\tilde{A} = cl(\bigcup_1^\infty \tilde{A}_n) = \bigcup_1^\infty \tilde{A}_n \cup A$ has the required properties (note that each component of $\tilde{A}$ meets the finite subset A_1).

LEMMA 1.3: *Let X be a connected and locally continuum-con-*

nected metric space. Then every compact subset is contained in a continuum.

PROOF: Let A be a compact subset of X . There exists by Lemma 1.2 a compact set $\tilde{A} \supset A$ such that $\tilde{A}$ has finitely many components. It is easily seen that X is continuum-connected. Thus the components of $\tilde{A}$ may be connected together by the addition of a finite collection of subcontinua of X , thereby producing a continuum $B \supset \tilde{A} \supset A$.

LEMMA 1.4: *If X is a locally continuum-connected metric space, then every growth hyperspace $\mathcal{G}$ of X is locally path-connected.*

PROOF: Given $A \in \mathcal{G}$ and $\epsilon > 0$, choose $\delta > 0$ such that whenever $x \in A$ and $y \in X$ with $d(x, y) < \delta$, there exists a continuum in X of diameter less than ϵ connecting x and y . We claim that for any $B \in \mathcal{G}$ with $\rho(A, B) < \delta$, there exists a path $\sigma: I \rightarrow \mathcal{G}$ between A and B , with $\rho(A, \sigma(t)) < \epsilon$ for each t . We may assume by Lemmas 1.1. and 1.2 that each of A and B has finitely many components. Adding a finite collection of continua to $A \cup B$, which connect each component of A to B and each component of B to A , and all of which have diameter less than ϵ , we obtain an element $C \in \mathcal{G}_A(X) \cap \mathcal{G}_B(X)$ such that $\rho(A, C) < \epsilon$. Then paths between A and C , and B and C , given by Lemma 1.1, will provide the desired path.

LEMMA 1.5: *Let $\mathcal{D} \subset 2^X$ be compact and connected, and let $A \in \mathcal{D}$. Then $\bigcup \mathcal{D} \in \mathcal{G}_A(X)$.*

PROOF: Clearly, $\bigcup \mathcal{D}$ is a compact subset of X and contains A . We show that each component of $\bigcup \mathcal{D}$ meets A . Let $x \in D \in \mathcal{D}$ be given. For each $\epsilon > 0$ there exists an ϵ -chain $\{D_i\}$ in $\mathcal{D}$ between D and A , and therefore an ϵ -chain $\{q_i\}$ in $\bigcup \mathcal{D}$ between x and some point of A . Since A is compact, there exists $a \in A$ such that for each $\epsilon > 0$, there is an ϵ -chain in $\bigcup \mathcal{D}$ between x and a . Then x and a are in the same quasi-component, hence the same component, of $\bigcup \mathcal{D}$.

THEOREM 1.6: *If X is locally continuum-connected (connected and locally continuum-connected), then every growth hyperspace $\mathcal{G}$ of X is an ANR (AR). Conversely, if there exists a growth hyperspace $\mathcal{G}$ such that $\mathcal{G} \supset C(X)$ and $\mathcal{G}$ is an ANR (AR), then X is locally continuum-connected (connected and locally continuum-connected).*

PROOF: We use the Lefschetz–Dugundji characterization of metric

ANR's [9]: a metric space M is an ANR if and only if, for each open cover α of M , there exists an open refinement β such that every partial β -realization in M of a simplicial polytope K (with the Whitehead topology) extends to a full α -realization of K . Thus, let α be an open cover of $\mathcal{G}$, and assume that the elements of α are open metric balls, with respect to the Hausdorff metric on $\mathcal{G}$. Take an open star-refinement α' of α . By Lemma 1.4 there exists an open refinement β of α' such that each element of β is path-connected. Then every partial β -realization $f: L \rightarrow \mathcal{G}$ of a polytope K extends to a partial α -realization $g: L \cup K^1 \rightarrow \mathcal{G}$, where K^1 is the 1-skeleton of K . Using Lemma 1.5, we may extend g to a full α -realization $h: K \rightarrow \mathcal{G}$ by the following inductive procedure. Consider an n -simplex σ of K , $n \geq 2$, such that h has been defined over $bd\sigma$. Let $r: \sigma \rightarrow C(bd\sigma)$ be any extension of the natural injection $bd\sigma \rightarrow C(bd\sigma)$. Then define h over σ by setting $h(x) = \cup\{h(p): p \in r(x)\}$. Thus $\mathcal{G}$ is an ANR.

If additionally X is connected, then by Lemma 1.3 every compact subset of X is contained in a continuum. Thus for arbitrary $A, B \in \mathcal{G}$, there exists a continuum C containing $A \cup B$, and $C \in \mathcal{G}$. By Lemma 1.1, there exist paths in $\mathcal{G}_A(C)$ from A to C and in $\mathcal{G}_B(C)$ from B to C , hence a path in $\mathcal{G}$ between A and B . Thus $\mathcal{G}$ is path-connected. Since the argument of the preceding paragraph shows that $\mathcal{G}$ is always n -connected for $n \geq 1$, it follows that $\mathcal{G}$ is an AR.

Conversely, suppose there exists a growth hyperspace $\mathcal{G}$ of X such that $\mathcal{G} \supset C(X)$ and $\mathcal{G}$ is an ANR. Let $x \in X$ and a neighborhood U be given. Since $\mathcal{G}$ is locally path-connected, there exists a neighborhood V of x such that for each $y \in V$, there exists a path $f: I \rightarrow \mathcal{G}$ between $\{x\}$ and $\{y\}$ with each $f(t) \subset U$. By Lemma 1.5, $\cup\{f(t): t \in I\} \subset U$ is a continuum. Thus X is locally continuum-connected. And if $\mathcal{G}$ is an AR, and therefore connected, X must also be connected.

The ANR (AR) characterizations for the hyperspaces 2^X and $C(X)$ of a compact metric space X were obtained by Wojdyslawski [15]. These characterizations were extended to complete metric spaces by Tašmetov [13]. Independently, some partial results along these lines were announced by Borges [3].

The following examples show that for noncomplete metric spaces, the property of local continuum-connectedness lies strictly between local connectedness and local path-connectedness.

EXAMPLE 1.7: There exists a connected and locally connected subset of the plane which is not locally continuum-connected.

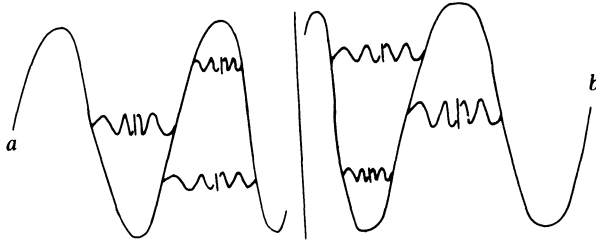
PROOF: There exist disjoint subsets A and B of the plane E^2 such

that every nondegenerate continuum in the plane meets both A and B ([10], p. 110). Thus A contains no nondegenerate subcontinuum, and is not locally continuum-connected. However, A is connected and locally connected. Suppose $A = A_1 \cup A_2$ is a separation. Then there exists a closed subset C of the plane separating A_1 and A_2 . Since C cannot be 0-dimensional, it contains a nondegenerate subcontinuum D . Then D must meet A , impossible. Thus A is connected, and the same argument applied locally shows that A is locally connected.

EXAMPLE 1.8: There exists a connected and locally continuum-connected subset of the plane which is not locally path-connected.

PROOF: We begin with the continuum

$$S = \{(x, \sin 1/x) : 0 < |x| \leq 1/\pi\} \cup \{(0, t) : |t| \leq 1\}.$$



A countable collection $\{S_i\}$ of progressively smaller copies of S is then fitted inside the individual loops of S as indicated, creating local continuum-connectedness on the limit segment $L = \{(0, t) : |t| \leq 1\} \subset S$. Then for each i , a countable collection $\{S_{ij}\}$ of copies of S is similarly fitted inside the loops of S_i . The infinite iteration of this procedure produces the desired space $X = S \cup (\cup \{S_i : i \geq 1\}) \cup (\cup \{S_{ij} : i, j \geq 1\}) \cup \dots$

X is connected and locally continuum-connected. However, X is not locally path-connected at any point on a limit segment such as L . It suffices to show that there exists no path in X between the endpoints $a = (-1/\pi, 0)$ and $b = (1/\pi, 0)$. Suppose there exists such a path σ . Then for some i (in fact, for infinitely many i), σ must contain a subpath σ_i in $S_i \cup (\cup \{S_{ij} : j \geq 1\}) \cup \dots$ between the corresponding endpoints a_i and b_i of S_i . By the same argument σ_i must contain a subpath σ_{ij} in some $S_{ij} \cup (\cup \{S_{ijk} : k \geq 1\}) \cup \dots$ between the endpoints a_{ij} and b_{ij} of S_{ij} . Thus the path σ must pass through each

member of some nested sequence $(S_i, S_{ij}, S_{ijk}, \dots)$. But this is impossible, since the limit point of such a sequence is not included in X .

2. Peano compactifications with locally non-separating remainders

Since $2^Y \approx Q$ for every non-degenerate Peano space Y , one way to study the hyperspace of a noncompact space X is to consider, when possible, a Peano compactification $\tilde{X}$ of X , and the corresponding Q -compactification $2^{\tilde{X}}$ of 2^X . The procedure works if the remainder $\tilde{X} \setminus X$ is sufficiently “nice”. In this section we specify the desired property of the remainder, and establish the conditions under which such a compactification exists.

DEFINITION: A subset A of X is *locally non-separating in X* if, for each nonempty connected open subset U of X , $U \setminus A$ is nonempty and connected.

Note that if A is locally non-separating, so is every subset of A . It is easily shown that if a locally connected space X has a connected open base $\{U_\alpha\}$ such that each $U_\alpha \setminus A$ is nonempty and connected, then A is locally non-separating.

The motivation for considering locally non-separating subsets comes from the following pair of results on positional properties of intersection hyperspaces. For $A_1, \dots, A_n \in 2^X$, we define the intersection hyperspaces $2^X(A_1, \dots, A_n) = \{F \in 2^X : F \cap A_i \neq \emptyset \text{ for each } i\}$ and $C(X; A_1, \dots, A_n) = \{F \in C(X) : F \cap A_i \neq \emptyset \text{ for each } i\}$. For any nondegenerate Peano space X , $2^X(A_1, \dots, A_n) \approx Q$, and $C(X; A_1, \dots, A_n) \approx Q$ if additionally X contains no free arcs [7]. A closed subset F of a metric space Y is a *Z-set* in Y if, for each compact subset K of Y and $\epsilon > 0$, there exists a map $\eta : K \rightarrow Y \setminus F$ with $d(\eta, id) < \epsilon$.

PROPOSITION 2.1: *Let A be a closed subset of a Peano continuum X . Then $2^X(A)$ is a Z-set in 2^X if and only if A is locally non-separating in X . More generally, for closed subsets $A, B_1, \dots, B_n$ of X , $2^X(A, B_1, \dots, B_n)$ is a Z-set in $2^X(B_1, \dots, B_n)$ if and only if A is locally non-separating in X and $B_i \setminus A$ is dense in B_i for each i .*

PROOF: Suppose A satisfies the stated conditions, and let $\epsilon > 0$ be given. We must construct a map $\eta : 2^X(B_1, \dots, B_n) \rightarrow 2^X(B_1, \dots, B_n) \setminus 2^X(A, B_1, \dots, B_n)$ such that $\rho(\eta, id) < \epsilon$. For each i , there exists a finite $\epsilon/3$ -net β_i for B_i such that $\beta_i \subset B_i \setminus A$. By [7], there

exists an “expansion” map $h: 2^X(B_1, \dots, B_n) \rightarrow 2^X(\beta_1, \dots, \beta_n)$ such that $\rho(h, id) \leq \epsilon/3$. And by [8], $2^X(\beta_1, \dots, \beta_n) \approx \text{inv lim } (2^{F_i}(\beta_1, \dots, \beta_n), f_i)$, where $\{F_i\}$ is a sequence of compact connected graphs in X , with each F_i containing $\beta_1 \cup \dots \cup \beta_n$ in its vertex set, and each bonding map $f_i: 2^{F_{i+1}}(\beta_1, \dots, \beta_n) \rightarrow 2^{F_i}(\beta_1, \dots, \beta_n)$ induced by a map $\varphi_i: F_{i+1} \rightarrow C(F_i)$ such that $\varphi_i(b) = \{b\}$ for each $b \in \beta_1 \cup \dots \cup \beta_n$. Thus for some i the projection map $p_i: 2^X(\beta_1, \dots, \beta_n) \rightarrow 2^{F_i}(\beta_1, \dots, \beta_n)$ satisfies $\rho(p_i, id) < \epsilon/3$.

Let $\mathcal{U}$ be a finite cover of F_i by connected open subsets of X with diameters less than $\epsilon/3$. There exists a subdivision SdF_i of F_i such that each simplex of SdF_i is contained in a member of $\mathcal{U}$. To each vertex v of SdF_i we assign a point $\kappa(v) \in \cap \{U \in \mathcal{U} : v \in U\} \setminus A$, with $\kappa(b) = b$ if $b \in \beta_1 \cup \dots \cup \beta_n$. Then κ may be extended to a map $\kappa: SdF_i \rightarrow X \setminus A$ such that, for each simplex σ of SdF_i , $\kappa(\sigma) \subset U \setminus A$ for some $U \in \mathcal{U}$ with $\sigma \subset U$ (we use the fact that each $U \setminus A$ is connected, locally connected, and locally compact, therefore path-connected). Thus $d(\kappa, id) < \epsilon/3$, and the induced map $k: 2^{F_i}(\beta_1, \dots, \beta_n) \rightarrow 2^{X \setminus A}(\beta_1, \dots, \beta_n)$ satisfies $\rho(k, id) < \epsilon/3$. The composition $kp_i h: 2^X(B_1, \dots, B_n) \rightarrow 2^{X \setminus A}(\beta_1, \dots, \beta_n) \subset 2^X(B_1, \dots, B_n) \setminus 2^X(A, B_1, \dots, B_n)$ satisfies $\rho(kp_i h, id) < \epsilon$, and $2^X(A, B_1, \dots, B_n)$ is a Z -set in $2^X(B_1, \dots, B_n)$.

Conversely, suppose the Z -set condition is satisfied. Then each $B_i \setminus A$ must be dense in B_i , otherwise $2^X(A, B_1, \dots, B_n)$ has a nonempty interior in $2^X(B_1, \dots, B_n)$. For each i , choose $b_i \in B_i \setminus A$. Given a neighborhood U of a point $y \in A$, let V be a connected open neighborhood of y such that $\bar{V} \subset U \setminus \{b_1, \dots, b_n\}$. We show that $V \setminus A$ is connected, thus A is locally non-separating. Suppose $V \setminus A = V_0 \cup V_1$ is a separation. There exists a continuum M in V such that $M \cap V_0 \neq \emptyset \neq M \cap V_1$. Let $\mathcal{F} = \{F \in 2^X(M) : F \setminus M = \{b_1, \dots, b_n\}\}$. Then $\mathcal{F}$ is homeomorphic to the connected hyperspace 2^M , and $\mathcal{F} \subset 2^X(B_1, \dots, B_n)$. For each $\epsilon > 0$ there exists a map $\eta: \mathcal{F} \rightarrow 2^X \setminus 2^X(A)$ with $\rho(\eta, id) < \epsilon$. If ϵ is sufficiently small, there exist elements $F_0, F_1 \in \mathcal{F}$ such that $\eta(F_0) \cap V_0 \neq \emptyset$ and $\eta(F_1) \cap V_0 = \emptyset$, and $\eta(F) \cap \partial V = \emptyset$ for every $F \in \mathcal{F}$. Then $\eta(\mathcal{F}) = \{\eta(F) : \eta(F) \cap V_0 \neq \emptyset\} \cup \{\eta(F) : \eta(F) \cap V_0 = \emptyset\}$ is a separation of the connected space $\eta(\mathcal{F})$, impossible.

PROPOSITION 2.2: Let $A, B_1, \dots, B_n$ be closed subsets of a Peano continuum X . Then $C(X; A, B_1, \dots, B_n)$ is a Z -set in $C(X; B_1, \dots, B_n)$ if and only if A is locally non-separating in X and $B_i \setminus A$ is dense in B_i for each i .

PROOF: The argument for obtaining the Z -set property is the exact

parallel of the corresponding argument in the proof of Proposition 2.1. For the converse, suppose $C(X; A, B_1, \dots, B_n)$ is a Z -set in $C(X; B_1, \dots, B_n)$. Actually, we only use the fact that $C(X; A, B_1, \dots, B_n)$ has empty interior in $C(X; B_1, \dots, B_n)$. It is immediate that each $B_i \setminus A$ must be dense in B_i , and $X \setminus A$ must be connected. Thus there exists a connected open set G in $X \setminus A$ such that $G \cap B_i \neq \emptyset$ for each i and $\bar{G} \cap A = \emptyset$. Given a neighborhood U of a point $y \in A$, let V be a connected open neighborhood of y such that $\bar{V} \subset U \setminus \bar{G}$, and choose $\epsilon > 0$ such that $N_\epsilon(\bar{V}) \subset U$. Let $\mathcal{W} = \{V \setminus A, G, W_1, W_2, \dots\}$ be an open cover of $X \setminus A$ such that each W_i is connected and has diameter less than ϵ . By connectedness of $X \setminus A$, we obtain a chain in $\mathcal{W}$ between $V \setminus A$ and G , which in turn leads to connected open sets H and W in $X \setminus A$ such that $H \supset G$, $H \cap V = \emptyset$, $H \cap W \neq \emptyset \neq W \cap V$, and $\text{diam } W < \epsilon$. Then $V \cup W \subset U$ is a connected open neighborhood of y , and we claim that $(V \cup W) \setminus A$ is connected. If there exists a separation $(V \cup W) \setminus A = V_0 \cup V_1$, with the connected set W contained in V_1 , then $(V \cup W \cup H) \setminus A = V_0 \cup (V_1 \cup H)$ is also a separation. However, there exists a continuum K in the connected open set $V \cup W \cup H$ which meets each B_i , and also meets the open sets V_0 and V_1 . Then K is in the interior of $C(X; A, B_1, \dots, B_n)$ in $C(X; B_1, \dots, B_n)$, impossible.

DEFINITION: A metric d for a space X has *Property S* if, for each $\epsilon > 0$, there exists a finite connected cover of X with mesh less than ϵ .

If X admits a metric with Property S, then X is locally connected. Without added conditions, the converse is not true (see Lemma 3.2 and Example 4.3).

DEFINITION: A metric d for a connected space X is *strongly connected* if, for each $x, y \in X$, $d(x, y) = \inf\{\text{diam } M : M \text{ is a connected subset containing } x \text{ and } y\}$.

A convex metric on a Peano continuum is an example of a strongly connected metric. If X admits a strongly connected metric, then X is locally connected. Conversely, the proof of the following lemma shows that every connected, locally connected metric space admits a strongly connected metric.

LEMMA 2.3: *Let X be a connected metric space which admits a metric with Property S. Then X admits a strongly connected metric with Property S.*

PROOF: Let d be a metric with Property S . Define a topologically equivalent metric d^* for X by $d^*(x, y) = \inf\{\text{diam } M : M \text{ is a connected subset of } X \text{ containing } x \text{ and } y\}$. It is easily verified that d^* is a metric function. Since $d^*(x, y) \geq d(x, y)$, every open set with respect to d is open with respect to d^* . The converse is easily established, using the local connectedness of X . And since the diameters of connected subsets are the same with respect to d and d^* , d^* is strongly connected and has Property S .

PROPOSITION 2.4: *A connected metric space X has a Peano compactification $\tilde{X}$ with a locally non-separating remainder $\tilde{X} \setminus X$ if and only if X admits a metric with Property S .*

PROOF: Suppose X admits a metric d with Property S . We may assume by Lemma 2.3 that d is also strongly connected. Then the completion $(\tilde{X}, \tilde{d})$ of (X, d) is the desired Peano compactification. That $(\tilde{X}, \tilde{d})$ is connected and has Property S follows from the same properties for (X, d) . And since a complete, totally bounded metric space is compact, $(\tilde{X}, \tilde{d})$ is a Peano compactification of (X, d) .

Given a nonempty connected open subset U of $\tilde{X}$, we show that the nonempty set $U \cap X$ is connected, thereby verifying that $\tilde{X} \setminus X$ is locally non-separating in $\tilde{X}$. Suppose $U \cap X = H \cup K$ is a separation. Since U is open in $\tilde{X}$, $U \cap X$ is dense in U , and $U \subset \bar{H} \cup \bar{K}$ (the closures are taken in $\tilde{X}$). We must have $\bar{H} \cap \bar{K} \cap U \neq \emptyset$, otherwise $U = (\bar{H} \cap U) \cup (\bar{K} \cap U)$ is a separation. Let $p \in \bar{H} \cap \bar{K} \cap U$. Choose $\delta > 0$ such that the 3δ -neighborhood of p lies in U , and choose points h and k of H and K , respectively, lying in the δ -neighborhood of p . Then $d(h, k) < 2\delta$, and since d is strongly connected there exists a connected subset M of X containing h and k , with $\text{diam } M < 2\delta$. Then M lies in the 3δ -neighborhood of p , therefore in U . Thus $M \subset U \cap X$ is a connected set meeting both H and K , and $H \cup K$ cannot be a separation of $U \cap X$.

Conversely, suppose X has a Peano compactification $\tilde{X}$ such that $\tilde{X} \setminus X$ is locally non-separating. Take any admissible metric $\tilde{d}$ on $\tilde{X}$, and let d be its restriction to X . For every connected open cover $\{U_i\}$ of $\tilde{X}$, $\{U_i \cap X\}$ is a connected cover of X . Since $(\tilde{X}, \tilde{d})$ has finite connected open covers with arbitrarily small mesh, so does (X, d) , and d has Property S .

3. Hyperspaces which are homeomorphic to $Q \setminus \text{point}$

LEMMA 3.1: *Let X be a connected, locally connected metric space, with compact subsets A and B such that $A \subset \text{int } B$. Then only finitely many components of the complement $X \setminus A$ meet $X \setminus B$.*

PROOF: Each component U of $X \setminus A$ must have a limit point in A , otherwise U is both open and closed in X . Thus if $U \setminus B \neq \emptyset$, we must have $U \cap \text{bd} B \neq \emptyset$. Suppose there exists an infinite sequence $\{U_i\}$ of distinct components of $X \setminus A$, each extending beyond B . Choose $y_i \in U_i \cap \text{bd} B$ for each i . By compactness of $\text{bd} B$, we may assume that $y_i \rightarrow y \in \text{bd} B$. Since y has a connected neighborhood in $X \setminus A$, the component of $X \setminus A$ containing y meets U_i for almost all i , contradicting our supposition that the U_i are distinct components.

LEMMA 3.2: *Every connected, locally connected, locally compact metric space admits a metric with Property S .*

PROOF: Let $\tilde{X} = X \cup \infty$ be the one-point compactification of such a space X . Then $\tilde{X}$ is metrizable, since X is separable metric. We claim that for any admissible metric d on $\tilde{X}$, the restriction of d to X has Property S (and therefore $\tilde{X}$ is a Peano continuum). Given $\epsilon > 0$, choose a compact subset $A \subset X$ such that the complement $X \setminus A$ lies in the ϵ -neighborhood of ∞ , and let $B \subset X$ be a compact neighborhood of A . Then by Lemma 3.1, only finitely many components of $X \setminus A$ extend beyond B . Thus a finite connected cover of B with mesh less than ϵ , together with the finite collection of components of $X \setminus A$ extending beyond B , provides a finite connected cover of X with mesh less than ϵ .

THEOREM 3.3: $2^X \approx Q \setminus \text{point}$ if and only if X is a connected, locally connected, locally compact, noncompact metric space. Similarly, $C(X) \approx Q \setminus \text{point}$ if and only if X satisfies the above conditions and contains no free arcs.

PROOF: Suppose X satisfies the stated conditions. By Lemma 3.2, X admits a metric with Property S , and by Proposition 2.4, X has a Peano compactification $\tilde{X}$ with locally non-separating remainder. Since X is locally compact it must be open in its compactification $\tilde{X}$, and the remainder $\tilde{X} \setminus X$ is closed. By Proposition 2.1, the intersection hyperspace $2^{\tilde{X}}(\tilde{X} \setminus X)$ is a Z -set in $2^{\tilde{X}}$. Thus $(2^{\tilde{X}}, 2^{\tilde{X}}(\tilde{X} \setminus X))$ and $(Q \times [0, 1], Q \times \{0\})$ are homeomorphic as pairs, and $2^X = 2^{\tilde{X}} \setminus 2^{\tilde{X}}(\tilde{X} \setminus X)$ is

homeomorphic to $Q \times (0, 1]$, which is homeomorphic to $Q \setminus \text{point}$ (since $\text{Cone } Q \approx Q$).

If in addition X contains no free arcs, then neither does $\tilde{X}$, and the hyperspaces $C(\tilde{X})$ and $C(\tilde{X}; \tilde{X} \setminus X)$ are copies of Q . By Proposition 2.2, $C(\tilde{X}; \tilde{X} \setminus X)$ is a Z -set in $C(\tilde{X})$, and it follows as above that $C(X) \approx Q \setminus \text{point}$.

Conversely, if either 2^X or $C(X)$ is homeomorphic to $Q \setminus \text{point}$, X must be a connected, locally connected metric space by Theorem 1.6. Since X has a closed imbedding into both 2^X and $C(X)$, X must be locally compact. Obviously, X is noncompact, and if $C(X) \approx Q \setminus \text{point}$, X contains no free arcs (otherwise $C(X)$ contains an open 2-cell).

4. Hyperspaces which are homeomorphic to 1^2

With the Hilbert cube Q coordinatized as $\Pi_1^\infty [0, 1]$, let $s = \Pi_1^\infty (0, 1) \subset Q$. Anderson [1] showed that s is homeomorphic to the Hilbert space $1^2 = \{(x_i) \in R^\infty : \sum_1^\infty x_i^2 < \infty\}$. Any subspace P of Q such that $(Q, P) \approx (Q, s)$ is called a *pseudo-interior* for Q , and its complement $Q \setminus P$ is a *pseudo-boundary*. A non-trivial example of a pseudo-boundary is the subset $\Sigma = \{(x_i) \in Q : 0 < \inf x_i \text{ and } \sup x_i < 1\}$. Kroonenberg [12] has given the following characterization for pseudo-boundaries, based on the original characterization by Anderson [2].

LEMMA 4.1: *Let $\{K_i\}$ be an increasing sequence of subsets of Q such that:*

- i) *each $K_i \approx Q$,*
- ii) *each K_i is a Z -set in Q ,*
- iii) *each K_i is a Z -set in K_{i+1} ,*
- iv) *for each $\epsilon > 0$, there exists a map $f : Q \rightarrow K_i$ for some i such that $d(f, id) < \epsilon$.*

Then $\bigcup_1^\infty K_i$ is a pseudo-boundary for Q .

THEOREM 4.2: *The following conditions are equivalent:*

- 1) *X has a Peano compactification $\tilde{X}$ such that $(2^{\tilde{X}}, 2^X) \approx (Q, s)$,*
- 2) *X has a Peano compactification $\tilde{X}$ such that $(C(\tilde{X}), C(X)) \approx (Q, s)$,*
- 3) *X is a topologically complete, separable, connected, locally connected, nowhere locally compact metric space which admits a metric with Property S.*

PROOF: Suppose X satisfies condition 3). Then by Proposition 2.4, X has a Peano compactification $\tilde{X}$ with a locally non-separating remainder. Let $\tilde{d}$ be a convex metric for $\tilde{X}$. Since X is topologically complete and nowhere locally compact, the remainder $\tilde{X} \setminus X$ must be a dense countable union $\bigcup_1^\infty F_i$ of closed, locally non-separating sets in $\tilde{X}$. We may assume that $F_i \subset F_{i+1}$ and F_i has empty interior in F_{i+1} , for each i . This can be arranged inductively as follows. Select a dense sequence $\{x_n\}$ in F_i , a sequence $\{y_n\}$ in $\tilde{X} \setminus F_i$ such that $\tilde{d}(x_n, y_n) < 1/n$ for each n , and a sequence $\{z_n\}$ in $(\tilde{X} \setminus X) \setminus F_i$ such that $\tilde{d}(y_n, z_n) < 1/n$ for each n . Then replace F_{i+1} by the compact set $F_i \cup F_{i+1} \cup \{z_n : n \geq 1\}$.

By Proposition 2.1, each intersection hyperspace $2^{\tilde{X}}(F_i)$ is a Z -set copy of Q in $2^{\tilde{X}}$, and each $2^{\tilde{X}}(F_i) = 2^{\tilde{X}}(F_i, F_{i+1})$ is a Z -set in $2^{\tilde{X}}(F_{i+1})$. Given $\epsilon > 0$, we claim there exists a map $f : 2^{\tilde{X}} \rightarrow 2^{\tilde{X}}(F_i)$ for some i , such that $\rho(f, id) \leq \epsilon$. For $D \in 2^{\tilde{X}}$, define $f(D)$ to be the closed ϵ -neighborhood of D in $\tilde{X}$ (with respect to the convex metric $\tilde{d}$). Suppose $f(2^{\tilde{X}}) \setminus 2^{\tilde{X}}(F_i) \neq \emptyset$ for each i . Then there exists a convergent sequence $y_i \rightarrow y$ in $\tilde{X}$ such that the ϵ -neighborhood of y_i is disjoint from F_i , for each i . It follows that the ϵ -neighborhood of y is disjoint from $\bigcup_1^\infty F_i = \tilde{X} \setminus X$, contrary to the fact that $\tilde{X} \setminus X$ is dense in $\tilde{X}$. Thus by Lemma 4.1, $\bigcup_1^\infty 2^{\tilde{X}}(F_i) = 2^{\tilde{X}} \setminus 2^X$ is a pseudo-boundary for $2^{\tilde{X}}$, and $(2^{\tilde{X}}, 2^X) \approx (Q, s)$.

The proof that $(C(\tilde{X}), C(X)) \approx (Q, s)$ is virtually the same as above, using Proposition 2.2.

Conversely, suppose either condition 1) or 2) is satisfied. Since s is a topologically complete, separable, nowhere locally compact metric AR, X must be a topologically complete, separable, connected, locally connected, nowhere locally compact metric space. We show that the remainder $\tilde{X} \setminus X$ is locally non-separating in $\tilde{X}$. For every connected open subset U of $\tilde{X}$, the hyperspace 2^U is a connected open subset of $2^{\tilde{X}}$. Since the pseudo-boundary $Q \setminus s$ is locally non-separating in Q , $2^{\tilde{X}} \setminus 2^X$ is locally non-separating in $2^{\tilde{X}}$. Thus $2^U \cap 2^X = 2^{U \cap X}$ is connected, and $U \cap X$ is connected. It follows from Proposition 2.4 that X admits a metric with Property S.

The first result of this type, $(2^Q, 2^s) \approx (C(Q), C(s)) \approx (Q, s)$, was obtained by Kroonenberg [12].

Using the very powerful Hilbert space characterization theorem of Toruńczyk [14], the author has recently shown that $2^X \approx C(X) \approx I^2$ for every topologically complete, separable, connected, locally connected, nowhere locally compact metric space X [5]. The following example illustrates the difference between this result and Theorem 4.2.

EXAMPLE 4.3: There exists a space X such that $2^X \approx C(X) \approx 1^2$, but X does not admit a metric with Property S .

PROOF: The space X is a countable union of copies of 1^2 meeting at a single point θ , and given the uniform topology at θ . X may be realized in 1^2 as follows. Let $N = \bigcup_1^\infty \alpha_i$ be a partition of the positive integers, with each α_i infinite, and for each i set $1_i^2 = \{(x_n) \in 1^2 : x_n = 0 \text{ if } n \notin \alpha_i\}$. Then $X = \bigcup_1^\infty 1_i^2 \subset 1^2$. Clearly, X is a closed, connected, locally connected, nowhere locally compact subset of 1^2 , thus $2^X \approx C(X) \approx 1^2$.

The argument that the space X does not admit a metric with Property S is easy. Consider any admissible metric d for X . For some $\delta > 0$, the δ -neighborhood (with respect to d) of θ in X must be contained in the neighborhood $\{x \in X : \|x\| < 1\}$ of θ . Now consider any connected cover of X with mesh less than δ . For each i , any element of the cover intersecting $\{x \in 1_i^2 : \|x\| \geq 1\}$ cannot contain θ , and must therefore lie in $1_i^2 \setminus \theta$. Hence the cover is infinite, and d does not have Property S .

REFERENCES

- [1] R.D. ANDERSON: Hilbert space is homeomorphic to the countable infinite product of lines. *Bull. Amer. Math. Soc.* 72 (1966) 515–519.
- [2] R.D. ANDERSON: On sigma-compact subsets of infinite-dimensional spaces, (unpublished manuscript).
- [3] C.R. BORGES: *Notices Amer. Math. Soc.* 22 (1975) 75T-G118 and 23 (1976), 731–54–10.
- [4] D.W. CURTIS: Growth hyperspaces of Peano continua. *Trans. Amer. Math. Soc.* 238 (1978) 271–283.
- [5] D.W. CURTIS: Hyperspaces homeomorphic to Hilbert space. *Proc. Amer. Math. Soc.* 75 (1979) 126–130.
- [6] D.W. CURTIS and R.M. SCHORI: 2^X and $C(X)$ are homeomorphic to the Hilbert cube. *Bull. Amer. Math. Soc.* 80 (1974) 927–931.
- [7] D.W. CURTIS and R.M. SCHORI: Hyperspaces which characterize simple homotopy type. *Gen. Top. and its Applications* 6 (1976) 153–165.
- [8] D.W. CURTIS and R.M. SCHORI: Hyperspaces of Peano continua are Hilbert cubes. *Fund. Math.* 101 (1978) 19–38.
- [9] J. DUGUNDJI: Absolute neighborhood retracts and local connectedness in arbitrary metric spaces. *Comp. Math.* 13 (1958) 229–246.
- [10] J.G. HOCKING and G.S. YOUNG: *Topology*. Addison-Wesley, 1961, Reading, Mass.
- [11] J.L. KELLEY: Hyperspaces of a continuum. *Trans. Amer. Math. Soc.* 52 (1942) 22–36.
- [12] N. KROONENBERG: Pseudo-interiors of hyperspaces. *Comp. Math.* 32 (1976) 113–131.
- [13] U. TASMETOV: On the connectedness of hyperspaces *Soviet Math. Dokl.* 15 (1974) 502–504.

- [14] H. TORUNCZYK: Characterizing Hilbert space topology (preprint).
[15] M. WOJDYSLAWSKI: Retracts absolus et hyperespaces des continus. *Fund. Math.* 32 (1939) 184–192.

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Added in proof

The condition iv) of the pseudo-boundary characterization Lemma 4.1 is insufficient, and should be replaced by the following condition iv)*: there exists a deformation $h: Q \times [0, 1] \rightarrow Q$, with $h(q, 0) = q$ for each $q \in Q$, such that for each $\epsilon > 0$, $h(Q \times [\epsilon, 1]) \subset K_i$ for some i . In the application of Lemma 4.1 contained in the proof of Theorem 4.2, this stronger condition is easily verified (the map f of $2^{\tilde{X}}$ is replaced by the deformation $h: 2^{\tilde{X}} \times [0, 1] \rightarrow 2^{\tilde{X}}$, where $h(D, t)$ is the closed t -neighborhood of D in $\tilde{X}$).