

# COMPOSITIO MATHEMATICA

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*Compositio Mathematica*, tome 40, n° 1 (1980), p. 79-85

[http://www.numdam.org/item?id=CM\\_1980\\_\\_40\\_1\\_79\\_0](http://www.numdam.org/item?id=CM_1980__40_1_79_0)

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## UNICITY OF THE LIE PRODUCT

Sebastian J. van Strien

### 1. Statement of the result

For a  $C^\infty$  manifold  $M$ ,  $\mathcal{X}(M)$  denotes the linear space of  $C^\infty$  vectorfields on  $M$ . Let  $\chi: \mathcal{X}(M) \times \mathcal{X}(M) \rightarrow \mathcal{X}(M)$  be a bilinear operator, defined for every  $n$  dimensional manifold  $M$ . This operator is called natural if for every smooth open embedding  $f: N \rightarrow M$  the following diagram commutes:

$$\begin{array}{ccc}
 \mathcal{X}(M) \times \mathcal{X}(M) - \chi & \rightarrow & \mathcal{X}(M) \\
 \downarrow f^* \times f^* & & \downarrow f^* \\
 \mathcal{X}(N) \times \mathcal{X}(N) - \chi & \rightarrow & \mathcal{X}(N)
 \end{array}$$

where  $M, N$  are  $C^\infty$  manifolds and  $f^*$  is the composition  $\mathcal{X}(M) \xrightarrow{r} \mathcal{X}(f(N))$

$\xrightarrow{(f^{-1})^*} \mathcal{X}(N)$ ,  $r$  the restriction operator, i.e.  $f^*X(x) = df(x)^{-1}(X(f(x)))$  for  $X \in \mathcal{X}(M)$ . In this note I shall prove that the Lie-product ( $[X, Y] = X \cdot Y - Y \cdot X$  for  $X, Y \in \mathcal{X}(M)$ ) is characterised by this property:

**THEOREM:** *Let  $\chi$  be a bilinear natural operator in the above sense, then there exists a constant  $\lambda \in \mathbb{R}$  such that  $\chi(X, Y) = \lambda \cdot [X, Y]$ , for all  $X, Y \in \mathcal{X}(M)$ .*

Palais and others [3], [4], [5] prove analogous results for operations on differential forms. Peetre [6] has a similar characterisation of linear (not bilinear) differential operators. The formal techniques are similar to those in [7]. I am indebted to my supervisor Prof. Floris Takens, for suggesting the problem and for his encouragement.

## 2. The proof

The naturality of  $\chi$  implies that it is a local operator, i.e. for  $U$  open in  $M$

$$\chi(X, Y) \Big|_U = \chi(X|_U, Y|_U).$$

Furthermore if  $U, V \subset M$ ,  $U, V$  diffeomorphic and  $\chi(X, Y) = \lambda \cdot [X, Y]$  for some constant and all  $X, Y \in \mathfrak{X}(U)$ , then also  $\chi(X, Y) = \lambda \cdot [X, Y]$  for all  $X, Y \in \mathfrak{X}(V)$ . Therefore I may assume  $M = \mathbb{R}^n$ . It is sufficient to prove

$$\chi(X, Y)(0) = \lambda \cdot [X, Y](0), \quad \forall X, Y \in \mathfrak{X}(\mathbb{R}^n),$$

because  $\chi$  commutes with translations. Of course naturality implies

$$(1) \quad f_*\chi(X, Y)(0) = \chi(f_*X, f_*Y)(0)$$

for every diffeomorphism  $f$  and every  $X, Y \in \mathfrak{X}(\mathbb{R}^n)$ .

The main step in the proof is  $\chi(X, Y)(0) = \chi(j^1X(0), j^1Y(0))(0)$ . (Where, for  $s \in \mathbb{N}$ ,  $j^sX(p)$  is the polynomial vectorfield of degree  $s$  corresponding to the  $s$ -jet of  $X$  in  $p$ , that is, the first  $s$  terms of the Taylor expansion of  $X$  in  $p$ .) In lemma 1 I use naturality to prove this for polynomial vectorfields. In lemmas 2 and 3 this is shown for arbitrary smooth vectorfields, by proving  $\chi(X, Y)(0) = 0$  if  $X(p)$  or  $Y(p)$  has in  $p = 0$  a zero of sufficiently high order.

In lemma 4 I show that there exist constants  $\gamma_1, \dots, \gamma_4$  such that:

$$\begin{aligned} \chi(X, Y)(0) = & \gamma_1 \cdot \nabla_X Y(0) + \gamma_2 \cdot \nabla_Y X(0) + \gamma_3 \cdot ((\operatorname{div} Y)(0)) \cdot X(0) \\ & + \gamma_4 \cdot ((\operatorname{div} X)(0)) \cdot Y(0). \end{aligned}$$

$$\left( \text{Where } \nabla_X Y = \sum X_j \frac{\partial Y_i}{\partial x_j} \frac{\partial}{\partial x_i}, \text{ if } X = \sum X_i \frac{\partial}{\partial x_i}, Y = \sum Y_i \frac{\partial}{\partial x_i}. \right)$$

In these lemmas I use the naturality property, but only with affine diffeomorphisms  $f$  in equation (1).

Finally in the proof of the theorem one needs non-linear diffeomorphisms  $f$  in (1) to show that the constants  $\gamma_1, \dots, \gamma_4$  satisfy  $\gamma_1 = -\gamma_2$ ,  $\gamma_3 = \gamma_4 = 0$ ; i.e.:  $\chi(X, Y) = \gamma_1[X, Y]$ .

LEMMA 1: *For monomial vectorfields*

$$X(x_1, \dots, x_n) = x_1^{\alpha_1} \dots x_n^{\alpha_n} \frac{\partial}{\partial x_i},$$

$$Y(x_1, \dots, x_n) = x_1^{\beta_1} \dots x_n^{\beta_n} \frac{\partial}{\partial x_j},$$

$\chi(X, Y)(0) = 0$  if  $\sum \alpha_i + \sum \beta_i \neq 1$ .

PROOF: Let

$$\begin{aligned} \chi(X, Y)(0) &= \chi \left( x_1^{\alpha_1} \dots x_n^{\alpha_n} \frac{\partial}{\partial x_i}, x_1^{\beta_1} \dots x_n^{\beta_n} \frac{\partial}{\partial x_j} \right) (0) \\ &= c_1 \frac{\partial}{\partial x_1} \Big|_0 + \dots + c_n \frac{\partial}{\partial x_n} \Big|_0. \end{aligned}$$

Define a diffeomorphism by  $\Phi(x) = \lambda \cdot x$ ,  $\lambda \neq 0$ . Then

$$\Phi_* X = \lambda^{-\sum \alpha_i + 1} \cdot X, \quad \Phi_* Y = \lambda^{-\sum \beta_i + 1} \cdot Y,$$

hence, using (1),

$$\Phi_*(\chi(X, Y)) = \chi(\Phi_* X, \Phi_* Y) = \lambda^{-\sum \alpha_i - \sum \beta_i + 2} \cdot \chi(X, Y).$$

However, the left side at 0 is equal to  $\lambda \cdot \chi(X, Y)(0)$ . This proves the lemma.

LEMMA 2: *For  $X$  a  $C^\infty$  vectorfield, there exists a  $C^\infty$  vectorfield  $\bar{X}$  and sequences  $p_s \rightarrow 0$ ,  $q_s \rightarrow 0$  such that: (1)  $\bar{X} \mid U_s = (X - j^s X(p_s)) \mid U_s$ ,  $U_s$  a neighbourhood of  $p_s$ . (2)  $\bar{X} \mid V_s = 0$ ,  $V_s$  a neighbourhood of  $q_s$ .*

PROOF: In fact lemmas 2 and 3 use a classical theorem of E. Borel and a technique of J. Peetre [6]. Take for example  $p_s = (1/s, 0, \dots, 0)$ ,  $q_s = -p_s$ . Define a smooth function  $\alpha: \mathbb{R}^n \rightarrow \mathbb{R}$  such that

$$\alpha(x) = \begin{cases} 1 & \text{for } \|x\| \leq \frac{1}{2} \\ 0 & \text{for } \|x\| \geq 1. \end{cases}$$

and  $\bar{X}_s = X - j^s X(p_s)$ , for every  $s \in \mathbb{N}$ . Choose  $\epsilon_s > 0$  so small that

$$\frac{1}{s+1} + \epsilon_{s+1} < \frac{1}{s} - \epsilon_s$$

so that

$$\left\| \alpha \left( \frac{x - p_s}{\epsilon_s} \right) \tilde{X}_s \right\|_i < 2^{-i} \quad \text{for } i = 1, \dots, s-1;$$

$$\|f\|_i = \sup_{\substack{|p|=i \\ x \in \mathbb{R}^n}} |D^p f(x)|.$$

Then

$$\bar{X} = \sum_s \alpha \left( \frac{x - p_s}{\epsilon_s} \right) \tilde{X}_s$$

converges and has the desired properties.

LEMMA 3: For all  $X, Y \in \mathcal{X}(\mathbb{R}^n)$

$$\chi(X, Y)(0) = \chi(j^1 X(0), j^1 Y(0))(0).$$

PROOF: This lemma can also be immediately deduced from Peetre [6], because  $X \rightarrow \chi(X, Y)$  is a local linear operator. But for the sake of completeness an elementary proof will be given here. Take  $\bar{X}, \bar{Y}$  as in lemma 2.

First I shall prove that for all  $p \in \mathbb{R}^n$ :

$$(2) \quad \chi \left( x_1^{\alpha_1} \dots x_n^{\alpha_n} \frac{\partial}{\partial x_i}, Y \right) (p) = \chi \left( x_1^{\alpha_1} \dots x_n^{\alpha_n} \frac{\partial}{\partial x_i}, j^1 Y(p) \right) (p).$$

$\chi(Z, \bar{Y})(0) = 0$  for every  $Z \in \mathcal{X}(\mathbb{R}^n)$ , because  $\chi(Z, \bar{Y})(q_s) = 0$  and  $q_s \rightarrow 0$ . Furthermore for  $a = (a_1, \dots, a_n)$   $x_1^{\alpha_1} \dots x_n^{\alpha_n}$  can be considered as a polynomial in  $x_1 - a_1, \dots, x_n - a_n$  and using lemma 1:

$$\chi \left( x_1^{\alpha_1} \dots x_n^{\alpha_n} \frac{\partial}{\partial x_i}, j^t Y(p_s) \right) (p_s) = \chi \left( x_1^{\alpha_1} \dots x_n^{\alpha_n} \frac{\partial}{\partial x_i}, j^1 Y(p_s) \right) (p_s)$$

for every  $t \in \mathbb{N}$ .

But, for every  $s$ ,  $j^1 Y(p_s)$  is a linear combination of

$$\frac{\partial}{\partial x_j}, x_k \frac{\partial}{\partial x_l}, j, k, l = 1, 2, \dots, n.$$

Since any linear operator on a finite dimensional vectorspace is

continuous:

$$\chi \left( x_1^{\alpha_1} \dots x_n^{\alpha_n} \frac{\partial}{\partial x_i}, j^1 Y(p_s) \right) (p_s) \rightarrow \chi \left( x_1^{\alpha_1} \dots x_n^{\alpha_n} \frac{\partial}{\partial x_i}, j^1 Y(0) \right) (0),$$

for  $s \rightarrow \infty$ . This together implies that the limit of:

$$\begin{aligned} \chi \left( x_1^{\alpha_1} \dots x_n^{\alpha_n} \frac{\partial}{\partial x_i}, Y \right) (p_s) &= \chi \left( x_1^{\alpha_1} \dots x_n^{\alpha_n} \frac{\partial}{\partial x_i}, \bar{Y} \right) (p_s) \\ &\quad + \chi \left( x_1^{\alpha_1} \dots x_n^{\alpha_n} \frac{\partial}{\partial x_i}, j^1 Y(p_s) \right) (p_s) \end{aligned}$$

is

$$\chi \left( x_1^{\alpha_1} \dots x_n^{\alpha_n} \frac{\partial}{\partial x_i}, j^1 Y(0) \right) (0).$$

Translation gives (2) for any  $p \in \mathbb{R}^n$ . Therefore:

$$\chi(X, Y)(p_s) = \chi(\bar{X}, \bar{Y})(p_s) + \chi(j^1 X(p_s), j^1 Y(p_s))(p_s)$$

which goes to  $\chi(j^1 X(0), j^1 Y(0))(0)$  for  $s \rightarrow \infty$ .

Compare this with the proof of the continuity of local operators on vectorfields in [2], XVIII. 13. problem 1.

LEMMA 4: *There are constants  $\gamma_1, \dots, \gamma_4$  such that*

$$\begin{aligned} \chi(X, Y)(0) &= \gamma_1 \cdot \nabla_X Y(0) + \gamma_2 \cdot \nabla_Y X(0) + \gamma_3 \cdot ((\operatorname{div} Y)(0)) \cdot X(0) \\ &\quad + \gamma_4 \cdot ((\operatorname{div} X)(0)) \cdot Y(0). \end{aligned}$$

PROOF: Lemmas 1 and 3 imply that  $\chi$  can be written as:

$$(3) \quad \chi(X, Y)(0) = (M_1(dY(0))) \cdot X(0) + (M_2(dX(0))) \cdot Y(0)$$

with  $M_i$  linear maps from the  $(n \times n)$ -matrices to the  $(n \times n)$ -matrices. The lemma is proved when I show that for certain constants  $\gamma_1, \gamma_3$ :

$$(4) \quad M_1(A) = \gamma_1 \cdot A + \gamma_3 \cdot \operatorname{Tr}(A) \cdot I.$$

for all matrices  $A$ .

Now take  $Y(0) = 0$ ,  $A = dY(0)$ ,  $f(x) = L \cdot x$  ( $L$  a linear invertible map) and use naturality (1) in equation (3). This implies:

$$(5) \quad M_1(L^{-1} \cdot A \cdot L) = L^{-1} \cdot M_1(A) \cdot L.$$

Let  $L$  run over all diagonal and permutation matrices and deduce from (5) that there exist constants  $\gamma_1, \gamma_3$  such that (4) is true for all diagonal matrices  $A$ . Therefore (4) is true for all diagonalisable matrices  $A$ . But every matrix is a sum of diagonalisable matrices. This proves (4).

*Proof of the theorem*

a) The constants in lemma 4 satisfy  $\gamma_1 = -\gamma_2, \gamma_3 = -\gamma_4$ . That is:

$$(6) \quad \chi(X, Y) = \gamma_1 \cdot [X, Y] + \gamma_3 \cdot ((\operatorname{div} Y)X - (\operatorname{div} X)Y).$$

To show this, it is now sufficient to prove  $\chi$  is antisymmetric, i.e. that  $\chi(X, X) = 0 \forall X \in \mathcal{X}(\mathbb{R}^n)$ .

If  $X(0) = 0$ , then lemmas 1 and 3 give  $\chi(X, X)(0) = 0$ .

If  $X(0) \neq 0$  the flow-box theorem [1] gives a local diffeomorphism  $\varphi: (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$  such that

$$\varphi_* X(0) = \frac{\partial}{\partial x_1}: \quad \varphi_* \chi(X, X)(0) = \chi\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}\right)(0) = 0$$

and again  $\chi(X, X)(0) = 0$ .

b) For  $n = 1$ :  $(\operatorname{div} Y)X - (\operatorname{div} X)Y = [X, Y]$  and we are done.

c) If  $n \geq 2$ : The operator  $(X, Y) \rightarrow (\operatorname{div} Y)X - (\operatorname{div} X)Y$  does not commute with every diffeomorphism  $\varphi$  and therefore  $\gamma_3 = 0$  in equation (6). To see this, take

$$X = \sum_{i=1}^{\alpha} X_i(x_1, \dots, x_{\alpha}) \frac{\partial}{\partial x_i}, \quad Y = \sum_{i=\alpha+1}^n Y_i(x_{\alpha+1}, \dots, x_n) \frac{\partial}{\partial x_i}$$

and a (non-measure preserving) local diffeomorphism  $\tilde{\varphi}_1: (\mathbb{R}^{\alpha}, 0) \rightarrow (\mathbb{R}^{\alpha}, 0)$  such that  $\operatorname{div} X(0) = \operatorname{div} Y(0) = 0$ ,  $Y(0) \neq 0$  and  $\operatorname{div}((\varphi_1)_* X) \neq 0$ .

Define  $\varphi(x_1, \dots, x_n) = (\tilde{\varphi}_1(x_1, \dots, x_{\alpha}), x_{\alpha+1}, \dots, x_n)$ , then  $\operatorname{div}(\varphi_* X)(0) \neq 0, \operatorname{div}(\varphi_* Y) = 0$ , that is  $(\operatorname{div} Y)X - (\operatorname{div} X)Y$  does not commute with this  $\varphi$ .

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(Oblatum 26-I-1978 & 4-X-1978)

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