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ORDINAL INVARIANTS IN TOPOLOGY–II

Sequential Order of compactifications

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Abstract

We prove a general theorem (using CH) from which the following result is deduced; If X is any noncompact zero-dimensional separable metrizable space and if α is any countable ordinal number, there is a sequential compactification of X with sequential order α. In particular, there exists a compact Hausdorff sequential space with any pre-fixed ordinal $\alpha \leq \omega_1$ as its sequential order.

This is then used to answer completely a question posed by Arhangel'skii and Franklin concerning the interrelations of sequential order and k-order. It is also used to give a complete answer to a question of Rajagopalan regarding sequential order and the spaces S_n.

The main corollary (Corollary 5 in §4) was partially announced in [3]. This answers a question of Arhangelskii and Franklin posed in [1]. It has also been announced in [4] in a weaker form, together with answers to many other questions of [1]. This paper is a sequel to [3] wherein two previous questions of [1] have been answered. The answer to the question posed by Rajagopalan in [7] obtained here (§4, corollary 6) has been announced in [5]. Further, corollary 1 in §4 answers Problem C.3 of [6; page 218].

The construction technique described in the first three sections of this paper, is by itself interesting. In addition to the positive results (Theorem of §3 and Corollaries in §4) proved by using it, we believe that it may be useful for providing interesting counter-examples. The nucleus of its idea is due to Isbell; see the space ψ described in [2, page 79]. Roughly, we may describe that our technique does to topological spaces lying in a large class, what Isbell's method does to discrete spaces.

The paper is divided into four short sections. In §1, we describe this

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construction in such a generality that is neither too complicated nor too weak. In §2 we append to this construction by giving a specific way, so that it may be easy to compute the sequential order of the extension, that is thus constructed. In §3 the main result is proved in a fairly general form. The last section concerns the three posed questions.

The notions like sequential spaces, sequential order, k-order, spaces S_n, scattered spaces, derived length, etc. are not defined here, though they should have been. The reader is referred to [3] or [6] for their definitions and properties.

§1

Let X be a space satisfying the following six conditions:

i) locally compact
ii) Hausdorff
iii) zero-dimensional
iv) sequential
v) every countably compact closed subset is compact.
vi′) not pseudocompact.

These conditions, though numerous, are not stringent; there are plenty of spaces satisfying them. For example, X may be any space obtained by deleting a limit point from a compact zero-dimensional metrizable space.

We now describe a process of constructing an extension X_* of X which retains all the above six properties and which, as we shall see later, helps to increase the sequential order of X without losing these fair properties.

In the presence of zero-dimensionality, it is known that vi) is equivalent to

vi) There exists a pairwise disjoint countably infinite family of clopen (that is, both closed and open) subsets of X covering X.

It is this form of vi) that will be more helpful for us in the sequel. In view of this, we have a decomposition of our spaces X in the form $X = \Sigma_{n=1}^{\infty} X_n$ where each X_n is a space satisfying the six conditions i) to vi).

Now we fix a positive integer n and consider X_n. We construct an extension $X_n(1)$ of X in a special way. For this, we write $X_n = \Sigma_{i=1}^{\infty} X_{n;i}$, where each $X_{n;i}$ satisfies i) to vi). This is done, by doing to X_n, what we did above for X.

We start with a family $\underline{F}$ of subsets of X_n satisfying the five conditions A) to E) to be mentioned below. The extension $X_n(1)$ to be constructed below, should be, for a greater clarity, denoted as $X_n(1, \underline{F})$, because it depends on the family $\underline{F}$. When $\underline{F}$ is changed to some other family $\underline{F}'$ (satisfying those five conditions) the extension $X_n(1, \underline{F}')$ is different from $X_n(1, \underline{F})$; in fact, at times it may even happen that the two extensions are non-homeomorphic. See [8]. Notwithstanding this, for convenience in notation, we omit the mention of the family and simply write the extension as $X_n(1)$.

The conditions imposed on $\underline{F}$ are the following:

A) Every member of $\underline{F}$ is clopen in X_n.

B) Any two distinct members of $\underline{F}$ intersect in a compact set.

C) $X_{n,i}$ belongs to $\underline{F}$ for every positive integer i.

D) No member of $\underline{F}$ is compact.

E) $\underline{F}$ is maximal with respect to the above four properties.It will be useful to note that E) can be equivalently stated as

E′) If V is a noncompact clopen subset of X_n and if $V \cap W$ is compact for every W in $\underline{F}$, then V is in $\underline{F}$.

The existence of such a family can be proved by an easy application of Zorn's lemma.

Then we let the disjoint union of X_n and $\underline{F}$ be the underlying set of the space $X_n(1)$ that we now define. If V is an element of $\underline{F}$, it is on one hand an element of $X_n(1)$ and on the other hand a subset of X_n and hence of $X_n(1)$. This may create some confusion in our later discussions. To avoid this, we use the notation V^* for V, when it is considered as an element of $X_n(1)$; we simply write it as V when we consider it as a subspace of X_n or $X_n(1)$. Thus for example,

$$X_n(1) = X_n \cup \{V^*: V \in E\}.$$

Now we describe the topology of $X_n(1)$. It is best described in terms of neighbourhoods. If x is in X_n, then its neighbourhood system in X_n is declared as its neighbourhood base in $X_n(1)$ also. If V^* is in $X_n(1)$, a neighbourhood base at V^* is given by

$$\{\{V^*\} \cup V \backslash K : K \text{ compact in } X_n.\}$$

It can be shown that what we consider, is the unique topology on $X_n(1)$ such that the following four conditions hold:

a) every compact set is closed.

b) X_n is open in $X_n(1)$ retaining its topology.

c) for every V in $\underline{F}$, the set $\{V^*\} \cup V$ is compact.

d) for every V in $\underline{F}$, the set $\{V^*\} \cup V$ is open in $X_n(1)$.

It is also describable as the strongest topology on $X_n(1)$ satisfying the conditions a), b) and c) given above.

We finally let

$$X_* = \sum_{n=1}^{\infty} X_n(1).$$

We shall presently show that X_* satisfies the six conditions i) to vi) mentioned in the beginning.

Claim 1: X_* is locally compact.

Because of conditions C) and D) we have $\{V \cup \{V^*\}: V \in \underline{F}\}$ is an open cover for $X_n(1)$; by C), every member of this open cover is compact. Hence $X_n(1)$ is locally compact. It follows that X_* is also so.

Claim 2: X_* is Hausdorff.

It suffices to prove that each $X_n(1)$ is Hausdorff. Let x and y be distinct elements of $X_n(1)$ and let us consider three cases.

Suppose both x and y are in X_n. Since X_n is open in $X_n(1)$ retaining its topology and since the topology of X_n is Hausdorff, x and y can be separated by disjoint open sets in $X_n(1)$.

Let x and y both belong to $\underline{F}$. Let them be V^* and W^*, where V and W are in $\underline{F}$. Then $\{V^*\} \cup (V \backslash W)$ and $\{W^*\} \cup (W \backslash V)$ are disjoint neighbourhoods of x and y respectively. Here we use condition B) to obtain that $V \cap W$ is compact.

Suppose $x \in X_n$ and $y = V^*$ for some V in $\underline{F}$. Here we consider two subcases. If x is in V, choose any compact neighbourhood K of x in X_n; then $\{V^*\} \cup V \backslash K$ and K are disjoint neighbourhoods of y and x respectively. If x is not in V, then $\{V^*\} \cup V$ and $X \backslash V$ are disjoint neighbourhoods of y and x respectively.

Claim 3: X_* is zero-dimensional.

As before, it suffices to prove that each $X_n(1)$ is zero-dimensional. First, we observe that if $V \in F$, then $\{V^*\} \cup V$ is clopen in $X_n(1)$. Next one can prove that $\{\{V^*\} \cup V: V \in \underline{F}\} \cup$ (any base for X_n) is a base for $X_n(1)$. Since X_n is zero-dimensional and locally compact, it admits a base of compact open sets. It follows that $X_n(1)$ is zero-dimensional.

Claim 4: X_* is sequential.

We first prove that for a locally compact Hausdorff space Y the following are equivalent:

1) the one-point-compactification of Y is sequential.

2) Y is sequential and every countably compact closed subset of Y is compact.

Let (1) hold and let A be a countably compact closed subset of Y. Let ∞ be the extra point in the one-point-compactification of Y. If a sequence in A converges to ∞, then the set of this sequence is infinite, discrete and closed in A, contradicting the countable compactness of A. Hence A is sequentially closed in $Y \cup \{\infty\}$ and hence closed by 1). Therefore A is compact. Y is sequential, since sequentialness is open-hereditary.

Conversely, let 2) hold and let $Y \cup \{\infty\}$ be the one-point-compactification of Y. Let A be sequentially closed in $Y \cup \{\infty\}$. If $A \backslash \{\infty\}$ is countably compact, then by assumption, it is compact. Clearly then, A is closed in $Y \cup \{\infty\}$. If $A \backslash \{\infty\}$ is not countably compact, it contains an infinite discrete subset B closed in Y. Then B can be shown to contain the set of a sequence converging to ∞. Since A is sequentially closed, ∞ is in A. Also $A \backslash \{\infty\}$ is sequentially closed in Y and hence closed in Y. Thus in both cases, A is closed in $Y \cup \{\infty\}$.

We next observe that 2) is closed-hereditary. In particular, because X_n satisfies it, it holds for V for every V in $\underline{F}$. Therefore $\{V^*\} \cup V$ is sequential. Thus there is an open cover for $X_n(1)$ every member of which, is sequential. Hence $X_n(1)$ itself for each n, and hence X_*, is sequential.

Claim 5: Every countably compact closed subset of X_* is compact.

Let F be a countably compact closed subset of X_*. Then $F \cap X_n(1)$ is nonempty only for a finite number of values of n. Hence, to prove the compactness of F, it suffices to prove that $F \cap X_n(1)$ is compact, for a general n. Now $F \cap X_n(1)$ is a countably compact closed subset of $X_n(1)$. It is easily seen that $X_n(1) \backslash X_n$ is discrete and closed. Therefore so is the set $(F \cap X_n(1)) \backslash X_n$. On the other hand it is countably compact. Therefore it is finite. Since $X_n(1)$ is locally compact and zero-dimensional (see claims 1 and 3), we can choose a compact open set W containing this finite set. Then $(F \cap X_n(1)) \backslash W$ is countably compact and closed in X_n. Therefore, by condition v) it is compact. Now $F \cap X_n(1)$ is a closed subset of the compact set $W \cup ((F \cap X_n(1) \backslash W)$. Therefore it is compact.

Claim 6: X_* is not pseudocompact.

By its very definition, X_* is a disjoint sum of infinite number of spaces. Hence it satisfies vi′) which is equivalent to non-pseudocompactness among zerodimensional Hausdorff spaces.

Before proceeding to the next section, we want to record the following fact: X is a dense subspace of X_*. Therefore the one point compactification of X_* is a compactification of X also. This compactification satisfies the five conditions i) to v).

§2

We assume continuum hypothesis, throughout the remaining sections of the paper. Under one more condition on X, we prove now that the family $\underline{F}$ discussed in §1, can be so chosen that the sequential closure of X will be contained in X_* in any Hausdorff extension of X_*. This result is needed for our later purposes. c denotes the cardinality of the continuum, ω the first infinite ordinal number and ω_1 the first uncountable ordinal number and N the set of all natural numbers.

LEMMA 1: *Let X be a space satisfying the seven conditions i) to vii) where i) to vi) are the ones listed in* §1 *and the last one is: vii) its cardinality is $\leq c$.*
Let $X = \Sigma_{n=1}^{\infty} X_n$ be a decomposition of X into a countably infinite number of noncompact clopen subsets. Such a decomposition always exists. See vi′) *in* §1. *Then there is a family $\underline{F}$ of subsets of X satisfying the following five conditions*:

A′) *Every member of $\underline{F}$ is clopen in X.*

B′) *Any two distinct members of $\underline{F}$ intersect in a compact set.*

C′) *X_n is in $\underline{F}$ for every positive integer n.*

D′) *No member of $\underline{F}$ is compact.*

F) *Every infinite discrete closed subset of X meets some member of $\underline{F}$ in an infinite set.*

[Note: The first four conditions are analogues of A), B), C), D) of §1. The new condition F) is crucial.]

PROOF: Let $\underline{C}$ be the collection of all infinite subsets A of X such that $A \cap X_n$ contains at most one element for every $n \in N$. Then $\underline{C}$ is contained in the family of all countable subsets of X. Hence $|\underline{C}| \leq c$, because $|X| \leq c$. On the other hand if we fix a point x_n in X_n for every n in N, then $\{x_n : n \in B\}$ is a member of $\underline{C}$ whenever $B \subset N$ is infinite. Since the set of all infinite subsets of N has cardinality c, it follows that $|\underline{C}| \geq c$. Thus we have $|\underline{C}| = c$.

Let $A_1, A_2 \ldots, A_\alpha, \ldots$ be a well-ordering of $\underline{C}$ to the order type of ω_1. We now define sets $W_1, W_2, \ldots, W_\alpha, \ldots$ by induction.

Look at A_1. For every positive integer n such that $A_1 \cap X_n$ is nonempty, choose a compact open neighbourhood $W_{1,n}$ of the unique point of $A_1 \cap X_n$ such that $W_{1,n} \subset X_n$. Let then $W_1 = \cup \{W_{1,n} : n \in N, A_1 \cap X_n \neq \phi\}$.

Suppose α is a countable ordinal number and suppose that we have defined clopen subsets W_β for every $\beta < \alpha$. To define W_α, look at A_α and consider two cases. If $A_\alpha \cap W_\beta$ is infinite for some $\beta < \alpha$, then let $W_\alpha = W_\beta$ for the least such β. If $A_\alpha \cap W_\beta$ is finite for every $\beta < \alpha$, we proceed to define W_α as indicated below. Because α is countable, it is possible to rewrite the members of the set $\{W_\beta : \beta < \alpha\}$ in the form of a (finite or infinite) sequence $W^1, W^2, \ldots, W^n, \ldots$ [We have only to set up a bijection between α and ω; it does not matter what this bijection is; we omit the repetitions in forming this sequence.] Using the two facts (a) $A_\alpha \cap W^n$ is finite for every n in N and (b) $A_\alpha \cap X_n$ is nonempty for an infinity of values of n, we can construct a strictly increasing sequence $r_1, r_2, \ldots r_n, \ldots$ of natural numbers such that (c) $A_\alpha \cap W^n \cap X_r$ is empty for every $r \geq r_n$ and (d) $A_\alpha \cap X_{r_n}$ is nonempty. Then for every natural number i, we choose a compact open subset $W_{\alpha,i}$ of X_{r_i} such that

$$A_\alpha \cap X_{r_i} \subset W_{\alpha,i} \subset X_{r_i} \setminus \left(\bigcup_{j=1}^{i} W_j \right).$$

Such a choice is possible because, $A_\alpha \cap X_{r_i}$ is a singleton set disjoint with the clopen set $\cup_{j=1}^{i} W_j$ and contained in the clopen set X_{r_i}. Finally let $W_\alpha = \cup_{i=1}^{\infty} W_{\alpha,i}$.

Having defined W_α thus for every $\alpha < \omega_1$ we make the following observations and assertions:

(a) For every $\alpha < \omega_1$, and for every $n < \omega$, the set $W_\alpha \cap X_n$ is compact and open.

(b) For every $\alpha < \omega_1$, the set $W_\alpha \cap X_n$ is nonempty for an infinity of values of n.

(c) For every $\alpha < \omega_1$, the set $A_\alpha \cap W_\alpha$ is infinite.

(d) If α and β are two countable ordinal numbers with $W_\alpha \neq W_\beta$, then $W_\alpha \cap W_\beta$ is compact.

Of these four statements, we prove only (d), the others being easy.

Choose the two ordinal numbers α_1 and β_1 such that

$W_\alpha = W_{\alpha_1}$

$W_\alpha \neq W_r$ for every $r < \alpha_1$

$W_\beta = W_{\beta_1}$

$W_\beta \neq W_r$ for every $r < \beta_1$

Of the two numbers β_1 and α_1, anything may be smaller, say $\beta_1 < \alpha_1$.

Then by the definition of W_{α_1}, we have $W_{\alpha_1} = \cup_{i=1}^{\infty} W_{\alpha_{1,i}}$ where each $W_{\alpha_{1,i}}$ is compact and $W_{\alpha_{1,i}} \cap W_{\beta_1}$ is empty except for a finite number of values of i. Therefore $W_{\alpha_1} \cap W_{\beta_1}$ is contained in a finite union of sets of the form $W_{\alpha_{1,i}}$ and hence is compact.

We now define $\underline{F} = \{W : W = W_\alpha \text{ for some } \alpha < \omega_1\} \cup \{X_n : n < \omega\}$. We claim that $\underline{F}$ satisfies the required five conditions.

A′) Every member of $\underline{F}$ is clopen in X. (It follows from (a) that each W_α is clopen in X; it is given that each X_n is clopen in X.)

B′) Any two distinct members of $\underline{F}$ intersect in a compact set. (Combine (a) and (d).)

C′) $X_n \in \underline{F}$ for every natural number n, by the very definition of $\underline{F}$.

D′) No member of $\underline{F}$ is compact. (The X_n's are non-compact, by hypothesis. The W_α's are non-compact because of (b).)

F) Every infinite discrete closed subset of X meets some member of $\underline{F}$ in an infinite set. For, let F be one such set. If $F \cap X_n$ is infinite for some n in N, we are done, since X_n is in $\underline{F}$. If $F \cap X_n$ is finite for every n, then form a set A by choosing one point from each of those sets that are nonempty, among the sets in $\{F \cap X_n : n \in N\}$. Then $A \in \underline{C}$ and so there is $\alpha < \omega_1$ such that $A = A_\alpha$. Then by (c), $A_\alpha \cap W_\alpha$ is infinite. We are through, after observing that $W_\alpha \in \underline{F}$ and $A_\alpha \subset F$.

The proof of the lemma is now complete.

LEMMA 2: *If $\underline{F}$ is as in the previous lemma, then $\underline{F}$ is maximal with respect to the four properties* A′), B′), C′) *and* D′).

PROOF: Suppose W is a noncompact clopen subset of X. Then W is not countably compact, since X satisfies v). Therefore W contains an infinite discrete closed subset, say A. Because $\underline{F}$ satisfies F), we have $A \cap W_1$ is infinite for some W_1 in $\underline{F}$. The set $A \cap W_1$ is discrete and closed, because A is so. Thus $W \cap W_1$ contains a noncompact closed subset and hence, is itself non compact. Therefore $\underline{F} \cup \{W\}$ does not satisfy B′). Hence the assertion.

REMARK: Now we go back to the space X of §1. We recall that we wrote it as $\Sigma_{n=1}^{\infty} X_n$ where each X_n satisfies the six conditions i) to vi).

We also had a decomposition $X_n = \Sigma_{i=1}^{\infty} X_{n,i}$ for each n. We apply the first lemma of this section, with $X_n = \Sigma_{i=1}^{\infty} X_{n,i}$ in the place of $X = \Sigma_{n=1}^{\infty} X_n$. Then we would have got a family $\underline{F}$ of subsets of X_n such that the five conditions A) to E) of §1 hold for $\underline{F}$ and further such that F') every infinite discrete closed subset of X_n meets some member of $\underline{F}$ is an infinite set.

We construct $X_n(1)$ with such a family $\underline{F}$.

PROPOSITION: *Let $X_n(1)$ be constructed as above. Let Y be any Hausdorff space containing $X_n(1)$. Then no sequence from X_n can converge to any point of $Y \backslash X_n(1)$.*

PROOF: Suppose a sequence from X_n converges to a point y of $Y \backslash X_n(1)$. Let A be the set of this sequence. Then since Y is Hausdorff, $A \cup \{y\}$ is closed.

Also, A is an infinite discrete closed subset of X_n. Hence by our choice of $\underline{F}$ (See condition F)), there is a member V of $\underline{F}$ such that $V \cap A$ is infinite. Look at the point V^* in $X_n(1)$. This must be a limit point of $V \cap A$, since $V \cap A$ cannot be contained in any compact set.

This contradicts the observation made in the previous paragraph that $A \cup \{y\}$ is closed.

REMARK: We let $X_* = \Sigma X_n(1)$ where $X_n(1)$ is as above.

TO SUM UP: Let X be a space satisfying the seven conditions i) to vii). Let X_* be an extension of X constructed as above. Then X_* has the following properties.

(1) X_* also satisfies conditions i) to vii)

(2) X is an open dense subspace of X_*

(3) $X_* \backslash X$ is discrete.

(4) In any Hausdorff extension of X_*, the sequential closure of X (that is, the set of all limits of sequences in X) is precisely X_*.

§3

Starting from X with certain properties, we constructed an extension X_* retaining these properties. Therefore we can repeat the process and construct $(X_*)_*$, $((X_*)_*)_*$ and so on. We wish to do it transfinitely up to ω_1 times. The limit ordinals offer some difficulty. The most natural way is to consider direct limits of the system consisting of previous spaces, when we encounter limit ordinals. But then there is no guarantee that properties like non-pseudo-compactness, are preserved. Unless this is guarenteed, we can not proceed ahead by again applying our process. The main purpose of this section is to overcome this difficulty, with a compromise that instead of having ω_1 steps, we shall have α steps, where α is any ordinal $< \omega_1$ fixed before.

As a standing assumption for this section, we take that α, β, γ, δ, denote general countable ordinal numbers. X is a space satisfying the seven conditions i) to vii).

When $\beta \leq \alpha$, we construct an extension $X(\beta, \alpha)$ of X as follows:

Consider the ordinal number $\omega^{\alpha+1}$. There is a collection $\{X_\gamma : \gamma < \omega^{\alpha+1}\}$ indexed by the set of $\omega^{\alpha+1}$ such that

a) X_γ is infinite and clopen in X for each γ

b) any two distinct members of this collection are disjoint

c) The union of the members of this collection is the whole of X.

In other words, $X = \Sigma_{\gamma<\omega^{\alpha+1}} X_\gamma$.

Such a splitting exists because $\omega^{\alpha+1}$ is countable and X is non-pseudo compact. [See condition vi′) of §1.].

Once such a partition of X is fixed, for each pair (δ, η) of ordinals $\leq\alpha$, define the subset $Z_{\delta,\eta}$ of X as follows:

$$Z_{\delta,\eta} = \sum\{X_\gamma : \omega^\eta\delta \leq \gamma < \omega^\eta(\delta+1)\}.$$

In the sequel, we will often talk of the sum of various $Z_{\delta,\eta}$'s for a fixed η and varying δ.

In such occasions, it is understood that δ varies over all those countable ordinal numbers such that $\omega^\eta\cdot(\delta+1) \leq \omega^{\alpha+1}$.

Having fixed these notations, we proceed to define $X(\beta, \alpha)$, by induction on β (fixing α at present). We first define

$$X(1, \alpha) = \sum_\delta (Z_{\delta,1})_*.$$

This is meaningful, since $Z_{\delta,1}$ can be easily proved to satisfy conditions i) to vii).

Suppose as induction hypothesis that we have defined $X(\gamma, \alpha)$ for every $\gamma < \beta_0$ such that the following hold:

a) Each $X(\gamma, \alpha)$ satisfies the seven conditions i) to vii).

b) If $\gamma_1 < \gamma_2 < \beta_0$, there is a natural one-to-one continuous open map i_{γ_1,γ_2} from $X(\gamma_1, \alpha)$ to $X(\gamma_2, \alpha)$; further $i_{\gamma_1,\gamma_2} \circ i_{\gamma_2,\gamma_3} = i_{\gamma,\gamma_3}$ holds whenever meaningful.

c) $\Sigma_\delta \overline{i_{0,\eta}(Z_{\delta,\eta})} = X(\eta, \alpha)$ for every $\eta < \beta_0$. This needs some explanation. We assume that $X(0, \alpha) = X$. Therefore by b), $i_{0,\eta}$ is a homeomorphism from X onto an open subspace of $X(\eta, \alpha)$. Therefore $i_{0,\eta}(Z_{\delta,\eta})$ is an open subspace of $X(\eta, \alpha)$ for each δ. The demand is that their closures are pairwise disjoint and exhaust the whole of $X(\eta, \alpha)$.

Before proceeding further, we pause a while to observe that our definition of $X(1, \alpha)$ satisfies the above three conditions (by taking $\beta_0 = 2$) : a$_1$) X and X$(1, \alpha)$ satisfy conditions i) to vii)

b_1) We take $i_{0,1}$ to be the sum (over δ) of the inclusion maps $Z_{\delta,1}$ in $(Z_{\delta,1})_*$. Then it is a homeomorphism from X onto an open subspace of $X(1,\alpha)$.

c_1) $X(1,\alpha)=\Sigma_\delta\overline{(Z_{\delta,1})}$ since $\overline{Z_{\delta,1}}$ is nothing but $(Z_{\delta,1})_*$.

Our purpose now is to define the space $X(\beta_0,\alpha)$ and the inclusion maps i_{γ,β_0} from $X(\gamma,\alpha)$ to $X(\beta_0,\alpha)$ for each $\gamma<\beta_0$, in such a way that analogues of a), b), c) hold for this bigger system also. We consider two cases.

If β_0 is a limit ordinal, we observe that $\{X(\gamma,\alpha),\ i_{\gamma_1,\gamma_2}: \gamma<\alpha,\ \gamma_1\leq\gamma_2<\alpha\}$ is a direct limit system. The limit space is denoted by $X(\beta_0,\alpha)$; the maps i_{γ,β_0} for $\gamma<\beta_0$ are naturally defined then.

If β_0 is a non-limit ordinal, then $\beta_0=\eta+1$ for some ordinal number η. Then the space $X(\eta,\alpha)$ has already (by induction hypothesis) been defined. We define

$$X(\eta+1,\alpha)=\sum_\delta \overline{(i_{0,\eta}(Z_{\delta,\eta+1}))}_*.$$

where the closures are taken in $X(\eta,\alpha)$. This is meaningful because by a), the space $\overline{i_{0,\eta}(Z_{\delta,\eta+1})}$ is a clopen subspace of $X(\eta,\alpha)$ (for any considered value of δ) and hence satisfies conditions i) to vii).

We now show that if $X(\beta_0,\alpha)$ is added to the system, conditions analagous to a), b), c) still hold good:

To prove a) for this larger system, we have to show that $X(\beta_0,\alpha)$ satisfies i) to vii). For this we make a claim that the first five of these properties are left invariant by any one of the following construction processes; the last two of the properties are easily verifiable.

(1) Forming the direct limit of such spaces over a diagram of a countable well-ordered set, provided each map in the direct limit system in injective, open and continuous.

(2) Forming disjoint topological sums.

(3) Forming the extension $X\longrightarrow X_*$ described in the previous sections.

(4) Forming clopen subspaces.

b) is easily proved.

c) follows from the argument below: If β_0 is a limit ordinal, then $\overline{i_{0,\beta_0}(Z_{\delta,\beta_0})}$ (the closure being taken in $X(\beta_0,\alpha)$ has the property that its preimage under i_{γ,β_0} is clopen in $X(\gamma,\alpha)$) for each $\gamma<\beta_0$. Therefore it is itself clopen in $X(\beta_0,\alpha)$.

If β_0 is a non-limit ordinal, c) holds by the very definition of $X(\beta_0,\alpha)$ and the fact that Y is always dense in Y_*.

The construction of $X(\beta,\alpha)$ is now complete.

THEOREM: *Let X be any space satisfying the seven conditions* i) *to* vii). *Let α be any countable ordinal number. Let $X(\alpha, \alpha)$ be constructed as above. Then*

1) *$X(\alpha, \alpha)$ also satisfies conditions* i) *to* vii).

2) *X is homeomorphic to an open dense subspace of $X(\alpha, \alpha)$.*

3) *$X(\alpha, \alpha)\backslash X$ is a scattered space of derived length $\leq \alpha$.*

4) *The sequential order of $X(\alpha, \alpha)$ is at least $\max(\beta, \alpha)$ and at most $\beta + \alpha$, where β is the sequential order of X.*

5) *In particular, if X is discrete, then the sequential order of $X(\alpha, \alpha)$ is exactly α.*

PROOF:

1) has been proved as a part of the construction of spaces $X(\beta, \alpha)$.

2) The map $i_{0,\alpha}$ serves as a homeomorphism from $X = X(0, \alpha)$ onto an open dense subspace of $X(\alpha, \alpha)$.

3) This follows from the fact that for every ordinal $\beta < \alpha$, it is true that $i_{\beta+1,\alpha}(X(\beta + 1,\alpha))\backslash i_{\beta,\alpha}(X(\beta, \alpha))$ is discrete.

This fact follows from the already proved result that $Y_*\backslash Y$ is discrete for every Y for which Y_* is defined.

4) Let σ be the sequential order of $X(\alpha, \alpha)$. Then we want to prove the three inequalities i) $\sigma \geq \alpha$ ii) $\sigma \geq \beta$ and iii) $\sigma \leq \beta + \alpha$. Let us denote the subspace $i_{\beta,\alpha}(X(\beta, \alpha))$ of $X(\alpha, \alpha)$ simply by $X(\beta, \alpha)$ for conveniences in the following proof. Then we have an increasing sequence $X = X(0, \alpha) \subset X(1, \alpha) \subset \cdots \subset X(\beta, \alpha) \subset \cdots \subset X(\alpha, \alpha)$ of subsets of $X(\alpha, \alpha)$. The set X is dense in $X(\alpha, \alpha)$. We can prove easily by appealing to earlier results (mainly the last proposition of §2) that for every $\beta < \alpha$, the set of all sequential limits of $X(\beta, \alpha)$ is contained in $X(\beta + 1, \alpha)$. It is immediate that at least α steps are required to catch the whole of $X(\alpha, \alpha)$ through sequential closures, starting from X. Hence we have $\sigma \geq \alpha$.

Since X is an open subspace of $X(\alpha, \alpha)$ and since open subspaces always have a smaller sequential order, the inequality $\sigma \geq \beta$ is easily proved.

To prove the third inequality, we make the following stronger claim: Let Y be any sequential space written as a union $Y = A \cup B$ where A is open and B is its complement. Further let B be scattered. Then the sequential order is at most the sum "sequential order of A + derived length of B".

This can be proved by transfinite induction on the derived length of B. The first step there, is the following result, whose proof may be found in [6, chapter 6]: If X is a sequential space, $x \in X$, V is a neighbourhood of x, α is an ordinal number and if the sequential order at any point of V is $\leq\alpha$, then the sequential order at x is at most $\alpha+1$.

We leave the other details of the proof of iii) to the reader.

5) If X is discrete, we have $\beta=0$. Therefore $\max(\beta,\alpha)=\beta+\alpha$. It follows from 4) that the sequential order of $X(\alpha,\alpha)$ is exactly α.

§4

In this final section, we derive some important corollaries of the theorem of §3. Note that CH has been used in its proof. Throughout this section, the one-point compactification of a locally compact non compact Hausdorff space Y will be denoted by $\tilde{Y}$ and the unique point of $\tilde{Y}\setminus Y$ will be denoted by ∞.

REMARK: The following observation on the sequential order of $\tilde{Y}$ will be helpful. Let Y be such that $\tilde{Y}$ is sequential. (See the proof of claim 4 in §1). Then it is always true that $\sigma(Y)\leq\sigma(\tilde{Y})\leq\sigma(Y)+1$ where σ denotes sequential order. Examples can be given to show that i) $\sigma(\tilde{Y})$ may be equal to $\sigma(Y)$ and ii) $\sigma(\tilde{Y})$ may be equal to $\sigma(Y)+1$. What is useful for us is the following fact. If Y is a disjoint sum of infinite number of spaces, then the sequential order at ∞ in Y is always 1; therefore $\sigma(\tilde{Y})=\sigma(Y)$.

With this back-ground, we now proceed to state the corollaries.

COROLLARY 1: *Let α be any ordinal $\leq\omega_1$. There exists a compact Hausdorff sequential space with sequential order α. [It is well-known conversely that the sequential order can never exceed ω_1.]*

PROOF: For $\alpha=0$, this is obvious, by considering a finite discrete space.

For $0<\alpha<\omega_1$, this follows from the theorem of §3 and the remark above, when we consider $(N(\alpha,\alpha))\tilde{}$ where N is the set of natural numbers with discrete topology.

For $\alpha=\omega_1$, consider Y where $Y=\Sigma_{0<\alpha<\omega_1}N(\alpha,\alpha)$. Then every countably compact closed subspace of Y is compact, since this property holds for each $N(\alpha,\alpha)$. Hence $\tilde{Y}$ is sequential. By the remark above, $\sigma(\tilde{Y})=\sigma(Y)=\mathrm{Sup}_{0<\alpha<\omega_1}\sigma(N(\alpha,\alpha))=\mathrm{Sup}_{0<\alpha<\omega_1}\alpha=\omega_1$.

COROLLARY 2: *Let X be any space satisfying the conditions i) to vii). Let α be any countable ordinal number. Then X admits a sequential compactification whose sequential order is $\geq\alpha$.*

PROOF: Consider $(X(\alpha, \alpha))\tilde{}$.

COROLLARY 3: *Let X be any noncompact space. Then the set S of all countable ordinal numbers that arise as sequential orders of zero dimensional sequential compactifications of X with cardinality $\leq c$, is either empty or cofinal in the set of all countable ordinal numbers.*

PROOF: Suppose S is not empty. Then there is a space Z which is a zero dimensional sequential compactification of X such that $|Z| \leq c$ and such that the sequential order of Z is countable. Since X is noncompact, there is at least one point in $Z \backslash X$. Let Y be the space obtained from Z by deleting one of the points of $Z \backslash X$. Then it is easy to prove that Y satisfies the seven conditions i) to vii). Now for each countable ordinal number α, the space $(Y(\alpha, \alpha))\tilde{}$ is a compactification of X with required properties, and its sequential order is $\geq \alpha$.

COROLLARY 4: *Let X be any noncompact zero-dimensional separable metrizable space. Let α be any countable ordinal number ≥ 1. Then X admits a sequential compactification with sequential order exactly α.*

PROOF: First, we can embed X in the cantor set K as a nonclosed subspace, take a point x of $\bar{X} \backslash X$ and put $Y = \bar{X} \backslash \{x\}$. (Here the closures are taken in K.) It is easy to verify that Y automatically satisfies the seven conditions i) to vii).

Consider $(Y(\alpha, \alpha))^{\sim}$. Its sequential order = sequential order of $Y(\alpha, \alpha)$. This is (by the Theorem of §3) at least α and at most $1 + \alpha$. Therefore if α is infinite, then it is exactly α (since, then $1 + \alpha = \alpha$). Hence the assertion is proved for infinite ordinals. Consider the sequence of spaces Y, $Y(1, 1)$, $Y(2, 2), \ldots, Y(N, N), \ldots$ let s_n be the sequential order of $Y(n, n)$ for each $n = 0, 1, 2, \ldots$ Then we have the following three facts:

i) for each $n \geq 0$, s_n is either n or $n + 1$. (This follows from assertion 4) of Theorem of §3). ii) $s_{n+1} \leq s_n + 1$ for each $n \geq 0$. (To prove this, first one can show that if α is any ordinal number, then $Y(\alpha + 1, \alpha + 1)$ can be thought of as $Z(1, 1)$ where $Z = Y(\alpha, \alpha)$; then apply assertion 4) of Theorem). iii) $s_0 = 1$. These three facts imply that 1, 2, 3, 4, ... is a subsequence of $s_0, s_1, s_2, \ldots$ (in fact, (s_n) is same as (n) after deleting one term, if necessary). Hence the assertion is proved.

COROLLARY 5: *Let α and β be two ordinal numbers such that $1 \leq \alpha \leq \beta \leq \omega_1$. Then there is a Tychonoff sequential space $X(\alpha, \beta)$ such that its k-order is α and sequential order is β.*

PROOF: By Corollary 1, there is a compact Hausdorff sequential space $K(\beta)$ with sequential order β. We have already established in [3] that there exists a Tychonoff sequential space $X(\alpha)$ for which the k-order and the sequential order are both equal to α. We let $X(\alpha, \beta) = K(\beta) + X(\alpha)$. This disjoint topological sum has the required property.

PROPOSITION: *S_2 cannot be embedded in a compact Hausdorff sequential space.*

PROOF: Let if possible X be a compact Hausdorff sequential space containing S_2. Without loss of generality, we may assume that S_2 is dense in X. Let us fix the following notations: For each $n = 1, 2, \ldots, S^n$ is the nth column of S_2 forming a convergent sequence; ∞ is the unique point of S_2 that is not of first countability; K is the set of all non-isolated points of S_2.

First, note that ∞ must be in the closure of the set $X \backslash S_2$. For if V is a compact neighbourhood of ∞ in X, then V cannot be contained in S_2.

Therefore there is a countable subset C of $X \backslash S_2$ having ∞ in its closure. (Here we use the fact that every sequential space is a c-space in the following sense: Whenever some point x is in the closure of a set A, it is true that x is in the closure of some countable subset of A.) Let $C = \{c_1, c_2, \ldots, c_n \ldots\}$. For each positive integer n, choose a neighbourhood V_n of c_n in X such that $\bar{V}_n$ does not meet the compact set K. Let $W_n = V_n \backslash (\cup_{i=1}^{n} S^i)$. Then W_n is a neighbourhood of c_n for every n. (because $\cup_{i=1}^{n} S^i$ is compact). Let $W = \cup_{n=1}^{\infty} W_n$. Then W is an open set containing C. Therefore $\infty \in \overline{(W \cap S_2)}$. But $W \cap S_2$ meets each column S^n in a finite set of isolated points. This is a contradiction. We have thus proved the stronger assertion that S_2 cannot be embedded in a compact Hausdorff c-space.

COROLLARY 6: *Given any two natural numbers n and m with $n \geq 2$, there is a compact Hausdorff sequential space with sequential order exactly m, not containing any copy of S_n.*

PROOF: Combine the above Proposition with Corollary 1.

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