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UPPER BOUNDS FOR AN IWASAWA INVARIANT

Frank Gerth III*

1. Introduction

Let k_0 be a finite extension of $\mathbf{Q}$, the field of rational numbers, and let K be a $\mathbf{Z}_\ell$ -extension of k_0 (that is, K is a Galois extension of k_0 , and $\text{Gal}(K/k_0)$ is topologically isomorphic to the additive group of the ℓ -adic integers $\mathbf{Z}_\ell$). Let the intermediate fields be denoted by $k_0 \subset k_1 \subset k_2 \subset \cdots \subset k_n \subset \cdots \subset K$, where k_n/k_0 is a cyclic extension of degree ℓ^n . Let σ be a topological generator of $\text{Gal}(K/k_0)$, and let $\sigma_n = \sigma|_{k_n}$. Then $\text{Gal}(k_n/k_0)$ is generated by σ_n , and we shall sometimes denote $\text{Gal}(k_n/k_0)$ by $\langle \sigma_n \rangle$. We define $\tau_n = \sigma_n^{\ell^{n-1}}$, and we note that $\text{Gal}(k_n/k_{n-1}) = \langle \tau_n \rangle$.

Next we let A_n denote the ℓ -class group of k_n for all n (that is, A_n is the Sylow ℓ -subgroup of the ideal class group of k_n). For any finite group C , we let $|C|$ denote the order of C . Then from the theory of $\mathbf{Z}_\ell$ -extensions (see [4]), $|A_n| = \ell^{e_n}$ with

$$(1) \quad e_n = \mu \ell^n + \lambda n + \nu$$

for n sufficiently large, where μ, λ, ν are integers (called the Iwasawa invariants of K/k_0) with $\mu \geq 0$ and $\lambda \geq 0$. In general it is difficult to compute μ, λ, ν , and we usually do not know how large n must be in order for equation 1 to be valid. Our goal in this paper is to specify an upper bound for μ based on the number of ramified primes in K/k_0 and on $|A_{n_1}|$ for a particular n_1 . Before stating our main theorem, we recall some facts about $\mathbf{Z}_\ell$ -extensions (cf. [4]). It is known that only finitely many primes ramify in K/k_0 (in fact, all the ramified primes are above the rational prime ℓ). If s_n denotes the number of primes of

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k_n that are ramified over k_0 , then $s_{n+1} \geq s_n$ for all n , and there is an integer n_0 such that $s_n = s_{n_0}$ for all $n \geq n_0$. We let s denote s_{n_0} , and we observe that the s primes of k_{n_0} which are ramified over k_0 are also totally ramified in K/k_{n_0} . We are now ready to state our main theorem.

THEOREM 1.: *Let K/k_0 be a $\mathbf{Z}_\ell$ -extension, where k_0 is a finite extension of $\mathbf{Q}$. Let s be defined as above, and let n_1 be chosen so that $l^{n_1} > s - 1$. If μ is the Iwasawa invariant of K/k_0 , then*

$$\mu \leq e_{n_1}/(l^{n_1} - s + 1),$$

where $|A_{n_1}| = \ell^{e_{n_1}}$, and A_{n_1} is the ℓ -class group of k_{n_1} .

REMARK: In [3] Greenberg shows that μ is “locally bounded” on certain $\mathbf{Z}_\ell$ -extensions of k_0 . Our Theorem 1 can be used to provide explicit local bounds for μ .

2. Preliminary results

We let the notation be the same as in section 1. We shall use multiplicative notation in each A_n , and the action of $\text{Gal}(k_n/k_0)$ shall be expressed by exponentiation. For $j > i \geq 0$, we define $N_{i,j}: A_j \rightarrow A_i$ to be the map induced by the norm map from k_j to k_i , and we define $J_{j,i}: A_i \rightarrow A_j$ to be the map induced by the inclusion map of k_i into k_j . For each $n > 0$, we define

$$T_n = 1 + \sigma_n^{\ell^{n-1}} + \cdots + \sigma_n^{(\ell-1)\ell^{n-1}}. \quad (2)$$

We note that $a^{T_n} = J_{n,n-1}(N_{n-1,n}(a))$ for each $a \in A_n$. We now list several facts that are proved in [2].

LEMMA 1: $T_n \equiv (1 - \sigma_n)^{(\ell-1)\ell^{n-1}} \pmod{(\ell \mathbf{Z}_\ell[\langle \sigma_n \rangle])}$.

LEMMA 2: *Let $R = \mathbf{Z}_\ell[\langle \sigma_n \rangle]/(T_n)$. Then R is a principal ideal domain. Next let A be a finite abelian ℓ -group, and define $\text{rank } A = \dim_{\mathbf{F}_\ell}(A/A^\ell)$, where $\mathbf{F}_\ell$ is the finite field with ℓ elements. If A is also a cyclic R module, and $\text{rank } A = r < (\ell - 1)\ell^{n-1}$, then A is an elementary abelian ℓ -group with $\text{rank} = r$.*

LEMMA 3: *For $j > i \geq n_0$, $N_{i,j}$ is surjective.*

The following lemma is a consequence of [5], Lemma 4.

LEMMA 4: Let $A_n^{(\tau_n)} = \{a \in A_n \mid a^{\tau_n} = a\}$. Then

$$|A_n^{(\tau_n)}| \leq |A_{n-1}| \cdot \ell^{s-1}.$$

3. Proof of Theorem 1

Let the notation be the same as in sections 1 and 2. Let $V_n = A_n/A_n^\ell$. We note that V_n may be viewed as a finite dimensional vector space over $\mathbf{F}_\ell$, and $(1 - \sigma_n)$ is a nilpotent linear transformation on V_n . In fact

$$(1 - \sigma_n)^{\ell^n} \equiv 1 - \sigma_n^{\ell^n} \equiv 0 \pmod{(\ell \mathbf{Z}_\ell[\langle \sigma_n \rangle])}.$$

Then from linear algebra

$$V_n = \prod_{i=1}^{z_n} M_{n,i}$$

where $\ell^n \geq \text{rank } M_{n,1} \geq \text{rank } M_{n,2} \geq \cdots \geq \text{rank } M_{n,z_n} \geq 1$;

$z_n = \text{rank } V_n / V_n^{1-\sigma_n}$; and

$M_{n,i}$ is an elementary abelian ℓ -group which is also a cyclic $\mathbf{Z}_\ell[\langle \sigma_n \rangle]$ module.

If we let $\alpha_{n,i} = \text{rank } M_{n,i}$, then $M_{n,i}^{(1-\sigma_n)^{\alpha_{n,i}}} = \{1\}$ but $M_{n,i}^{(1-\sigma_n)^{\alpha_{n,i}-1}} \neq \{1\}$. Also the integers $\alpha_{n,1}, \dots, \alpha_{n,z_n}$ are uniquely determined by V_n .

Now we let $\bar{N}_{i,j}: V_j \rightarrow V_i$ be the map induced by $N_{i,j}$, and we let $\bar{J}_{j,i}: V_i \rightarrow V_j$ be the map induced by $J_{j,i}$. Since $N_{i,j}$ is surjective for $j > i \geq n_0$ by Lemma 3, then $\bar{N}_{i,j}$ is surjective for $j > i \geq n_0$.

Unless otherwise noted, we shall assume $n \geq n_2$, where $n_2 = \max(n_0, n_1)$. (See Theorem 1 for the definition of n_1 .) Let $y_{0,n}$ denote the number of $\alpha_{n,i}$ with $\alpha_{n,i} = \ell^n$. (Note: If $\alpha_{n,1} < \ell^n$, then $y_{0,n} = 0$.) We claim that $y_{0,n+1} \leq y_{0,n}$. In fact, if $M_{n+1,i}$ is a cyclic $\mathbf{Z}_\ell[\langle \sigma_{n+1} \rangle]$ module factor of V_{n+1} with $\text{rank} = \ell^{n+1}$, then letting $M = \bar{N}_{n,n+1}(M_{n+1,i})$, we see that $\bar{J}_{n+1,n}(M) = \bar{J}_{n+1,n}(\bar{N}_{n,n+1}(M_{n+1,i})) = M_{n+1,i}^{T_{n+1}}$. Since

$$T_{n+1} \equiv (1 - \sigma_{n+1})^{(\ell-1)\ell^n} \pmod{(\ell \mathbf{Z}_\ell[\langle \sigma_{n+1} \rangle])}$$

by Lemma 1, then $\text{rank } \bar{J}_{n+1,n}(M) = \ell^{n+1} - (\ell-1)\ell^n = \ell^n$. Then it is easy to see that M is a cyclic $\mathbf{Z}_\ell[\langle \sigma_n \rangle]$ module factor of V_n with

rank $= \ell^n$. So for every cyclic module factor of V_{n+1} with rank $= \ell^{n+1}$, there is a corresponding cyclic module factor of V_n with rank $= \ell^n$. Then we see that $y_{0,n+1} \leq y_{0,n}$. Since each $y_{0,n} \geq 0$, there are integers $\gamma_0 \geq 0$ and $f_0 \geq n_2$ such that $y_{0,n} = \gamma_0$ for all $n \geq f_0$.

Next let $y_{1,n}$ denote the number of $\alpha_{n,i}$ with $\alpha_{n,i} = \ell^n - 1$. By arguments similar to those above, $y_{1,n+1} \leq y_{1,n}$ for all $n \geq f_0$, and there are integers $\gamma_1 \geq 0$ and $f_1 \geq f_0$ such that $y_{1,n} = \gamma_1$ for all $n \geq f_1$. Similarly if $y_{j,n}$ (for $1 < j \leq s-1$) denotes the number of $\alpha_{n,i}$ with $\alpha_{n,i} = \ell^n - j$, then there are nonnegative integers $\gamma_2, \dots, \gamma_{s-1}$ and $g = f_{s-1} \geq \dots \geq f_2 \geq f_1$ such that $y_{j,n} = \gamma_j$ for all $n \geq f_j$. We observe that $V_g = \prod_{i=1}^{z_g} M_{g,i}$ with

$$\alpha_{g,i} = \begin{cases} \ell^g & \text{for } i \leq \gamma_0 \\ \ell^g - 1 & \text{for } \gamma_0 < i \leq \gamma_0 + \gamma_1 \\ \vdots & \\ \ell^g - s + 1 & \text{for } \sum_{j=1}^{s-2} \gamma_j < i \leq \sum_{j=1}^{s-1} \gamma_j \end{cases}$$

and $\alpha_{g,i} < \ell^g - s + 1$ for $i > \sum_{j=1}^{s-1} \gamma_j$. We let y denote $\sum_{j=1}^{s-1} \gamma_j$, and we let M_g denote the product of all $M_{g,i}$ with $\alpha_{g,i} < \ell^g - s + 1$. So $V_g = (\prod_{i=1}^y M_{g,i}) \times M_g$. For each $n > g$, we define inductively $M_{n,i}$ ($1 \leq i \leq y$) and M_n such that $\bar{N}_{n-1,n}(M_{n,i}) = M_{n-1,i}$; $\bar{N}_{n-1,n}(M_n) = M_{n-1}$; $V_n = (\prod_{i=1}^y M_{n,i}) \times M_n$; and $\alpha_{n,i}$ satisfies the same equation as $\alpha_{g,i}$ (for $1 \leq i \leq y$) except with n replacing g . For $n \geq g$, we then define inductively $\mathbf{Z}_\ell[\langle \sigma_n \rangle]$ modules $B_n \supseteq A_n^\ell$ so that the image of B_n in V_n is M_n and $N_{n,n+1}(B_{n+1}) = B_n$. We note that $|A_n/B_n| = |V_n/M_n| = \ell^{\beta_n}$ for $n \geq g$, where

$$\beta_n = \gamma_0 \ell^n + \gamma_1 (\ell^n - 1) + \dots + \gamma_{s-1} (\ell^n - s + 1).$$

Our next step is to estimate $B_n^{\langle \tau_n \rangle} = \{a \in B_n \mid a^{\tau_n} = a\} = B_n \cap A_n^{\langle \tau_n \rangle}$. First we note that for $n > g$, $J_{n,n-1}(A_{n-1}) \subseteq A_n^{\langle \tau_n \rangle}$, and

$$((J_{n,n-1}(A_{n-1})) \cdot B_n)/B_n = ((J_{n,n-1}(N_{n-1,n}(A_n))) \cdot B_n)/B_n = (A_n^{T_n} \cdot B_n)/B_n.$$

Since $T_n \equiv (1 - \sigma_n)^{(\ell-1)\ell^{n-1}} \pmod{\ell \mathbf{Z}_\ell[\langle \sigma_n \rangle]}$ by Lemma 1, then

$$\begin{aligned} |(A_n^{T_n} \cdot B_n)/B_n| &= |A_n/B_n| \cdot [|A_n/(A_n^{T_n} \cdot B_n)]|^{-1} \\ &= \ell^{\beta_n} \cdot \ell^{-(\gamma_0 + \dots + \gamma_{s-1})(\ell-1)\ell^{n-1}} \\ &= \ell^{\beta_{n-1}}. \end{aligned}$$

Since $A_n^{(\tau_n)}/(B_n \cap A_n^{(\tau_n)}) \cong (A_n^{(\tau_n)} \cdot B_n)/B_n$, then $|A_n^{(\tau_n)}/(B_n \cap A_n^{(\tau_n)})| = |(A_n^{(\tau_n)} \cdot B_n)/B_n| \geq \ell^{\beta_{n-1}}$. Then

$$|B_n^{(\tau_n)}| = |B_n \cap A_n^{(\tau_n)}| \leq |A_n^{(\tau_n)}| \cdot \ell^{-\beta_{n-1}} \leq |A_{n-1}| \cdot \ell^{s-1} \cdot \ell^{-\beta_{n-1}}$$

by Lemma 4. Since $|A_{n-1}/B_{n-1}| = \ell^{\beta_{n-1}}$, then $|A_{n-1}| \cdot \ell^{-\beta_{n-1}} = |B_{n-1}|$. So $|B_n^{(\tau_n)}| \leq |B_{n-1}| \cdot \ell^{s-1}$ for $n > g$.

Now we let $W_n = B_n/B_n^\ell$. We can apply the same procedures to the W_n that we applied to the V_n . So we can find submodules C_n of the B_n containing B_n^ℓ , nonnegative integers $\delta_0, \delta_1, \dots, \delta_{s-1}$, and an integer $r \geq g$ such that $N_{n,n+1}(C_{n+1}) = C_n$ for $n \geq r$, and $|B_n/C_n| = \ell^{\epsilon_n}$ for $n \geq r$, where

$$\epsilon_n = \delta_0 \ell^n + \delta_1(\ell^n - 1) + \dots + \delta_{s-1}(\ell^n - s + 1).$$

Also $|C_n^{(\tau_n)}| \leq |C_{n-1}| \cdot \ell^{s-1}$ for $n > r$. We note that for $n \geq r$, $|A_n/C_n| = \ell^{\theta_n}$ with

$$\theta_n = (\gamma_0 + \delta_0)\ell^n + (\gamma_1 + \delta_1)(\ell^n - 1) + \dots + (\gamma_{s-1} + \delta_{s-1})(\ell^n - s + 1).$$

We can then consider C_n/C_n^ℓ and repeat the above procedures. Eventually we obtain submodules H_n of the A_n and an integer $n_3 \geq r$ such that $N_{n,n+1}(H_{n+1}) = H_n$ for $n \geq n_3$, and H_n/H_n^ℓ has no cyclic $\mathbf{Z}_\ell[\langle \sigma_n \rangle]$ module factor with rank $\geq \ell^n - s + 1$ for $n \geq n_3$. (Remark: Our procedure terminates after a finite number of steps because $|A_{n_1}|$ is finite and $|\bar{N}_{n_1,n}(M)| \geq \ell^{\ell^{n_1-s+1}} \geq \ell$ for each cyclic $\mathbf{Z}_\ell[\langle \sigma_n \rangle]$ module factor M with rank $\geq \ell^n - s + 1$.) Also $|A_n/H_n| = \ell^{\omega_n}$ for $n \geq n_3$, where

$$\omega_n = \psi_0 \ell^n + \psi_1(\ell^n - 1) + \dots + \psi_{s-1}(\ell^n - s + 1)$$

for some nonnegative integers $\psi_0, \psi_1, \dots, \psi_{s-1}$. Furthermore $|H_n^{(\tau_n)}| \leq |H_{n-1}| \cdot \ell^{s-1}$ for $n > n_3$. We observe that

$$\omega_n = (\psi_0 + \psi_1 + \dots + \psi_{s-1})\ell^n - (\psi_1 + 2\psi_2 + \dots + (s-1)\psi_{s-1}).$$

We let $\omega = \psi_0 + \psi_1 + \dots + \psi_{s-1}$ and $\psi = \psi_1 + 2\psi_2 + \dots + (s-1)\psi_{s-1}$. So $\omega_n = \omega \ell^n - \psi$. Now by equation 1, $|A_n| = \ell^{e_n}$ with $e_n = \mu \ell^n + \lambda n + \nu$ for n sufficiently large. Since $|A_n/H_n| = \ell^{\omega_n}$, then $|H_n| = \ell^{e'_n}$ for n sufficiently large, where

$$(3) \quad e'_n = \mu' \ell^n + \lambda' n + \nu'$$

with $\mu' = \mu - \omega$, $\lambda' = \lambda$, and $\nu' = \nu + \psi$. Also, from the structure of A_n/H_n , routine calculations show that

$$|A_{n_1}| \geq |N_{n_1, n_3}(A_{n_3})/N_{n_1, n_3}(H_{n_3})| = \ell^{\omega \ell^{n_1-1} - \psi} \geq \ell^{\omega(\ell^{n_1-1} - s + 1)}.$$

So if $|A_{n_1}| = \ell^{e_{n_1}}$, then $\omega \leq e_{n_1}/(\ell^{n_1} - s + 1)$. To complete the proof of Theorem 1, it suffices to show that $\mu' = 0$. Then $\mu = \omega \leq e_{n_1}/(\ell^{n_1} - s + 1)$.

So we consider H_n with the following properties: H_n/H_n^ℓ has no cyclic $\mathbf{Z}_\ell[\langle \sigma_n \rangle]$ module factor with rank $\geq \ell^n - s + 1$ for $n \geq n_3$; $N_{n, n+1}(H_{n+1}) = H_n$ for $n \geq n_3$; and $|H_n^{\langle \tau_n \rangle}| \leq |H_{n-1}| \cdot \ell^{s-1}$ for $n > n_3$. We let $P_n = H_n/H_n^\ell$. Now

$$\begin{aligned} P_n/P_n^{1-\tau_n} &= (H_n/H_n^\ell)/(H_n/H_n^\ell)^{1-\tau_n} \cong H_n/(H_n^\ell \cdot H_n^{1-\tau_n}) \\ &\cong (H_n/H_n^{1-\tau_n})/(H_n/H_n^{1-\tau_n})^\ell. \end{aligned}$$

Since $N_{n-1, n}(H_n) = H_{n-1}$ for $n > n_3$, then $H_n/\ker N_{n-1, n} \cong H_{n-1}$. Also it is easy to see that $\ker N_{n-1, n} \supseteq H_n^{1-\tau_n}$ and $|H_n/H_n^{1-\tau_n}| = |H_n^{\langle \tau_n \rangle}|$. It then follows that $|\ker N_{n-1, n}/H_n^{1-\tau_n}| \leq \ell^{s-1}$. Since H_{n-1}/H_{n-1}^ℓ is the direct product of cyclic $\mathbf{Z}_\ell[\langle \sigma_{n-1} \rangle]$ modules with ranks $< \ell^{n-1} - s + 1$ for $n > n_3$, then $(H_n/\ker N_{n-1, n})/(H_n/\ker N_{n-1, n})^\ell$ is the direct product of cyclic $\mathbf{Z}_\ell[\langle \sigma_n \rangle]$ modules with ranks $< \ell^{n-1} - s + 1$ for $n > n_3$. Then $(H_n/H_n^{1-\tau_n})/(H_n/H_n^{1-\tau_n})^\ell$ and hence $P_n/P_n^{1-\tau_n}$ are direct products of cyclic $\mathbf{Z}_\ell[\langle \sigma_n \rangle]$ modules with ranks $< \ell^{n-1}$ for $n > n_3$. Since $1 - \tau_n = 1 - \sigma_n^{\ell^{n-1}} \equiv (1 - \sigma_n)^{\ell^{n-1}} \pmod{\ell \mathbf{Z}_\ell[\langle \sigma_n \rangle]}$, we must have $P_n^{1-\tau_n} = \{1\}$. So

$$\begin{aligned} |P_n| &= |P_n/P_n^{1-\tau_n}| = |(H_n/H_n^{1-\tau_n})/(H_n/H_n^{1-\tau_n})^\ell| \\ &= \frac{|H_n/\ker N_{n-1, n}| \cdot |\ker N_{n-1, n}/H_n^{1-\tau_n}|}{|(H_n/H_n^{1-\tau_n})^\ell|} \\ &\leq \frac{|H_{n-1}| \cdot |\ker N_{n-1, n}/H_n^{1-\tau_n}|}{|H_{n-1}^\ell|} \\ &\leq |P_{n-1}| \cdot \ell^{s-1}. \end{aligned}$$

Next we note that $H_n^{T_n} \subset H_n^\ell$ since

$$T_n \equiv (1 - \sigma_n)^{(\ell-1)\ell^{n-1}} \equiv (1 - \tau_n)^{\ell-1} \pmod{\ell \mathbf{Z}_\ell[\langle \sigma_n \rangle]}, \quad \text{and } P_n^{1-\tau_n} = \{1\}.$$

On the other hand, $H_n/H_n^{T_n}$ is a finitely generated module over the principal ideal domain $\mathbf{Z}_\ell[\langle \sigma_n \rangle]/(T_n)$. Since

$$\begin{aligned}
(H_n/H_n^{T_n})/(H_n/H_n^{T_n})^\ell &\cong H_n/(H_n^\ell \cdot H_n^{T_n}) \\
&\cong (H_n/H_n^\ell)/(H_n/H_n^\ell)^{T_n} \\
&= P_n/P_n^{T_n} = P_n,
\end{aligned}$$

and P_n is the direct product of cyclic modules with ranks $< \ell^{n-1} \leq (\ell-1)\ell^{n-1}$, then $H_n/H_n^{T_n}$ is the direct product of cyclic modules over $\mathbf{Z}_\ell[\langle \sigma_n \rangle]/(T_n)$ with ranks $< (\ell-1)\ell^{n-1}$. So $H_n/H_n^{T_n}$ is an elementary abelian ℓ -group by Lemma 2. Hence $H_n^{T_n} \supseteq H_n^\ell$. Thus $H_n^\ell = H_n^{T_n} = J_{n,n-1}(N_{n-1,n}(H_n)) = J_{n,n-1}(H_{n-1})$. So $|H_n^\ell| \leq |H_{n-1}|$. Then

$$|H_n| = |H_n/H_n^\ell| \cdot |H_n^\ell| = |P_n| \cdot |H_n^\ell| \leq |P_{n-1}| \cdot \ell^{s-1} \cdot |H_{n-1}|.$$

Let $t_{n-1} = \text{rank } P_{n-1}$. Then $|H_n| \leq |H_{n-1}| \cdot \ell^{t_{n-1}+s-1}$ for $n > n_3$. So $|H_{n_3+1}| \leq |H_{n_3}| \cdot \ell^{t_{n_3}+s-1}$;

$$\begin{aligned}
|H_{n_3+2}| &\leq |H_{n_3+1}| \cdot \ell^{t_{n_3+1}+s-1} \leq (|H_{n_3}| \cdot \ell^{t_{n_3}+s-1}) \cdot \ell^{(t_{n_3}+s-1)+s-1} \\
&= |H_{n_3}| \cdot \ell^{2t_{n_3}+3(s-1)},
\end{aligned}$$

and by induction, $|H_{n_3+i}| \leq |H_{n_3}| \cdot \ell^v$, where $v = it_{n_3} + (s-1)(i+1)(i)/2$ for $i \geq 1$. Now by equation 3, $|H_{n_3+i}| = \ell^{e'_{n_3+i}}$ with $e'_{n_3+i} = \mu' \ell^{n_3+i} + \lambda'(n_3+i) + \nu'$ for i sufficiently large. We note that $\lim_{i \rightarrow \infty} (e'_{n_3+i}/\ell^{n_3+i}) = \mu'$. From our above calculations, $e'_{n_3+i} \leq e'_{n_3} + it_{n_3} + (s-1)(i+1)(i)/2$. Then it is easy to see that $\lim_{i \rightarrow \infty} (e'_{n_3+i}/\ell^{n_3+i}) = 0$. So $\mu' = 0$. This is what we wanted to show, and hence the proof of Theorem 1 is complete.

4. Some special cases

The following corollaries are immediate consequences of Theorem 1.

COROLLARY 1: *Let K/k_0 be a $\mathbf{Z}_\ell$ -extension of k_0 , where k_0 is a finite extension of $\mathbf{Q}$. Assume that there is only one ramified prime in K/k_0 , and it is totally ramified. If $|A_0| = \ell^{e_0}$, where A_0 is the ℓ -class group of k_0 , then the Iwasawa invariant μ of K/k_0 satisfies $\mu \leq e_0$.*

REMARK: Corollary 1 is also proved in [2].

COROLLARY 2: *Let K/k_0 be a $\mathbf{Z}_\ell$ -extension of k_0 , where $[k_0:\mathbf{Q}] = 2$. If ℓ splits in k_0 , and both primes above ℓ are totally ramified in K/k_0 , then $\mu \leq e_1/(\ell-1)$.*

REMARK: Let K be the $\mathbf{Z}_\ell$ -extension of k_0 contained in the field obtained by adjoining to k_0 all ℓ -power roots of unity. Assume that k_0 is an imaginary quadratic extension of $\mathbf{Q}$. In [1] Ferrero proves that $\mu \leq e_0$ for this special type of $\mathbf{Z}_\ell$ -extension. Also if k_0 is any abelian extension of $\mathbf{Q}$ and $\ell = 2$ or 3 , Ferrero shows that $\mu = 0$ for this special type of $\mathbf{Z}_\ell$ -extension.

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