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SIMULTANEOUS RESOLUTION OF RATIONAL SINGULARITIES

Jonathan M. Wahl*

Abstract

Let $\text{Spec } R$ be a rational surface singularity over $\mathbb{C}$. Generalizing work of Brieskorn, Artin, and others, we prove there is a smooth irreducible component A of the moduli space of $\text{Spec } R$, consisting of deformations which resolve simultaneously in a family after Galois base change. Further, the group is a direct product of Weyl groups associated to -2 configurations in the graph of R . We also prove that for a determinantal singularity, A consists of the determinantal deformations.

0. Introduction

Let R be a two-dimensional normal local ring over $\mathbb{C}$ with a rational singularity at the closed point, and $X \rightarrow \text{Spec } R$ the minimal resolution. The simplest examples are those of embedding dimension $e = 3$, the rational double points (or RDP's). These are the Kleinian singularities $\mathbb{C}^2/G$, where $G \subset \text{SL}(2, \mathbb{C})$ is a finite subgroup; they are called A_n (G cyclic), D_n (binary dihedral), E_6 (binary tetrahedral), E_7 (binary octahedral), and E_8 (binary icosahedral). The exceptional fibre E in X is a configuration of non-singular rational curves, of self-intersection -2 , whose (weighted) dual graph is the Dynkin diagram of the corresponding simple Lie algebra.

Brieskorn discovered ([5], [6], [7]) a relationship between the deformation theory of such an R and the Weyl group W of the Lie algebra. We say a deformation $\mathcal{V} \rightarrow T$ *resolves simultaneously* if there

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is a smooth map $\mathcal{X} \rightarrow T$, factoring through $\mathcal{V}$, such that for each $t \in T$, $\mathcal{X}_t \rightarrow \mathcal{V}_t$ is a (minimal) resolution of singularities. Atiyah observed in 1958 [4] that the versal (analytic) deformation of A_1 resolves simultaneously after a $\mathbb{Z}/2$ -base change (i.e., $x^2 + y^2 + z^2 = t^2$ resolves simultaneously).

THEOREM (Brieskorn [7]): *The versal deformation of a rational double point resolves simultaneously after a Galois base change, with group W .*

This was proved independently by Tjurina [19] and (for A_n) Kas [13], using Brieskorn's earlier work.

There is a more precise picture of how the simple algebraic groups $G = \mathrm{SL}(n)$, $\mathrm{Sp}(n)$, etc., themselves come into play, and not just the Weyl groups [1], [7]. The idea (due to Grothendieck) is to study the subregular elements of G ; in particular, one looks at the singular locus of $G \rightarrow T/W$ ($T =$ maximal torus), sending an element of G to the conjugacy class of its semi-simple part. (If $G = \mathrm{SL}(n)$, this sends a matrix to its characteristic polynomial).

Artin and Schlessinger [2] generalized part of Brieskorn's result (and also a result of Huikeshoven [12]) to rational singularities of higher multiplicity, and made it more algebraic; however, one must work in a suitably localized algebraic category (e.g., algebraic spaces, or local henselian schemes).

THEOREM [2]: *There is a smooth space Res parametrizing deformations of $\mathrm{Spec} R$ with simultaneous resolution, and a finite map $\Phi: \mathrm{Res} \rightarrow \mathrm{Def}$ into the deformation space, whose image is an irreducible component A of Def . ($A =$ Artin component).*

When $e = 3$ or 4 , then Def is smooth, hence Φ is surjective. However, Pinkham [16] showed that for the cone over $\mathbb{P}^1 \rightarrow \mathbb{P}^4$ ($e = 5$), Def has one- and three-dimensional components; every deformation is a smoothing, but simultaneous resolution takes place on only the second component.

The main purpose of this paper is to prove

THEOREM 1: *$\Phi: \mathrm{Res} \rightarrow \mathrm{Def}$ is Galois onto A , with group $W = \prod W_i$, the product of the Weyl groups associated to the maximal connected -2 configurations in the graph of R . In particular, A is smooth.*

This had been conjectured by Burns–Rapoport [8] and Wahl [21].

The first authors had noticed that each -2 curve gives an automorphism of $\text{Res} \rightarrow \text{Def}$ (an “elementary transformation”). We proved that the dimension of the kernel of the tangent space map of $\text{Res} \rightarrow \text{Def}$ is the number of -2 curves; in particular, if there are no -2 ’s, then $\text{Res} \xrightarrow{\sim} A$ [20]. It is recent work of J. Lipman [15] which completes the proof.

The idea of the proof is rather simple. First, interpret Res as the deformation space of X ([2], 4.6). Next, blow down the -2 configurations in X to rational double points, obtaining $X \rightarrow V \rightarrow \text{Spec } R$. This gives blowing-down maps

$$\text{Res} = \text{Def}(X) \rightarrow \text{Def}(V) \rightarrow \text{Def}(R) = \text{Def}.$$

Third, using Brieskorn’s rational double point theorem and Burns–Wahl [9] on the relation of local to global deformations, one deduces $\text{Def}(X) \rightarrow \text{Def}(V)$ is Galois and surjective, with group W . Therefore, it remains only (!) to show $\text{Def } V$ injects into $\text{Def}(R)$ (i.e., $\text{Def } V$ is the Artin component). In an earlier version of this paper (cf. [24]), we used a cohomological argument to prove Theorem 1 in case the fundamental cycle has multiplicity 1 at the -3 curves, e.g., for determinantal or quotient singularities. Lipman proves the injectivity directly; the point is that V is “canonically” obtained from $\text{Spec } R$, even after deformation of each.

There is one case where the result is more concrete.

THEOREM 2: *Let R be determinantal, of multiplicity d , hence defined by the 2×2 minors of a $2 \times d$ matrix. Then the Artin component is the versal determinantal deformation.*

That is, the deformations of R corresponding to perturbations of the entries of the defining matrix form an irreducible component of Def , equal to A . First, we observe that determinantal deformations yield deformations of V , owing to the simple construction of V in this case [23]. Then, we recognize the determinantal nature of R from a morphism $X \rightarrow \mathbb{P}^1 \times \mathbb{P}^{d-1}$; standard obstruction theory shows this map lifts to each deformation $\bar{X}$ of X , whence $\Gamma(\mathcal{O}_{\bar{X}})$ is also determinantal.

In a forthcoming paper [25], we study the finer structure of $\text{Res} \rightarrow A$, especially the irreducible components of the discriminant locus, and the fact that the monodromy group over A is W .

In §1, we define the action of W on Res ; our treatment there is influenced by a letter from E. Horikawa. We outline a proof of

Lipman's result in §2, while §3 discusses determinantal rational singularities.

§1. The action of W on Res

(1.1) We assume known the basic facts about rational singularities ([3], [14]). Moduli spaces are minimally versal deformation spaces. $\text{Spec } R$, and moduli spaces like Res and Def , are assumed local henselian or local analytic spaces, in order to avoid the non-separatedness of Res as an algebraic space.

(1.2) Interpreting Res as $\text{Def } X$, there is a blowing-down map $\text{Def } X \rightarrow \text{Def } V$ arising from $X \rightarrow V$ ([17], [9]). Denote by $p_1, \dots, p_r$ the RDP's on V , and by $S_1, \dots, S_r$ their moduli spaces. The composition $\text{Def } X \rightarrow \text{Def } V \rightarrow \prod S_i$ factors via $\prod Z_i$, where $Z_i \rightarrow S_i$ is the $\text{Res} \rightarrow \text{Def}$ map for the RDP p_i [9]. (Thinking of Z_i as the deformation space of some neighborhood U_i of the exceptional fibre of p_i in X , one has simply that deformations of X give deformations of each U_i).

THEOREM 1.3: *The diagram*

$$\begin{array}{ccc} \text{Def } X & \rightarrow & \prod Z_i \\ \downarrow & & \downarrow \\ \text{Def } V & \rightarrow & \prod S_i \end{array}$$

is cartesian, all spaces are smooth, the horizontal maps are smooth, and the vertical maps are Galois (and surjective), with group $W = \prod W_i$.

PROOF: The cartesian property is [9], 2.6 (it is assumed there that V is projective, but this is not needed for the proof). All spaces are obviously smooth (all global H^{2i} 's and local T^{2i} 's vanish). The top map is smooth by [9], 2.14; the bottom, because it is surjective on the tangent spaces:

$$\text{Ext}_V^1(\Omega_V^1, \mathcal{O}_V) \rightarrow H^0(T_V^1) \rightarrow H^2(\theta_V) = 0.$$

Finally, Brieskorn's RDP theorem gives the Galois property of the right-hand map; since the diagram is cartesian, these automorphisms (and the Galois property) pull back to $\text{Def } X \rightarrow \text{Def } V$.

(1.4) Thus, $\text{Res}/W = \text{Def } X/W \xrightarrow{\sim} \text{Def } V$. Now, $W = \prod W_i$ is generated by reflections; we give a more geometric picture of the

action of a reflection σ corresponding to E_1 , a -2 curve on X . This is the “elementary operation” of [8], §7, or [11], Appendix B.

First, let $Z \rightarrow S$ be $\text{Res} \rightarrow \text{Def}$ for a single -2 curve (A_1 -singularity), with $\mathcal{X} \rightarrow Z$ the total family. Here, $\dim Z = \dim S = 1$, and $\dim \mathcal{X} = 3$. Then the exceptional curve $E \subset X \subset \mathcal{X}$ has normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ [11]. Let $p: \mathcal{Y} \rightarrow \mathcal{X}$ be the blow-up of E ; then $p^{-1}(E) \rightarrow E$ is $\pi_1: \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 = E$, and the normal bundle of $p^{-1}(E)$ in $\mathcal{Y}$ is $(- \text{diagonal})$. By [10], $p^{-1}(E)$ may be blown down in the direction of the other ruling, yielding $\mathcal{Y} \rightarrow \mathcal{X}' \rightarrow Z$. By functoriality, $\mathcal{X}' \rightarrow Z$ is obtained from $\mathcal{X} \rightarrow Z$ by an automorphism $\sigma: Z \rightarrow Z$, of order 2 by construction.

For a general rational singularity, let V_1 be the space obtained from X by contracting E_1 , and consider the (cartesian) diagram

$$\begin{array}{ccc} \text{Def } X & \rightarrow & Z \\ \downarrow & & \downarrow \\ \text{Def } V_1 & \rightarrow & S. \end{array}$$

Let $R_1 \subset \text{Def } X$ be the fibre over the origin of Z ; R_1 is a smooth, codimension 1 subvariety corresponding to deformations of X to which E_1 lifts. If $\mathcal{X} \rightarrow \text{Def } X$ is the total space, and $\mathcal{X}_1 \rightarrow R_1$ is the induced deformation, one has a relative effective Cartier divisor $\mathcal{E} \subset \mathcal{X}_1$, which lifts E_1 . Blow up $\mathcal{E} \subset \mathcal{X}$, then blow down in another direction as before, yielding again the reflection σ .

(1.5) Note that $\sigma(R_1) = R_1$. In fact, for an RDP, the R_i 's correspond to hyperplanes left fixed by a basis of the positive roots when one views W as acting on the usual complex vector space as a reflection group. To see the other hyperplanes as subvarieties of Res , and for the generalization to all rational singularities, see [25].

(1.6) Curiously, the Weyl group of E_8 cannot appear unless R is the RDP E_8 . That is, an E_8 -configuration in the graph of a rational singularity is necessarily the entire graph. In light of Lemma 1.7 below, this is because the fundamental cycle of E_8 has multiplicity ≥ 2 at every component.

LEMMA 1.7: *Inside a rational configuration, suppose L is a reduced, connected curve intersecting an irreducible E_1 in one point; let E_2 be the curve in L with $E_1 \cdot E_2 = 1$. Then the multiplicity of E_2 in Z_L , the fundamental cycle of L , is 1.*

PROOF: We must have $(Z_L + E_1) \cdot (Z_L + E_1 + K) \leq -2$. Since

$Z_L \cdot (Z_L + K) = E_1 \cdot (E_1 + K) = -2$, we deduce $Z_L \cdot E_1 \leq 1$. This implies the result.

§2. Lipman's theorem

(2.1) As mentioned in the introduction, the main theorem will follow from the injectivity of the blowing-down map $\text{Def } V \rightarrow \text{Def } R$. The usual functorial argument shows that it suffices to prove injectivity on the tangent spaces, since $H^0(\theta_V) \simeq \theta_R$ ([22], 1.12). We will sketch (a slight variant of) Lipman's proof.

THEOREM 2.2 (Lipman [15]): *Def V injects into Def R .*

PROOF: We show first that $V = \text{Proj} \bigoplus H^0(X, \omega_X^{\otimes n})$ (as schemes over $\text{Spec } R$). Letting $f: X \rightarrow V$, we have $f_*\omega_X = \omega_V$ (dualizing differentials on V), so $H^0(X, \omega_X^{\otimes n}) = H^0(V, \omega_V^{\otimes n})$. We show ω_V is very ample for $V \rightarrow \text{Spec } R$. By [14], 12.1, it suffices to show that $(\omega_V \cdot F_1) > 0$, for each exceptional curve F_1 in V . Since $f^*\omega_V = \omega_X$ (V has only RDP's), this intersection number is $(\omega_X \cdot E_1)$, where E_1 is a non-2 curve in X , hence is positive. Using the surjectivity $\Gamma(\omega_X^{\otimes m}) \otimes \Gamma(\omega_X^{\otimes n}) \rightarrow \Gamma(\omega_X^{\otimes(m+n)})$ ([14], 7.3), the claim now follows.

Next, if $U = X - E = \text{Spec } R - \{m\}$, we have

$$(2.2.1) \quad H^0(X, \omega_X^{\otimes n}) = H^0(U, \omega_U^{\otimes n}), \quad n = 0, 1$$

$$(2.2.2) \quad H^0(X, \omega_X^{\otimes n}) = \text{Im}(\phi_n: H^0(U, \omega_U)^{\otimes n} \rightarrow H^0(U, \omega_U^{\otimes n})), \quad n \geq 1.$$

(2.2.1) follows from the exact sequence of local cohomology, since $H_E^1(\mathcal{O}_X) = 0$ (Grauert–Riemenschneider – see [20], Theorem A), and $H_E^1(\omega_X) = 0$ (dual to $H^1(\mathcal{O}_X) = 0$). For (2.2.2), we use

$$(2.2.3) \quad \begin{array}{ccc} H^0(X, \omega_X)^{\otimes n} & \longrightarrow & H^0(X, \omega_X^{\otimes n}) \\ \downarrow & & \downarrow \\ H^0(U, \omega_U)^{\otimes n} & \xrightarrow{\phi_n} & H^0(U, \omega_U^{\otimes n}). \end{array}$$

The top row is surjective as above, and the right map is injective, whence (2.2.2). Putting everything together gives that V is computable canonically from U , viz.

$$(2.2.4) \quad V = \text{Proj}(\bigoplus \text{Im } \phi_n)$$

Let $\bar{V}$ be a deformation of V over $D = \text{Spec } \mathbb{C}[\epsilon]/\epsilon^2$, and $\omega_{\bar{V}}$ the

relative canonical sheaf of $\bar{V}$ over D (this is the unique lifting of the line bundle ω_V to $\bar{V}$). We claim $\bar{V} \simeq \text{Proj} \bigoplus H^0(\bar{V}, \omega_{\bar{V}}^{\otimes n})$; since a map between deformations is automatically an isomorphism, it suffices to show $H^0(\bar{V}, \omega_{\bar{V}}^{\otimes n})$ is D -flat, all $n \geq 0$. But $R^1 f_*(\omega_X^{\otimes n}) = 0$ and $H^1(X, \omega_X^{\otimes n}) = 0$ ([14], 7.3), so $H^1(V, \omega_V^{\otimes n}) = 0$. Therefore, $H^0(\bar{V}, \omega_{\bar{V}}^{\otimes n})$ is D -flat, by [22], 0.4.4.

If $\bar{V}$ blows down to a trivial deformation, then, as in [22], §1, the induced deformation $\bar{U}$ of U is trivial. Let $\omega_{\bar{U}} = \Omega_{\bar{U}/D}^2$. The barred analogues of (2.2.1)–(2.2.3) are still true (use $H^0(\bar{V}, \omega_{\bar{V}}^{\otimes n})$ instead of $H^0(X, \omega_X^{\otimes n})$), again using [22], Theorem 0.4. Therefore, $\bar{V} \simeq \text{Proj}(\bigoplus \text{Im } \bar{\phi}_n)$. But since $\bar{U}$ is a trivial deformation, $\bar{\phi}_n$ is a product, hence $\bar{V}$ is trivial. This completes the proof.

(2.3) One can identify directly the kernel of the tangent space map of $\text{Def } V \rightarrow \text{Def } R$ as $\text{Ext}_V^1(\Omega_{V/R}^1, \mathcal{O}_V)$. If $f: X \rightarrow V$, this can be recomputed as $\text{Ext}_X^1(f^* \Omega_{V/R}^1, \mathcal{O}_X)$, and a (non-obvious) reduction equates injectivity with $\text{Hom}_X(f^* \Omega_V^1, \mathcal{O}_Z(Z)) = 0$, where Z is the fundamental cycle. This should be viewed as a vanishing theorem analogous to those of [20]. After computing $f^* \Omega_V^1$ near the fibre of each RDP, we identified “easy cases” (cf. [21]) in which the theorem was true – determinantal singularities, and those with no -3 curves [24]. A more careful analysis of bad cycles gives the result if Z has multiplicity 1 at the -3 curves (e.g., for quotient singularities). Fortunately, Lipman’s theorem proves injectivity in complete generality, and without our long and complicated method.

§3. Determinantal deformations

(3.1) A determinantal rational singularity R has equations given by the 2×2 minors of a $2 \times d$ matrix, $d = \text{multiplicity of } R$ (see [23], §3 for a full discussion). There is a smooth subvariety (or subfunctor) Det of Def consisting of determinantal deformations; merely perturb arbitrarily the entries of the given matrix defining R . We will show $\text{Det} = A$, obtaining another proof of Theorem 2.2 in this case. Thus, Det is independent of the matrix used. Assume $d \geq 3$.

THEOREM 3.2: *For a determinantal rational singularity, $\text{Det} = A$; i.e., the determinantal deformations are exactly those which, after base change, simultaneously resolve in a family.*

PROOF: Let $X \rightarrow V \rightarrow \text{Spec } R$ be as usual. We show first that the inclusion $\text{Det} \subset \text{Def}$ factors via $\text{Def } V$, hence via A . Then we prove

that $\text{Def } X \rightarrow A$, which is surjective (as a map of spaces, not functors), factors via Det .

Recall V has a simple construction [23]. Assume R is defined (formally, say) by

$$(3.2.1) \quad \text{rk} \begin{pmatrix} f_1 & \cdots & f_d \\ g_1 & \cdots & g_d \end{pmatrix} \leq 1,$$

where f_i, g_i are in a power series ring of $d+1$ variables. Then V is the closure of the graph of the rational map $\text{Spec } R \rightarrow \mathbb{P}^1$ defined by the columns. In fact, $V \subset \text{Spec } R \times \mathbb{P}^1$ is defined by $sf_i = tg_i$, where s, t are homogeneous coordinates on $\mathbb{P}^1$. If now

$$\text{rk} \begin{pmatrix} F_1 & \cdots & F_d \\ G_1 & \cdots & G_d \end{pmatrix} \leq 1$$

defines a determinantal deformation $\text{Spec } \bar{R}$, we may use $sF_i = tG_i$ to define a deformation $\bar{V}$ of V . The verification that $\bar{V}$ is flat is done by using, e.g., that at least $d-1$ of the f_i 's have linearly independent leading forms ([23], 3.4). We omit the details. This shows $\text{Det} \subset A$.

On X , denote by E_0 the $-d$ curve, and by E_i ($i > 0$) the other -2 curves; recall E_0 has multiplicity 1 in the fundamental cycle Z . Pulling back $\mathcal{O}(1)$ from $X \rightarrow V \rightarrow \mathbb{P}^1$ gives an invertible sheaf $\mathcal{L}$ on X , with $\mathcal{L} \cdot E_0 = 1$, $\mathcal{L} \cdot E_i = 0$ ($i > 0$), and $\omega_X = \mathcal{L}^{\otimes(d-2)}$. Note that $h^1(\mathcal{L}) = 0$, $h^0(\mathcal{L} \otimes \mathcal{O}_Z) = 2$ (use Riemann–Roch).

Suppose that $Z \cdot E_0 < 0$; this means that the entries of the matrix (3.2.1) generate the maximal ideal of R , or the strict tangent cone is 0. Therefore, the rational map $\text{Spec } R \rightarrow \mathbb{P}^{d-1}$ (defined by the rows) is well-defined after one blow-up; in particular, there is a map $X \rightarrow \mathbb{P}^{d-1}$. Denote by $\mathcal{M}$ by pull-back of $\mathcal{O}(1)$. Since $\mathcal{M}$ is generated by its global sections, $h^1(\mathcal{M}) = 0$; also, $h^0(\mathcal{M} \otimes \mathcal{O}_Z) = d$. Now, the projectivized tangent cone $\text{Proj} \bigoplus H^0(Z, \mathcal{O}_Z(-nZ))$ embeds in $\mathbb{P}^1 \times \mathbb{P}^{d-1}$ (since $Z \cdot E_0 < 0$), so $\mathcal{O}_Z(-Z) \cong \mathcal{L} \otimes \mathcal{M} \otimes \mathcal{O}_Z$, whence $\mathcal{O}_X(-Z) \cong \mathcal{L} \otimes \mathcal{M}$ (recall that numerically equivalent line bundles are isomorphic). Thus, the map $\pi: X \rightarrow \mathbb{P}^1 \times \mathbb{P}^{d-1}$ has $\pi^*(\mathcal{O}(1) \otimes \mathcal{O}(1)) = \mathcal{O}(-Z)$. The maps

$$\Gamma(\mathbb{P}^1 \times \mathbb{P}^{d-1}, \mathcal{O}(n) \otimes \mathcal{O}(n)) \rightarrow \Gamma(X, \mathcal{O}_X(-nZ)) \subset \Gamma(X, \mathcal{O}_X)$$

give

$$(3.2.2) \quad C = \bigoplus_{n=0}^{\infty} \Gamma(\mathbb{P}^1 \times \mathbb{P}^{d-1}, \mathcal{O}(n) \otimes \mathcal{O}(n)) \rightarrow \Gamma(X, \mathcal{O}_X).$$

We claim (3.2.2) is surjective on completions, by showing the map of gr 's is surjective. The map of n^{th} graded pieces is

$$\Gamma(\mathbf{P}^1 \times \mathbf{P}^{d-1}, \mathcal{O}(n) \otimes \mathcal{O}(n)) \rightarrow \Gamma(Z, \mathcal{O}_Z(-nZ)).$$

But consider

$$\begin{array}{ccc} (\bigotimes^n \Gamma(\mathbf{P}^1, \mathcal{O}(1))) \otimes (\bigotimes^n \Gamma(\mathbf{P}^{d-1}, \mathcal{O}(1))) & \rightarrow & \bigotimes^n \Gamma(\mathcal{O}_Z(-Z)) \\ \downarrow \wr & & \downarrow \\ (\bigotimes^n \Gamma(\mathcal{L} \otimes \mathcal{O}_Z)) \otimes (\bigotimes^n \Gamma(\mathcal{M} \otimes \mathcal{O}_Z)) & & \\ \downarrow & & \downarrow \\ \Gamma(\mathbf{P}^1, \mathcal{O}(n)) \otimes \Gamma(\mathbf{P}^{d-1}, \mathcal{O}(n)) & \longrightarrow & \Gamma(\mathcal{O}_Z(-nZ)). \end{array}$$

The top and right maps are surjective, by [14], 7.3, whence so is the bottom. This proves the claim. Note C is the generic $2 \times d$ determinantal singularity, and (3.2.2) shows how to write the matrix (3.2.1) from $\pi: X \rightarrow \mathbf{P}^1 \times \mathbf{P}^{d-1}$.

But $\mathcal{L}$ and $\mathcal{M}$ lift uniquely to any deformation of X (since $h^1(\mathcal{O}_X) = h^2(\mathcal{O}_X) = 0$), and the sections of $H^0(\mathcal{L})$ and $H^0(\mathcal{M})$ lift as well (since their H^1 's are 0). Thus, the map $X \rightarrow \mathbf{P}^1 \times \mathbf{P}^{d-1}$ lifts, and (3.2.2) lifts after deformation (of course, C is rigid). This shows how to perturb the entries of the matrix (3.2.1) after deformation of X . (We use implicitly that a map between deformations is an isomorphism). Thus, $\text{Def } X$ maps into Det .

Next, suppose $Z \cdot E_0 = 0$; it is no longer true that the projectivized tangent cone embeds in $\mathbf{P}^1 \times \mathbf{P}^{d-1}$. Define inductively cycles L_j, B_j , where $L_0 = E, B_0 = Z$, and

(i) L_{j+1} = connected component of $\{E_i \subset L_j \mid B_j \cdot E_i = 0\}$ containing E_0

(ii) B_{j+1} = fundamental cycle of L_{j+1} .

Eventually, $B_k \cdot E_0 < 0$, some k . Let $Z_1 = B_0 + \cdots + B_k$. Then $Z_1 \cdot E_0 < 0$, $B_i \cdot B_j = 0$, $i \neq j$, so $h^0(\mathcal{O}_{Z_1}) = k + 1$, and $h^1(\mathcal{O}(-Z_1)) = 0$. (Use Lemma 1.7 to show $Z_1 \cdot E_j \leq 0$, all i ; e.g., end curves of L_{j+1} have multiplicity 1 in B_{j+1}). By construction, $k + 1$ = multiplicity of E_0 in Z_1 = number of blow-ups of $\text{Spec } R$ needed to drop the multiplicity (cf. [18]). Also, $H^0(\mathcal{O}(-Z_1)) = I$ is the complete ideal, of colength $k + 1$, generated by the entries of the matrix (3.2.1).

The rational map $\text{Spec } R \rightarrow \mathbf{P}^{d-1}$ is well-defined after $k + 1$ blow-ups of $\text{Spec } R$ (following the point of multiplicity d), hence there is a map $X \rightarrow \mathbf{P}^{d-1}$. Let $\mathcal{M}$ be the pull-back of $\mathcal{O}(1)$. Then $\mathcal{L} \otimes \mathcal{M}$ is the pull-back of $\mathcal{O}(1) \otimes \mathcal{O}(1)$ from $X \rightarrow \mathbf{P}^1 \times \mathbf{P}^{d-1}$; but by construction, this is $I\mathcal{O}_X$,

where I is the ideal generated by the entries of the matrix. Thus, $\mathcal{L} \otimes \mathcal{M} \simeq \mathcal{O}(-Z_1)$. The map $Z \rightarrow \mathbb{P}^1 \times \mathbb{P}^{d-1}$ again expresses the determinantal nature of the projectivized tangent cone. There is a map

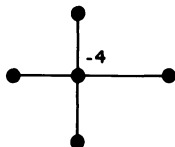
$$(3.2.4) \quad \bigoplus \Gamma(\mathbb{P}^1 \times \mathbb{P}^{d-1}, \mathcal{O}(n) \otimes \mathcal{O}(n)) \rightarrow \Gamma(X, \mathcal{O}_X)$$

so that the map on the n^{th} piece of the associated graded is

$$\Gamma(\mathcal{L}^n \otimes \mathcal{O}_Z) \otimes \Gamma(\mathcal{M}^n \otimes \mathcal{O}_Z) \rightarrow \Gamma(\mathcal{O}_Z(-nZ_1)) \simeq I^n / mI^n$$

(compare to the preceding). In fact, the completion of (3.2.4) maps onto $\mathbb{C} + I$, a subring of finite colength in R . Nonetheless, we proceed as before. Deformations of X carry the map into $\mathbb{P}^1 \times \mathbb{P}^{d-1}$ ($H^1(\mathcal{L}) = H^1(\mathcal{M}) = 0$), hence (3.2.4) deforms, and one again knows how to perturb the entries of the matrix defining R . This completes the proof of Theorem 3.2.

EXAMPLE (3.3) (See [23], 5.5): A particular rational singularity of multiplicity 4 with graph



may be written determinantly via the matrix

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_4 & x_1 + x_5^2 \end{pmatrix}$$

One computes that $\dim T_R^1 = 10$. The versal determinantal deformation, of dimension 8, is given by

$$(3.3.1) \quad \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 + t_1 + t_6 x_5 & x_3 + t_2 + t_7 x_5 & x_4 + t_3 + t_8 x_5 & x_1 + x_5^2 + t_4 + t_{10} x_3 \end{pmatrix}.$$

(Note t_{10} is the “equisingular” parameter). Another four-dimensional family (not obviously an irreducible component) can be read off the 2×2 minors of the symmetric 3×3 matrix:

$$(3.3.2) \quad \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 + t_5 + t_9 x_5 & x_4 \\ x_3 & x_4 & x_1 + x_5^2 + t_4 + t_{10} x_3 \end{pmatrix}.$$

Now, the one-parameter subfamily of (3.3.2) given by $t_4 = s^2$, $t_5 = s^3$, $t_9 = s$, $t_{10} = 0$, has a family of -4 singularities along the section $x_1 = x_2 = x_3 = x_4 = 0$, $x_5 = s$; also, it intersects (3.3.1) only at $s = 0$. But a check shows that (3.3.2) acts as the non-Artin component along the s -curve. By local versality of Def, and Pinkham's description [16] of the moduli space for -4 , it follows that there is another component, of dimension ≥ 6 , acting as the Artin component along the s -curve. In fact, a computation shows Def has 4, 6, and 8-dimensional components; the 6-dimensional one is singular, with smooth normalization.

REMARK (3.4): For a general determinantal singularity, Det is not an irreducible component of the moduli space; in fact, it will depend on which matrix representation is used. For instance, the moduli space of 4 lines through the origin in $\mathbb{C}^4$ is a cone over $\mathbb{P}^1 \times \mathbb{P}^3$ (hence irreducible, but not smooth). Note, incidentally, that this singularity is the affine form of the projectivized tangent cone in (3.3) above (i.e., set $x_5 = 0$ in the equation).

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