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HEREDITARY CIRCUIT SPACES

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(to Leon Mirsky on his sixtieth birthday)

In this paper we study a natural class of independence spaces, namely those in which each 'line' contains at least three points, i.e. those in which each independent pair of points is contained in a circuit of cardinality three. In the introductory section we outline some of the motivations for studying such spaces, and we survey known results concerning them. Section 2 contains some observations about their linear representability and their relation to projective spaces. The lattices of flats of independence spaces in general and of these special spaces in particular are briefly discussed in Section 3. In the final section we take some initial steps to determine 'how large' is the class of these spaces.

1. Introduction and preliminary results

1.1 A famous theorem formulated by Sylvester and proved by Gallai (see [7, p3]) states that, if a set of non-collinear points of real n-dimensional projective space is such that no line contains precisely two members of the set, then the set must be infinite. This theorem led U.S.R. Murty in 1968 [12, 13] to begin investigating a class of matroids (or finite independence spaces) with the property that every pair of independent points lies in a circuit of cardinality three (a 3-circuit). Murty referred to these as 'Sylvester matroids'. More recently, in [3, 6], the corresponding independence spaces have been called *hereditary circuit spaces* (hcs's). The motivation for this terminology is given in the following result. Recall that, in [4, 14], a *circuit space* is an independence space in which each basis lies in a circuit.

THEOREM 1.1: *Let $(E, \mathscr{E})$ be an independence space. Then the following three properties are equivalent*:

(i) *$(E, \mathscr{E})$ is an hcs (i.e. every independent 2-set lies in a 3-circuit)*;

(ii) *for each flat F of $(E, \mathscr{E})$ of finite rank at least 2, the restriction of $\mathscr{E}$ to F is a circuit space*;

(iii) *every finite independent set A with $|A| \geq 2$ lies in an $(|A|+1)$-circuit.* $\square$

The proof of this result is omitted because the equivalence of (ii) and (iii) is very straightforward and the proof of the equivalence of (i) and (iii) for matroids in [12, 13] carries over to this more general situation, with no restriction on the cardinality of E.

The following is another result due to Murty.

THEOREM 1.2: *An hcs of rank r contains at least $2^r - 1$ points.* $\square$

If the space contains exactly $2^r - 1$ points, then it is clear from Murty's proof of this result that each independent k-set lies in a *unique* $(k+1)$-circuit $(k \geq 2)$. In Theorem 2.4 below we give an alternative proof to Murty's that, in this situation, the space arises from a projective geometry over GF(2).

As stated in the introductory paragraph, in this paper we propose to take up the study of hcs's and to turn our attention to the problems listed there. For an introduction to independence spaces we refer the reader to [5, 10, 19]. Our notation is that of [10] and the term 'geometry' will be used as in [5] for an independence space in which every 2-set is independent. The symbol $\{\}_{\neq}$ will be used to indicate that the elements enclosed by the brackets are distinct. Throughout the rest of this paper, all spaces are assumed to have finite rank.

1.2 By describing a simple example, we now indicate how we came across hcs's. Carathéodory's classical theorem in an n-dimensional real vector space X states that, if a point is in the convex hull of a subset A of X, then it is in the convex hull of a subset of A of cardinality not exceeding $n+1$. With this in mind, given any closure operator [ ] on a set E, we call $A \subseteq E$ *Carathéodory-independent if* either $A = \emptyset$ or $\bigcup_{B \subset A} [B] \subset [A]$. In fact, Carathéodory's theorem asserts that, if [ ] denotes the usual convex-hull operator in a real vector space, then the Carathéodory-independent sets are precisely the affinely independent sets.

One of the most natural closure operators for us to consider in the present context is that induced from an independence space $(E, \mathscr{E})$ with $[A]$ equal to the flat spanned by A for each $A \subseteq E$. We now investigate under what conditions the Carathéodory-independent sets again coincide with the existing independent sets (i.e. members of $\mathscr{E}$).

THEOREM 1.3: *Let* $(E, \mathscr{E})$ *be an independence space and let* $\mathscr{E}_c$ *denote the collection of Carathéodory-independent subsets of E with respect to the closure operator* [ ] *in which* $[A]$ *is the flat spanned by* A *in* $(E, \mathscr{E})$. *Then*

(i) $\mathscr{E}_c \subseteq \mathscr{E}$;

(ii) $\mathscr{E}_c = \mathscr{E}$ *if and only if* $(E, \mathscr{E})$ *is an* hcs.

PROOF: (i) Assume that $A \notin \mathscr{E}$ and let A' be a basis of A. Then $A' \subset A$ and

$$[A] = [A'] = \bigcup_{B \subset A} [B];$$

and so $A \notin \mathscr{E}_c$.

(ii) Assume that $\mathscr{E}_c = \mathscr{E}$, and let $A \in \mathscr{E}$ with $|A| \geq 2$. Then $A \in \mathscr{E}_c$, and so there exists $x \in [A] \setminus \bigcup_{B \subset A} [B]$. Clearly $A \cup \{x\}$ is a circuit of $(E, \mathscr{E})$; and $(E, \mathscr{E})$ is an hcs.

Conversely, let $(E, \mathscr{E})$ be an hcs and let $A \in \mathscr{E}$. Then either $A = \emptyset$ $(\in \mathscr{E}_c)$, or $A = \{x\}$ say (and

$$[A] = [\{x\}] \supset [\phi] = \bigcup_{B \subset A} [B],$$

so that $A \in \mathscr{E}_c$) or $|A| \geq 2$. In this last case, there exists x such that $A \cup \{x\}$ is a circuit, and evidently $x \in [A] \setminus \bigcup_{B \subset A} [B]$ and so again $A \in \mathscr{E}_c$. Hence $\mathscr{E} \subseteq \mathscr{E}_c$ and, by (i), $\mathscr{E} = \mathscr{E}_c$. □

1.3 We conclude this introductory section with some ideas needed later. Let r, s be non-negative integers with $r \leq s$ and let $(E, \mathscr{E})$ be an independence space of rank s. Its *truncation at r* is the independence space $(E, \mathscr{E}')$, where $\mathscr{E}'$ is the set of those members of $\mathscr{E}$ of cardinality at most r. An independence space which is the truncation of another space of strictly greater rank will be called a *truncated space.* Evidently truncated spaces are necessarily circuit spaces; though the converse is false [14]. Hcs's are obviously closed under the operation of truncation but, of course, not under restriction. Slightly less obvious is the following elementary fact used later.

LEMMA 1.4: *The contraction of an hcs is itself an hcs.*

PROOF: Let $(E, \mathscr{E})$ be an hcs, let $F \subseteq E$ and let $Y \in \mathscr{E}_{\otimes E \setminus F}$ (the contraction of $\mathscr{E}$ away from F) with $|Y| \geq 2$. Then, if B is any basis of

$\mathscr{E} \mid F$, it follows that $Y \cup B \in \mathscr{E}$ and $Y \cup B \cup \{x\}$ is a circuit of $\mathscr{E}$ for some $x \in E$. Since $B \cup \{x\} \notin \mathscr{E} \mid F$, therefore $x \in E \backslash F$. Further, for each $y \in Y$, $((Y \cup \{x\}) \backslash \{y\}) \cup B \in \mathscr{E}$ and $(Y \cup \{x\}) \backslash \{y\} \in \mathscr{E}_{\otimes E \backslash F}$. Thus $Y \cup \{x\}$ is a circuit of $\mathscr{E}_{\otimes E \backslash F}$, and $(E \backslash F, \mathscr{E}_{\otimes E \backslash F})$ is an hcs. $\square$

With each independence space $(E, \mathscr{E})$ there is associated in an obvious and natural way a geometry, geom$(E, \mathscr{E})$; its construction is described in [14]. We observe in passing that $(E, \mathscr{E})$ is an hcs if and only if geom $(E, \mathscr{E})$ is an hcs.

Finally, we mention that an independence space which is not an hcs may still have the property that, for some fixed $k > 2$, every independent k-set lies in a $(k+1)$-circuit. Indeed, if $E = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and the circuits of the rank-4 independence space $(E, \mathscr{E})$ are precisely $\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{1,2,7,8\}$, $\{1,3,5,7\}$, $\{1,3,6,8\}$, $\{1,4,5,8\}$, $\{1,4,6,7\}$ and their complements in E, then every independent 3-set lies in a 4-circuit, no independent 2-set lies in a 3-circuit, and no independent 4-set lies in a 5-circuit. It is interesting to compare this situation with that in Theorem 2.2 below.

2. Modular, projective and linear spaces

2.1 We begin this section with material which is well known, though in a somewhat different context; and our account is therefore brief. An independence space is generally called modular if its lattice of flats is modular. So if, as above, $[X]$ denotes the flat spanned by $X (\subseteq E)$ in $(E, \mathscr{E})$, then the space is modular if and only if $A \cap [B \cup C] = [(A \cap B) \cup C]$ for all flats A, B, C with $A \supseteq C$. Certainly not every hcs is modular; for example, the 9-point affine plane $E = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ with circuits $\{1,2,3\}$, $\{1,4,5\}$, $\{1,6,7\}$, $\{1,8,9\}$, $\{2,4,6\}$, $\{2,5,9\}$, $\{2,7,8\}$, $\{3,4,8\}$, $\{3,5,7\}$, $\{3,6,9\}$, $\{4,7,9\}$, $\{5,6,8\}$ and all 4-sets not containing one of these, is non-modular. (Note for future reference that these 3-circuits form a Steiner triple system on E.) In fact, the modular hcs's are very special indeed.

THEOREM 2.1: *Let $(E, \mathscr{E})$ be a geometry of rank at least 2. Then it is a projective geometry if and only if it is a modular hcs.* $\square$

This is an immediate consequence of the known result that the projective geometries are precisely the connected modular geometric independence spaces. A direct proof of Theorem 2.1 is straightforward, but the derivation of it from the result on connected spaces was pointed out to us by P. Vámos, and we are grateful to him for giving us access to his unpublished work [17] in which the latter result can be found. (Related results appear in [1, 2, 9].) Another of its consequences is the following theorem.

THEOREM 2.2: *Let $(E, \mathscr{C})$ be modular and of rank r, and let k be a fixed integer with $2 \leq k \leq r$. Then if each independent k-set lies in a $(k+1)$-circuit it follows that $(E, \mathscr{C})$ is an hcs.*

PROOF: It follows at once from the given condition that each independent 2-set is contained in a circuit, and hence that the space is connected. If it is a geometry, then the proof is complete. Otherwise, we invoke the remarks about geom$(E, \mathscr{C})$ in section 1.3. □

We refer to [17] again, or to [2, 9], for the result that $(E, \mathscr{C})$ is modular if and only if, given a circuit $\{x_1, \ldots, x_n\}_{\neq}$ $(n \geq 4)$, there exists $x \in E$ such that both of $\{x_1, \ldots, x_{n-2}, x\}$, $\{x_{n-1}, x_n, x\}$ are circuits. We make use of this in the next two results which involve uniqueness conditions.

LEMMA 2.3: *Let $(E, \mathscr{C})$ be such that each independent k-set is contained in a unique $(k+1)$-circuit (for each $k \geq 2$). Then $(E, \mathscr{C})$ is modular.*

PROOF: Let $\{x_1, \ldots, x_n\}_{\neq}$ be a circuit in $(E, \mathscr{C})$ with $n \geq 4$, and let $\{x_1, \ldots, x_{n-2}, x\}$, $\{x_{n-1}, x, y\}_{\neq}$ be circuits. Then, for each i with $1 \leq i \leq n-1$, the set $\{x_1, \ldots, x_{n-1}, y\}_{\neq}$ contains a circuit containing x_i. Since, however it contains a unique circuit (because $\{x_1, \ldots, x_{n-1}\} \in \mathscr{C}$), it follows that $\{x_1, \ldots, x_{n-1}, y\}$ is itself a circuit. The uniqueness of the n-circuit containing $\{x_1, \ldots, x_{n-1}\}$ shows that $y = x_n$ and that $\{x_1, \ldots, x_{n-2}, x\}$, $\{x_{n-1}, x_n, x\}$ are both circuits. Thus $(E, \mathscr{C})$ is modular. □

From this lemma, we deduce a companion result to Theorem 2.1.

THEOREM 2.4: *Let $(E, \mathscr{C})$ be a geometry of rank at least 2. It is a projective geometry over GF(2) if and only if each independent k-set is contained in a unique $(k+1)$-circuit $(k \geq 2)$.*

PROOF: Certainly each independent k-set of a projective geometry over GF(2) is contained in a unique $(k+1)$-circuit $(k \geq 2)$. The converse follows from Lemma 2.3 and Theorem 2.1 since the projective geometries over GF(2) are the only projective geometries in which each line (or flat of rank 2) contains just 3 points. □

2.2 Much interest has been aroused increasingly in recent years concerning the linear representation of independence spaces; and some remarks on the linear representability of hcs's would seem to be appropriate here. In [18] P. Vámos has shown that every independence space can be embedded in an hcs (his construction always yields an infinite space), and an immediate corollary is that not every hcs is linearly representable (This is, of course, also clear from a consideration of non-Desarguesian projective planes.) He has also demonstrated the existence of finite geometric hcs's which cannot be embedded in any projective space. We further note the immediate translation of Sylvesters theorem, namely that no finite hcs of rank exceeding 2 is linear representable over the reals. In [12] Murty raised the question of finding classes of independence spaces disjoint from the class of hcs's. Fairly simple direct proofs that this is the case for the class of transversal spaces of rank exceeding 2 and for the class of cotransversal spaces of rank exceeding 2 have been given in [6]. As Dr. Vámos kindly points out, since the transversal spaces and their duals are linearly representable over the reals (see, for example [8, 15]), an alternative proof of these last results for finite spaces can be obtained from Sylvester's theorem.

3. Lattices of flats

In this section we compare the 'lengths' and 'widths' of elements in a semi-modular lattice, and hence obtain a new characterization of those lattices which are isomorphic to the lattice of flats of an independence space. We are then able to compare this with a corresponding characterization of the lattices associated with hcs's.

Let Λ (or $(\Lambda, <)$) denote a lattice with inf and sup operations $\wedge$, $\vee$ respectively and which has no infinite chains. In particular, Λ has a null element 0. We recall that x *covers* y in Λ if $y < x$ and there is no $z \in \Lambda$ with $y < z < x$. The elements of Λ which cover 0 are the *atoms* of Λ. A *semi-modular* lattice is one without infinite chains in which, if x covers $x \wedge y$, then $x \vee y$ covers y. It is well known that, in such a lattice Λ,

$$\ell(x \vee y) + \ell(x \wedge y) \leq \ell(x) + \ell(y)$$

for each $x, y \in \Lambda$, where $\ell(x)$ denotes the usual *length* (or rank) of x

and is finite. (See, for example, [2, 16].) Hence, in a semi-modular lattice Λ,

$$\ell(x_1 \vee \cdots \vee x_n) \leq \ell(x_1) + \cdots + \ell(x_n)$$

for each $x_1, \ldots, x_n \in \Lambda$. In any lattice Λ we may also define the *width* $w(x)$ of $x \in \Lambda$ to be the number (not necessarily finite) of elements of Λ which are covered by x. In a semi-modular lattice Λ the atoms of Λ are the elements of length 1, and their width is also 1.

THEOREM 3.1: *For a semi-modular lattice Λ the following three properties are equivalent*:

(i) $w(x) \geq 2$ *whenever* $\ell(x) \geq 2$;

(ii) *for each* $x \in \Lambda \backslash \{0\}$, *there exist atoms* $x_1, \ldots, x_n$ *with* $x = x_1 \vee \cdots \vee x_n$;

(iii) $w(x) \geq \ell(x)$ *for each* $x \in \Lambda$.

PROOF: (i)$\Rightarrow$(ii). Assume (i), and let $x \in \Lambda \backslash \{0\}$. We prove (ii) by induction on $\ell(x)$, the case $\ell(x) = 1$ being immediate since x is then itself an atom. So assume that $\ell(x) > 1$ and that the result is known for elements of shorter length. Then, by (i), $w(x) \geq 2$ and so x covers $y, z \in \Lambda$ with $y \neq z$. Clearly $\ell(y)$, $\ell(z) < \ell(x)$ and so, by the induction hypothesis, there exist atoms $x_1, \ldots, x_n$ with

$$y = x_1 \vee \cdots \vee x_r; \quad z = x_{r+1} \vee \cdots \vee x_n.$$

Now $y < y \vee z \leq x$ and so, as x covers y, it follows that

$$x = y \vee z = x_1 \vee \cdots \vee x_n,$$

as required.

(ii)$\Rightarrow$(iii) Assume (ii) and let $x \in \Lambda$. If $x = 0$, then $\ell(x) = w(x) = 0$, and if x is an atom then $\ell(x) = w(x) = 1$. So let us suppose that x is neither the null element nor an atom. Then, by (ii), $x = x_1 \vee \cdots \vee x_n$ for some atoms $x_1, \ldots, x_n$ with n chosen to be minimal and $n \geq 2$. Now, if $1 \leq i \leq n$, then

$$y_i = x_1 \vee \cdots \vee x_{i-1} \vee x_{i+1} \vee \cdots \vee x_n \leq x$$

and, by the minimality of n, $y_i \neq x$. Hence $y_1, \ldots, y_n < x$. In fact, x covers each of $y_1, \ldots, y_n$. For, again by the minimality of n, $x_i \nleq y_i$ and so $y_i \wedge x_i = 0$. Thus x_i covers $y_i \wedge x_i$ and, by semimodularity, y_i is covered by $x_i \vee y_i$ $(= x_1 \vee \cdots \vee x_n = x)$. Clearly the $y_1, \ldots, y_n$ are distinct, for if $i \neq j$, then $x_i \leq y_j$ whereas $x_i \nleq y_i$. Hence x covers at least n elements of Λ, and

$$w(x) \geq n = \ell(x_1) + \cdots + \ell(x_n) \geq \ell(x_1 \vee \cdots \vee x_n) = \ell(x),$$

as required.

(iii) $\Rightarrow$ (i). Immediate. □

It is well known (see, for example, [5]) that the lattice Λ is isomorphic to the lattice of flats of an independence space if and only if it is a geometric lattice, i.e. if and only if it is semi-modular and each element is the supremum of a finite set of atoms. So the following corollary is immediate.

COROLLARY 3.2: *The lattice Λ is isomorphic to the lattice of flats of an independence space if and only if it is semi-modular and satisfies either of the conditions*:

(i) $w(x) \geq 2$ *whenever* $\ell(x) \geq 2$;

(ii) $w(x) \geq \ell(x)$ *whenever* $\ell(x) \geq 2$

(*in which case it satisfies both*). □

It is of interest to compare this corollary with the following result.

THEOREM 3.3: *The lattice Λ is isomorphic to the lattice of flats of an hcs if and only if it is semi-modular and satisfies either of the conditions*:

(i) $w(x) > 2$ *whenever* $\ell(x) \geq 2$;

(ii) $w(x) > \ell(x)$ *whenever* $\ell(x) \geq 2$

(*in which case it satisfies both*).

PROOF: Clearly condition (ii) implies condition (i). Assume first that Λ is a semi-modular lattice with $w(x) > 2$ whenever $\ell(x) \geq 2$. Then, in particular, by Corollary 3.2, Λ is (isomorphic to, and may be identified with) the lattice of flats of an independence space $(E, \mathscr{E})$. Let $\{a, b\}_{\neq} \in \mathscr{E}$. Then $\ell([\{a, b\}]) = 2$ and so $w([\{a, b\}]) > 2$. Hence $[\{a, b\}]$ covers $[\{a\}]$, $[\{b\}]$ and some other flat of the form $[\{c\}]$, where $\{c\} \in \mathscr{E}$. Since $[\{c\}] < [\{a, b\}]$ (whence $[\{c\}] \subseteq [\{a, b\}]$), it follows that $c \in [\{a, b\}]$ and $\{a, b, c\} \notin \mathscr{E}$. Also, as $[\{a\}]$, $[\{b\}]$ $[\{c\}]$ are all distinct, each of $\{a, b\}$, $\{b, c\}$, $\{a, c\}$ is in $\mathscr{E}$. Hence $\{a, b, c\}$ is a circuit in $\mathscr{E}$; and $(E, \mathscr{E})$ is an hcs.

Conversely, let us assume (again without loss of generality) that Λ *is* the lattice of flats of an hcs $(E, \mathscr{E})$. Certainly, Λ is semi-modular; and it only remains to prove that condition (ii) is satisfied. Suppose, then, that $x \in \Lambda$ is such that $\ell(x) = n \geq 2$. Then x is a flat of rank n of $\mathscr{E}$ spanned, say, by $\{a_1, \ldots, a_n\}_{\neq} \in \mathscr{E}$. Now there exists a circuit $A = \{a_1, \ldots, a_n, a_{n+1}\}_{\neq}$ in $(E, \mathscr{E})$ and, for each $\{i, j\}_{\neq} \subseteq \{1, 2, \ldots, n+1\}$,

$[A \backslash \{a_i, a_j\}]$ is a flat covered by x. Since, for $n \geq 2$, $\binom{n+1}{2} > n$, it follows that $w(x) > \ell(x)$. $\square$

4. The extent of the class of hcs's

4.1 It has already been observed that the unique Steiner triple system on 9 elements provides a simple example of a non-modular hcs (the non-triples are the bases). This is therefore not a projective space. It is straightforward to show that it is also non-truncated, but we omit the proof since Theorem 4.2 below is a stronger result.

LEMMA 4.1: *Let $(E, \mathscr{E})$ be a truncated space and let A, B be two of its bases. Then either there is an element $b \in B$ such that $A \cup \{b\}$ is a circuit or there is an element $x \in E$ such that $A \cup \{x\}$, $B \cup \{x\}$ are both circuits.*

PROOF: Let $(E, \mathscr{E})$ be the truncation at n of the independence space $(E, \mathscr{E}')$ of rank $n+1$, and let $B \cup \{x\}$ be a basis of $\mathscr{E}'$. Then, by basis-exchange in $\mathscr{E}'$, there is an element $b \in B \cup \{x\}$ such that $A \cup \{b\}$ is a basis of $\mathscr{E}'$. Since $A \cup \{b\}$ and $B \cup \{x\}$ are both circuits of $\mathscr{E}$, the result follows. $\square$

As is customary, we denote by $\mathrm{PG}(r-1, q)$ the projective geometry of rank r (dimension $r-1$) over $\mathrm{GF}(q)$.

THEOREM 4.2: *There exist geometric hcs's of every rank $r \geq 3$ which are neither projective spaces nor truncated spaces.*

PROOF: Consider $\mathrm{PG}(r-1, 3)$, whose elements are regarded as proportional r-ads. For convenience we write a typical element without parentheses or commas as $a_1 \ldots a_r$, where each a_i is 0, 1 or 2. Let $(E, \mathscr{E})$ be the space $\mathrm{PG}(r-1, 3)$ from which the one element $1 \ldots 1$ has been deleted, and where independence means linear independence. Certainly $(E, \mathscr{E})$ is an independence space of rank r which is not a projective space. Further, since every 2-set is independent and spans a line of at least 3 points, the space is also a geometric hcs. We use Lemma 4.1 to show that it is not truncated. Consider the bases

$$A = \{100\ldots0,\ 010\ldots0,\ 001\ldots0, \ldots, 000\ldots1\}$$

and

$$B = \{110\ldots0,\ 120\ldots0,\ 001\ldots0, \ldots, 000\ldots1\}$$

of $(E, \mathscr{C})$. If $A \cup \{x\}$ is a circuit, then each component of x is 1 or 2 (and, in particular, $x \notin B$). Also, if $B \cup \{x\}$ is a circuit, then one of the first two components of x must be 0. Hence A and B fail to satisfy Lemma 4.1; and so the space is non-truncated. □

4.2 In view of Th. 4.2, the class of geometric hcs's embraces spaces more general than projective spaces of their truncations. We now pose the natural question 'does there exist an hcs on a set E of any assigned finite positive cardinal?' The answer is trivially 'yes' if we permit dependent singletons or doubletons, or if we allow spaces of rank less than 3. So we should obviously seek to determine the cardinalities of those sets on which there may be constructed a geometric hcs of rank at least 3. If such a space exists on E, then its truncation at 3 is a geometric hcs of rank 3 on E. It is sufficient, therefore, to look for spaces of rank 3. Perhaps surprisingly, an answer to this question has been in the literature (in other terms) for about 25 years in the work of Th. Motzkin [11]. The spaces which he constructed were restrictions of projective planes over some GF(q), and his result is virtually equivalent to the following theorem.

THEOREM 4.3: *There exists a geometric hcs of rank* 3 *on a set of cardinality n if and only if $n = 7$ or $n \geq 9$.* □

We now attempt to determine the range of values of n for which there exist geometric hcs's of prescribed rank r (≥ 3) on sets of cardinality n. An integer n will be called *r-admissible* if there exists a geometric hcs of rank r on a set of cardinality n. We shall present a partial solution to the problem of finding the set of r-admissible numbers.

LEMMA 4.4: *If $(E, \mathscr{C})$ is a geometric hcs of rank r (≥ 2) in which at least one line contains* 4 *or more points, then $|E| \geq 2^r + 2^{r-2} - 1$.*

PROOF: The case $r = 2$ is trivially true, so let us assume that $r > 2$ and that the result is known for $r - 1$. Let $(E, \mathscr{C})$ be an hcs of rank r with a line ℓ containing 4 or more points. Now ℓ lies in a hyperplane H of $\mathscr{C}$, and the restriction of $(E, \mathscr{C})$ to H is a geometric hcs of rank $r - 1$. Therefore, by the induction hypothesis, $|H| \geq 2^{r-1} + 2^{r-3} - 1$. Now, for a fixed $x \in E \backslash H$, each line joining x to a point of H contains at least one further point. Therefore

$$|E| \geq 2(2^{r-1} + 2^{r-3} - 1) + 1 = 2^r + 2^{r-2} - 1. \quad \square$$

THEOREM 4.5: *Given $r \geq 2$, no integer n satisfying*

$$2^r - 1 < n < 2^r + 2^{r-2} - 1$$

is r-admissible.

PROOF: Again the result is trivial for $r = 2$, so let us assume that $r > 2$ and that the result is true for $r - 1$ but fails for r. Then there exists a geometric hcs $(E, \mathscr{C})$ of rank r with $|E| = n$, where

$$2^r - 1 < n < 2^r + 2^{r-2} - 1.$$

By Lemma 4.4, each line of $(E, \mathscr{C})$ contains exactly 3 points. Let $x \in E$, and let $\mathscr{C}'$ be the contraction of $\mathscr{C}$ away from $\{x\}$. Then $\{p, q\}_{\neq} \in \mathscr{C}'$ if and only if the line joining p and q in $(E, \mathscr{C})$ does not contain x. Therefore the geometric hcs naturally associated with $(E \backslash \{x\}, \mathscr{C}')$ contains precisely the same number of points as there are lines in $(E, \mathscr{C})$ containing x, namely $\frac{1}{2}(n-1)$. Also the rank of this associated space is $r - 1$. But then

$$2^{r-1} - 1 < \frac{n-1}{2} < 2^{r-1} + 2^{r-3} - 1;$$

which contradicts the assumption that the result holds for $r - 1$. $\square$

In the preliminaries to our main result in this section, Theorem 4.8, we use the notion of a '3-configuration' motivated by the 3 concurrent lines in projective space used in Motzkin's proof of Theorem 4.3. The configuration is more conveniently defined in an affine space, which is a projective space from which the points of one particular hyperplane ('at infinity') have been removed. We now define inductively a 3-configuration in an affine space $\mathrm{AG}(r-1, q)$ of rank r (≥ 3) over $\mathrm{GF}(q)$ ($q \geq 3$):

(1) In $\mathrm{AG}(2, q)$ a 3-configuration is the union of the points of 3 parallel lines.

(2) For $r > 3$, a 3-configuration in $\mathrm{AG}(r-1, q)$ consists of the union of the points of 3 parallel hyperplanes together with a 3-configuration, as defined for $\mathrm{AG}(r-2, q)$ in each of $q - 3$ other hyperplanes each parallel to the first 3.

Recall that $|\mathrm{AG}(r-1, q)| = q^{r-1}$, and note that, if $\Delta(r-1, q)$ is a 3-configuration in $\mathrm{AG}(r-1, q)$, then

$$|\Delta(r-1, q)| = 3\{q^{r-2} + (q-3)q^{r-3} + \cdots + (q-3)^{r-3}q\}.$$

This is easily established by induction since, for $r > 3$, $|\Delta(r-1, q)| = 3q^{r-2} + (q-3)|\Delta(r-2, q)|$.

THEOREM 4.6: *If $r \geq 3$ and $q \geq 5$, then every integer n which satisfies the inequalities*

$$|\Delta(r-1,q)| \leq n \leq q^{r-1}$$

is r-admissible. Indeed, there is a restriction of AG$(r-1,q)$ *of rank r and cardinality n which is an* hcs.

PROOF: For $r = 3$ this is readily checked (and the necessity for the condition $q \geq 5$ becomes clear). Let us assume that $r > 3$ and that the result is true for $r-1$, and let n satisfy

$$|\Delta(r-1,q)|(=3q^{r-2}+(q-3)|\Delta(r-2,q)|) \leq n \leq q^{r-1}.$$

Then $n = 3q^{r-2} + n_1 + \cdots + n_{q-3}$ for some n_i with $|\Delta(r-2,q)| \leq n_i \leq q^{r-2}$, $1 \leq i \leq q-3$. Take q distinct parallel hyperplanes $H_1, \ldots, H_q$ in AG$(r-1,q)$ and, for $1 \leq i \leq q-3$, apply the induction hypothesis to n_i and H_i to give $E_i \subseteq H_i$ with $|E_i| = n_i$ and such that the restriction of the AG$(r-2,q)$ on H_i to E_i is an hcs. Hence the restriction of AG$(r-1,q)$ to each E_i is an hcs. Now the restriction of AG$(r-1,q)$ to $E_1 \cup \cdots \cup E_{q-3} \cup H_{q-2} \cup H_{q-1} \cup H_q$ has cardinality n and rank r. Further, it is an hcs since each contributing set arises from an hcs and each line through two points from different contributing sets must meet one of H_{q-2}, H_{q-1}, H_q in an additional point. □

LEMMA 4.7: *If $r \geq 3$ and $2^k \geq 3(r-2)2^{r-2}$, then $2^{k(r-1)} > |\Delta(r-1,2^{k+1})|$.*

PROOF: Under the given conditions

$$\begin{aligned}2^{k(r-1)} - |\Delta(r-1,2^{k+1})| &= (2^k)^{r-1} - 3\{(2^{k+1})^{r-2} + (2^{k+1}-3)(2^{k+1})^{r-3} \\ &\quad + \cdots + (2^{k+1}-3)^{r-3}2^{k+1}\} \\ &> (2^k)^{r-1} - 3(2^{k+1})^{r-2}(r-2) \\ &= (2^k)^{r-2}\{2^k - 3(r-2)2^{r-2}\} \geq 0. \quad \square\end{aligned}$$

Lemma 4.7 shows that, for k satisfying $2^k \geq 3(r-2)2^{r-2}$, the intervals $[|\Delta(r-1,2^k)|, 2^{k(r-1)}]$ and $[|\Delta(r-1,2^{k+1})|, 2^{(k+1)(r-1)}]$ overlap; and this, together with Theorem 4.6 implies that any integer $n \geq |\Delta(r-1,2^k)|$ is r-admissible.

THEOREM 4.8: *Let $r \geq 3$ be given. Then there exists a geometric hcs of rank r on a set of n elements for each sufficiently large n.* □

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