COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 37, nº 3 (1978), p. 233-241
<http://www.numdam.org/item?id=CM_1978__37_3_233_0>

COMPOSITIO MATHEMATICA, Vol. 37, Fasc. 3, 1978, pag. 233–241.
Sijthoff & Noordhoff International Publishers–Alphen aan den Rijn
Printed in the Netherlands

ON THE ANALYTIC SOLUTION OF THE EQUATION OF FIFTH DEGREE

Mark L. Green*

1. Introduction

The topic of this paper is very classical, and there will be little that is original, other than the point of view. The first analytic solution of the fifth degree equation was by Hermite in 1858 using the modular equation, and there have been several others, beginning with one by Kronecker in the same year. The analytic solution presented here will be less explicit than these, the intention being to highlight the theoretical reasons why an analytic solution must exist.

To explain what is meant by an analytic solution, consider the one-parameter family of cubics $4x^3 - 3x + a = 0$ (any non-trivial cubic reduces to one of these by $x \to cx + d$) with which Hermite begins his paper. From the trigonometric identity

$$\sin \alpha = 3 \sin \frac{\alpha}{3} - 4 \sin^3 \frac{\alpha}{3}$$

it follows that if we choose an α so $\sin \alpha = a$, the three solutions of the cubic are $\sin \alpha/3$, $\sin(\alpha + 2\pi)/3$, $\sin(\alpha + 4\pi)/3$. Thus, giving an analytic solution consists of reducing the problem of solving a family of polynomial equations to that of inverting and evaluating certain analytic functions. In Hermite's solution, the modular equation plays the role of the trigonometric identity above.

In the next section, the classical proof that any fifth degree equation can be transformed to one of the form $x^5 + ax + b$ by a substitution of the form $x \to \Sigma_{i=0}^{4} a_i x^i$ will be given, but recast in algebro-geometric language. In section 3, a very beautiful Riemann surface of genus 4 mentioned in Klein's book on the icosahedron will be discussed

* The author gratefully acknowledges the support of the N.S.F.

whose uniformization will be shown to be equivalent to finding an analytic solution of equations having form $x^5 + ax + b$. In section 4, this curve of solutions will be shown to be identical with the stellated dodecahedron. In section 5, the uniformization of the curve of solutions by automorphic forms will be discussed. Section 6, which is based on Klein's book, discusses uniformization by Schwarzian differential equations.

2. The Jerrard-Bring reduction

The problem of solving the general equation of the fifth degree can be reduced to that of solving an equation of the form $x^5 + ax + b = 0$. More generally, the equation $\Sigma_{r=0}^{N} c_r x^r = 0$, $c_N = 1$, can be transformed, after solving certain auxiliary equations of degree at most four, to one with $c_{N-1} = c_{N-2} = c_{N-3} = 0$.

Consider the transformation $y = \Sigma_{n=0}^{4} a_n x^n$, with the a_n as yet to be determined. If $x_1, \ldots, x_N$ are the roots of the equation we wish to transform, it will suffice to solve the equation with roots $y_1, \ldots, y_N$ where $y_k = \Sigma_{n=0}^{4} a_n x_k^n$, as the x_k may then be obtained by solving a quartic.

The conditions we wish to impose on the new equation are that the first three elementary symmetric functions in $y_1, \ldots, y_N$ vanish. By equalities of Newton, an equivalent system of equations is

$$0 = \sum y_k$$

$$0 = \sum y_k^2$$

$$0 = \sum y_k^3.$$

Letting $s_n = \Sigma x_k^n$, substituting $y_k = \Sigma_{n=0}^{4} a_n x_k^n$ yields three equations H, Q, C homogeneous in $a_0, \ldots, a_4$ of degrees 1, 2, 3 respectively, with coefficients polynomials in the s_n, hence expressible as polynomials in the coefficients c_n of the original equation.

The problem is to find the coordinates of a point on the curve $H \cap Q \cap C$ in $\mathbb{P}_4$ by solving equations of degree at most four. Unfortunately, the curve has degree 6. This is gotten around by imposing an auxiliary equation. For simplicity, we work in the $\mathbb{P}_3$ determined by H, so $Q' = Q \cap H$ and $C' = C \cap H$ are a quadric and a cubic surface respectively. By solving a quadratic equation, we can find a hyperplane H' tangent to Q' and not containing the point $(1,0,0,0,0)$. Then $H' \cap Q'$ is a conic with a double point, hence a union of two lines, L_1 and L_2. The equations of these lines may be found by solving a quadratic equation. Now $H' \cap Q' \cap C' = (L_1 \cap C') \cup (L_2 \cap C')$. We can find a point on $L_1 \cap C'$ by solving a cubic equation, and this gives a point $(a_0, \ldots, a_4)$ on $H' \cap H \cap Q \cap C$ other than the trivial solution $(1,0,0,0,0)$. This transformation reduces the equation to the desired form.

REMARK: This method does *not* show that an equation of degree N can be transformed to one of the form $x^N + ax + b = 0$ by solving equations of degree at most $N-1$.

3. The curve of solutions of the family of equations $x^5 + ax + b = 0$

We will identify the equation $x^5 + ax + b$ with the equation $(\lambda x)^5 + a(\lambda x) + b = 0$, $\lambda \in \mathbb{C}^*$. This replaces a by a/λ^4 and b by b/λ^5. Thus, taking $(a,b) \neq (0,0)$, such a class of equations is represented by the point a^5/b^4 in $\mathbb{P}_1$.

If $x_1, \ldots, x_5$ are the roots of such an equation, our identification allows us to regard $x_1, \ldots, x_5$ as homogeneous coordinates of a point in $\mathbb{P}_4$. Let V be the curve of solutions of equations of this form. Thus, V is given by the equations $\Sigma_{k=1}^5 x_k = 0$, $\Sigma_{k=1}^5 x_k^2 = 0$, $\Sigma_{k=1}^5 x_k^3 = 0$. The natural branched covering $V \xrightarrow{\pi} \mathbb{P}_1$ is given by $\pi(x_1, \ldots, x_5) = \sigma_4(x_1, \ldots, x_5)^5/\sigma_5(x_1, \ldots, x_5)^4$ where σ_4, σ_5 are the elementary symmetric functions of degrees 4 and 5.

If one does the analogous thing for equations of degrees 2, 3, 4, the curve V is rational. The non-solvability by radicals of the equation of degree 5 seems to be intimately related to the fact V is non-rational.

In fact, V has genus 4. If we view V as sitting in the $\mathbb{P}_3$ determined by the linear equation $\Sigma\, x_i = 0$, it is the intersection of a quadric with a cubic. These meet transversally, for we observe the matrix of normal vectors

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 \end{pmatrix}$$

always has rank 3 along V, since none of our family of fifth degree equations has worse than a double root. It is now standard that V is of genus 4 and furthermore its embedding in the $\mathbb{P}_3$ above is the canonical embedding by abelian differentials.

The genus of V can also be calculated by applying the Riemann-Hurwitz formula if we determine the branch points of the covering $V \xrightarrow{\pi} \mathbb{P}_1$. We will need this later. There is a natural $\mathcal{S}_5$ action on V which permutes the roots, and π is just $V \to V/\mathcal{S}_5$, a 120-sheeted covering. The branch points are points $(x_1, \ldots, x_5)$ so that there exists a permutation $\sigma \in \mathcal{S}_5 - \{e\}$ and $\lambda \in \mathbb{C}^*$ so $(x_{\sigma(1)}, \ldots, x_{\sigma(5)}) = \lambda(x_1, \ldots, x_5)$. They may be enumerated as follows:

above $(0, 1)$, 24 branch points of order 5	(e.g. $(1, \zeta, \zeta^2, \zeta^3, \zeta^4)$ where $\zeta^5 = 1$).
above $(1, 0)$, 30 branch points of order 4	(e.g. $(1, i, -1, -i, 0)$)
above $(1, -4^4/5^5)$, 60 double points	(e.g. double roots).

Thus, By Riemann-Hurwitz, $2(120 + g - 1) = 24 \cdot 4 + 30 \cdot 3 + 60 = 246$; hence $g = 4$.

4. The stellated dodecahedron

I am grateful to Bruce Renshaw for suggesting the following relationship, which we worked out together.

Johannes Kepler constructed a number of generalizations of the five Platonic solids, presumably as a hedge lest more planets be discovered than the five known in his time. One of these, the stellated dodecahedron, is constructed as follows: Begin with a regular icosahedron. For each vertex, span the pentagon of five adjacent vertices by a new face. This done, throw away the original icosahedron. What is left is a self-intersecting regular polyhedron. This may be given the structure of a Riemann surface in a natural way (though Kepler omitted to do so) by passing a sphere through the vertices of the original icosahedron and projecting our new polyhedron onto this sphere from its center, which produces a 3-sheeted branched cover with one double point at each of the 12 vertices of the icosahedron. The resulting Riemann surface $V^{\#}$ is thus of genus 4.

It is a pretty fact that $V^{\#}$ and the surface V of the preceding section not only have the same genus, but are identical as Riemann surfaces. To see this, consider the branched covering gotten by taking the quotient of the action of the group of symmetries of the icosahedron ($\mathcal{A}_5$) on $V^{\#}$. This is a 60-sheeted covering. The symmetries of the icosahedron are of three types (aside from the identity):

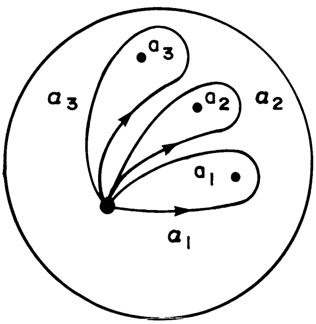
(I) Rotation by $2\pi/5$ about an axis of the sphere through 2 vertices.

(II) Rotation by $2\pi/3$ about an axis of the sphere joining the centers of two faces.

(III) Rotation by π about an axis of the sphere through the midpoints of two edges.

The branch points of our covering are 24 branch points of order 5 coming from fixed points of type I symmetries, all lying over two critical values in the quotient, and 30 double points coming from fixed points of type III symmetries and lying over a single point of the quotient. Thus $V^{\#}/\mathscr{A}_5 = \mathbb{P}_1$ by Riemann-Hurwitz, and there are three critical values, two having every preimage a quintuple point and one having every preimage a double point.

We note that for our other curve of genus 4, we can consider $V \to V/\mathscr{A}_5 = \mathbb{P}_1$, which also has three critical values exactly like those of $V^{\#} \to V^{\#}/\mathscr{A}_5$. We can make a projective change of coordinates on $\mathbb{P}_1$ so these critical values a_1, a_2, a_3 are the same in both cases. We will know $V = V^{\#}$ if the representations $\rho_1, \rho_2: \pi_1(\mathbb{P}_1 - \{a_1, a_2, a_3\}) \to \mathscr{S}_{60}$ are equivalent. Since by construction both representations factor through the (left) regular representation $\mathscr{A}_5 \xrightarrow{\nu} \mathscr{S}_{60}$, it suffices to show we have equivalent representations $\sigma_1, \sigma_2: \pi_1(\mathbb{P}_1 - \{a_1, a_2, a_3\}) \to \mathscr{A}_5$. If we take $\alpha_1, \alpha_2, \alpha_3$ generators for $\pi_1(\mathbb{P}_1 - \{a_1, a_2, a_3\})$ as pictured



with the relation $\alpha_1\alpha_2\alpha_3 = Id$, we must have α_1 and α_2 of order 5 and α_3 of order 2 if we are to get branch points of the prescribed orders. Up to equivalence of representations, there is only one way to do this. For label elements so that our even permutation of order 2 is $\alpha_3 = (23)(45)$. Now α_2 and $\alpha_2\alpha_3$ are both 5-cycles. Relabeling so $\alpha_2(1) = 2$ we see $\alpha_2(2) \neq 3$, hence relabeling $\alpha_2(2) = 4$. Now $\alpha_2(4) \neq 5$, so $\alpha_2(4) = 3$ and $\alpha_2(3) = 5$. Thus σ_1 and σ_2 are equivalent, and the same is true of ρ_1 and ρ_2. So $V^{\#} = V$.

5. Uniformization of the curve of solutions by automorphic forms

The general uniformization theorem guarantees that the upper half-plane is the universal cover of the curve of solutions V. We thus have

$$\begin{tikzcd}
\mathcal{H} \arrow[d, "\mu"] \arrow[dr, "(\psi_1,\ldots,\psi_5)"] \arrow[dd, bend right=60, "\phi"'] & \\
V \arrow[r, hook] \arrow[d, "\pi"] & \mathbb{P}_4 \\
\mathbb{P}_1 &
\end{tikzcd}$$

This shows in principle the existence of analytic solution in the sense discussed in the introduction. Unfortunately, the uniformization theorem is notoriously non-constructive. In this section, the functions ϕ and $\psi_1, \ldots, \psi_5$ will be identified as particular automorphic forms.

In $\mathcal{H}$, construct a non-euclidean triangle with interior angles $\pi/2$, $\pi/4$, $\pi/5$. Let $\tilde{\Gamma}$ be the subgroup of the conformal automorphisms of $\mathcal{H}$ consisting of products of an even number of inversions in the sides of this triangle. If r_1, r_2, r_3 are inversions in the three sides respectively, generators for $\tilde{\Gamma}$ are $R = r_2r_1$, $S = r_1r_3$, $T = r_2r_3$ with the relations $R^2 = S^5 = T^4 = I$, $RS = T$.

We consider $\Gamma \subset \tilde{\Gamma}$ the normal subgroup of relations with generator $(RS^{-2}RS^2)^2$. Then $\tilde{\Gamma}/\Gamma \simeq \mathcal{S}_5$ under the map $R \to (12)$, $S \to (12345)$, see Coxeter and Moser, *Generators and Relations for Discrete Groups*, p. 137.

We first note that $\mathcal{H} \to \mathcal{H}/\Gamma$ is a covering, i.e. that no element of Γ has a fixed point, If $\gamma \in \Gamma$ did, let $\sigma \in \tilde{\Gamma}$ take it to our base triangle. Then $\sigma\gamma\sigma^{-1} \in \Gamma$ as Γ is a normal subgroup of $\tilde{\Gamma}$ and has a fixed point in our base triangle which must be a vertex. Thus either R, S, or T is in Γ, which does not happen.

Now $\mathcal{H}/\Gamma \to \mathcal{H}/\tilde{\Gamma}$ comes from dividing out by the action of $\tilde{\Gamma}/\Gamma = \mathcal{S}_5$. It has three critical values a_1, a_2, a_3 and the representation $\pi_1(\mathbb{P}_1 - \{a_1, a_2, a_3\}) \to \mathcal{S}_5$ sends $\alpha_1, \alpha_2, \alpha_3$ to $R\Gamma$, $S\Gamma$, $T^{-1}\Gamma$, elements of order $2, 4, 5$ respectively whose product is the identity. Up to equivalence, we have already seen there is only one such representation, precisely the one associated to the branched covering π of the curve of solutions V over the $\mathbb{P}_1$ parametrizing our family of fifth degree equations. Thus $V = \mathcal{H}/\Gamma$ as a Riemann surface, and the map $\mathcal{H}/\Gamma \to \mathcal{H}/\tilde{\Gamma}$ is just π. We can summarize matters by the diagram

$$\begin{tikzcd}
\mathcal{H} \arrow[d, "\mu"] \arrow[dr, "(\psi_1,\ldots,\psi_5)"] \arrow[dd, bend right=60, "\phi"'] & \\
\mathcal{H}/\Gamma = V \arrow[r, hook] \arrow[d, "\pi"] & \mathbb{P}_4 \\
\mathcal{H}/\tilde{\Gamma} = \mathbb{P}_1 &
\end{tikzcd}$$

We should now be able to represent $\psi_1, \ldots, \psi_5$ as automorphic forms. As V is canonically embedded in a hyperplane of $\mathbb{P}_4$ by holomorphic 1-forms, we know we can take $\psi_1, \ldots, \psi_5$ to be automorphic forms with respect to Γ of weight -2.

If $\omega_1, \ldots, \omega_5$ are the holomorphic 1-forms on V corresponding to $\psi_1, \ldots, \psi_5$, and if $\sigma \in \mathcal{S}_5$, $\sigma: V \to V$, then because V is canonically embedded we see $\sigma^*\omega_i = sgn(\sigma)\omega_{\sigma^{-1}(i)}$. Thus, ω_1 is the unique (up to a constant) holomorphic 1-form on V left invariant under the $\mathcal{A}_4$ of even permutations leaving the first element fixed. Put another way, $V \to V/\mathcal{A}_4$ has 12 sheets and 6 double points of the form $(0, 1, i, -1, -i)$, hence $V/\mathcal{A}_4$ has genus 1 and ω_5 is the pullback of its unique homorphic 1-form. Let $\Gamma_i^{\#}$ be the pre-image of the $\mathcal{A}_4$ having the ith element fixed under the map $\tilde{\Gamma} \to \tilde{\Gamma}/\Gamma = \mathcal{S}_5$. Then ψ_1 may be taken to be any $\Gamma_1^{\#}$-automorphic form of weight -2.

We then have

$$\phi(z) = \frac{\sigma_4(\psi_1(z), \ldots, \psi_5(z))^5}{\sigma_5(\psi_1(z), \ldots, \psi_5(z))^4}$$

and these automorphic forms and functions give the analytic solution we seek.

6. Schwarzian differential equations

It is possible to directly uniformize a certain class of branched covers of $\mathbb{P}_1$ by a classical procedure of Schwarz, which was an antecedent of Poincaré's general program of doing uniformization via second order differential equations. This gives another way of obtaining the map $\mathcal{H} \xrightarrow{\phi} \mathbb{P}_1$ of section 5 – in fact better yet, one explicitly obtains ϕ^{-1}.

If we have integers ν_1, ν_2, ν_3, all > 1, with $(1/\nu_1) + (1/\nu_2) + (1/\nu_3) < 1$, there exists a non-euclidean triangle in $\mathcal{H}$ with angles π/ν_1, π/ν_2, π/ν_3, unique up to the action of $SL(2, \mathbb{R})$ once we label the vertices. Let Γ be the subgroup of the conformal automorphisms of $\mathcal{H}$ consisting of products of an even number of inversions in the sides of the triangle. The map $\mathcal{H} \xrightarrow{\phi} \mathcal{H}/\Gamma = \mathbb{P}_1$ has three critical values, which we take to be $1, 0, \infty$ so the preimages are all branched of orders ν_1, ν_2, ν_3 respectively.

The inverse function η to ϕ is multiple-valued, the ambiguity coming from the representation $\pi_1(\mathbb{P}_1 - \{0, 1, \infty\}) \to SL(2, \mathbb{R})/\{\pm I\}$.

Schwarz discovered the derivative operator bearing his name

$$[\eta]_z = \frac{\eta'''}{\eta'} - \frac{3}{2}\left(\frac{\eta''}{\eta'}\right)^2 \quad z = \text{a variable in } \mathbb{P}_1.$$

The *Schwarzian derivative* has the properties

(I) $[\phi]_w = [\phi]_z(dz/dw)^2 + [z]_w$ any holomorphic $w(z)$.

(II) $[z]_w = 0$ if $w = \dfrac{az+b}{cz+d}$ $\quad a, b, c, d \in \mathbb{C}$.

(III) $\left[\dfrac{a\phi+b}{c\phi+d}\right]_z = [\phi]_z \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})$.

By (III), the function $[\eta]_z$ is well-defined globally on $\mathbb{P}_1$. The singularities of $[\eta]_z$ determine it, and these occur only at $\{0, 1, \infty\}$. Following Klein, a local calculation at 0 and 1 shows

$$[\eta]_z = \frac{\nu_1^2 - 1}{2\nu_1^2(z-1)^2} + \frac{A}{z-1} + \frac{\nu_2^2 - 1}{2\nu_2^2 z^2} + \frac{B}{z} + C$$

and $[\eta]_z$ must have singularity at ∞ with leading term $(\nu_3^2 - 1)/2\nu_3^2(1/z)^2$. This determines A, B, C and we get

$$[\eta]_z = \frac{\nu_1^2 - 1}{2\nu_1^2(z-1)^2} + \frac{\nu_2^2 - 1}{2\nu_2^2 z^2} + \frac{\dfrac{1}{\nu_1^2} + \dfrac{1}{\nu_2^2} - \dfrac{1}{\nu_3^2} - 1}{2(z-1)z}.$$

If we write $\eta = y_1/y_2$, where y_1, y_2 are a basis of solutions of the second order equation

$$y'' + py' + qy = 0$$

we get

$$[\eta]_z = 2q - \tfrac{1}{2}p^2 - p'$$

and conversely if this equation holds, η is a ratio of solutions of $y'' + py' + qy = 0$.

The reason for reducing to these second order differential equations with regular singular points is that they turn out to be recognizable O.D.E.'s. For any equation of the form

$$y'' + \frac{y'}{z(1-z)}[(1 - \alpha - \alpha') - (1 + \beta + \beta')z]$$
$$+ \frac{y}{z^2(1-z)^2}[\alpha\alpha' - (\alpha\alpha' + \beta\beta' - \gamma\gamma')z + \beta\beta' z^2] = 0$$

where $\alpha, \ldots, \gamma' \in \mathbb{C}$ and $\alpha + \alpha' + \cdots + \gamma' = 1$ is a *hypergeometric equation*. These were considered by Gauss and Riemann, and have as solution the function $P\left(\begin{smallmatrix}\alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma'\end{smallmatrix}, z\right)$ defined by Riemann, *Math. Werke*, pp. 62–82. Taking $\nu_1 = 2$, $\nu_2 = 4$, $\nu_3 = 5$ we have

$$\begin{pmatrix}\alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma'\end{pmatrix} = \begin{pmatrix}\frac{1}{8} & \frac{1}{10} & \frac{1}{4} \\ -\frac{1}{8} & -\frac{1}{10} & \frac{3}{4}\end{pmatrix}$$

by equating coefficients.

(Oblatum 18-X-1976)

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