NICOLE ARTHAUD

On Birch and Swinnerton-Dyer’s conjecture for elliptic curves with complex multiplication. I


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ON BIRCH AND SWINNERTON-DYER'S CONJECTURE FOR
ELLiptic Curves WITH COMPLEX MULTIPLICATION. I.

Nicole Arthaud

Introduction

Let $K$ be an imaginary quadratic field, and $E$ an elliptic curve with complex multiplication by the ring of integers of $K$. Assume that $E$ is defined over a finite extension $F$ of $K$, and let $L(E/F, s)$ be the Hasse-Weil zeta function of $E$ over $F$. Deuring has proven that $L(E/F, s)$ can be analytically continued over the whole complex plane, by identifying it with a product of Hecke $L$-series with Grössencharacters (see [7], Theorem 7.42). The conjecture of Birch and Swinnerton-Dyer asserts that $L(E/F, s)$ has a zero at $s = 1$ of order equal to $g_F$, the rank of the group $E(F)$ of points of $E$ with coordinates in $F$. Recently, Coates and Wiles [4] made some progress on a weak form of this conjecture. Namely, they showed that if $K$ has class number 1 and $F = K$, then $g_F > 1$ implies that $L(E/F, s)$ does indeed vanish at $s = 1$. The aim of the present paper is to extend Coates and Wiles' proof to the case in which $K$ has class number 1, $E$ is still defined over $K$, but the base field $F$ is now an arbitrary finite abelian extension of $K$.

**Theorem 1:** Let $K$ be an imaginary quadratic field with class number 1, and $E$ an elliptic curve defined over $K$, with complex multiplication by the ring of integers of $K$. If $F$ is a finite abelian extension of $K$ such that $E$ has a point of infinite order with coordinates in $F$, then $L(E/F, s)$ vanishes at $s = 1$.

In a subsequent, but considerably more technical, paper [1] in preparation, we shall prove an analogous result when (i) no restriction is made on the class number of $K$, (ii) the base field $F$ is again supposed to be an abelian extension of $K$, and finally (iii) the torsion
points of $E$ are assumed to generate over $F$ an abelian extension of $K$ (see Theorem 7.44 of [7] for a necessary and sufficient condition for (iii) to be valid for $E$). Since the methods of [4] depend crucially on the explicit knowledge of class field theory for abelian extensions of $K$, there seems to be little hope at present of proving results like Theorem 1 without hypotheses (ii) and (iii) above.

The broad outlines of the proof of Theorem 1 follow fairly closely the arguments in [4]. However, there are some significant and interesting innovations in dealing with an arbitrary finite abelian extension of $K$ as base field. In particular, certain partial Hecke $L$-functions with Grössencharacters play a natural role in the proof. This is in striking analogy with the theory of cyclotomic $\mathbb{Z}_p$-extensions, where the values of partial $L$-functions formed with characters of finite order give the coefficients of Stickelberger ideals (see [2]). Also, we have simplified the proof of [4] in several cases (cf. the proof of Theorem 19).

In conclusion, I wish to thank John Coates for his guidance with this work.

1. Notation

To a large extent, we follow the notation of [4]. Thus $K$ will denote an imaginary quadratic field with class number 1, lying inside the complex field $\mathbb{C}$, and $\mathcal{O}$ the ring of integers of $K$. As in the Introduction, $E$ will be an elliptic curve defined over $K$, whose ring of endomorphisms is isomorphic to $\mathcal{O}$. We fix a Weierstrass model for $E$ where $g_2, g_3$ belong to $\mathcal{O}$, and where the discriminant of (1) is divisible only by the primes of $K$ where $E$ has a bad reduction, and (possibly) by the primes of $K$ above 2 and 3. Let $\wp(z)$ be the associated Weierstrass function, $L$ the period lattice of $\wp(z)$, and $\xi(z) = (\wp(z), \wp'(z))$. Choose $\Omega \in L$ such that $L = \Omega \mathcal{O}$. We identify $\mathcal{O}$ with the endomorphism ring of $E$ in such a way that the endomorphism corresponding to $\alpha \in \mathcal{O}$ is given by $\xi(z) \mapsto \xi(\alpha z)$. If $\alpha \in \mathcal{O}$, we write $E_\alpha$ for the kernel of the endomorphism $\alpha$ of $E$. Let $\psi$ be the Grössencharacter of $E$ over $K$ as defined in [7], §7.8. We denote the conductor of $\psi$ by $f$, and write $f$ for some fixed generator of $f$.

Let $F$ be an arbitrary finite abelian extension of $K$, which will be fixed for the rest of the paper. We write $S$ for the finite set consisting of 2, 3, and all rational primes $q$ which have a prime factor in $K$,
which is either ramified in $F$, or at which $E$ has a bad reduction. Henceforth, $p$ will denote a rational prime, which splits in $K$, and which does not belong to the finite exceptional set $S$. We write $\varphi$ and $\tilde{\varphi}$ for the factors of $p$ in $K$, and put $\pi = \psi(\varphi)$. Thus, by the definition of $\psi$, $\pi$ is a generator of the ideal $\varphi$. Finally, let $\ell$ denote the least common multiple of the conductor of $\psi$ and the conductor of $F/K$.

2. Computation of conductors

We now compute the conductors of various abelian extensions of $K$ which occur in the proof of Theorem 1. The arguments are similar to those in §2 of [4]. If $\alpha \in \mathcal{O}$, recall that $E_{\alpha}$ is the group of $\alpha$-division points on $E$.

**Lemma 2:** Let $\mathfrak{h} = (h)$ be any multiple of the conductor of $\psi$. Then $K(E_{\mathfrak{h}})$ is the ray class field of $K$ modulo $\mathfrak{h}$.

**Proof:** By the classical theory of complex multiplication, the ray class field modulo $\mathfrak{h}$ is contained in $K(E_{\mathfrak{h}})$. To prove the converse, we use the notation and results of Shimura [7]. Let $U(\mathfrak{h})$ be the subgroup of the idèle group of $K$ as defined on p. 116 of [7], and let $x$ be any element of $U(\mathfrak{h})$ with $x_{\infty} = 1$. Since the conductor of $\psi$ divides $\mathfrak{h}$, it follows from Shimura's reciprocity law (cf. the proof of Lemma 3 in [4]) that the Artin symbol $[x, K]$ fixes $E_{\mathfrak{h}}$. Thus $K(E_{\mathfrak{h}})$ is contained in the ray class field modulo $\mathfrak{h}$, and the proof of the lemma is complete.

Recall that $\ell$ is the least common multiple of the conductor of $\psi$, and the conductor of $F/K$. Also, $p$ is any rational prime, not in $S$, which splits in $K$, say $(p) = \varphi \tilde{\varphi}$.

**Lemma 3:** For each $n \geq 0$, the conductor of $F_n = F(E_{g^{n+1}})$ over $K$ is equal to $\ell_n = \ell \varphi^{n+1}$. Moreover, if $\mathcal{R}_n$ denotes the ray class field of $K$ modulo $\ell_n$, then $\mathcal{R}_n$ is the compositum of $F_n$ and $H = K(E_{g})$, and $F_n \cap H = F$.

**Proof:** Let $g_n$ denote the conductor of $F_n/K$. Since $F_n \subseteq K(E_{g^{n+1}})$, and the conductor of this latter field is $\ell_n = \ell \varphi^{n+1}$ by Lemma 2, we conclude that $g_n$ divides $\ell_n$. On the other hand, it is clear that the conductor of $F$ over $K$ divides $g_n$. Also, as $E$ has a good reduction everywhere over $F_n$ (see Theorem 2 of [4]), the Grössencharacter of $E$ over $F_n$ must be unramified. As the Grössencharacter of $E$ over $F_n$ is the composition of the norm map from $F_n$ to $K$ with $\psi$, it follows
that the conductor $f$ of $\psi$ divides $g_n$. Combining these last two facts, we conclude that $g$ divides $g_n$. But $p'^{n+1}$ divides $g_n$ because $F_n$ contains the ray class field modulo $p'^{n+1}$. As $(\rho, g) = 1$ by hypothesis, we deduce that $g_n = f_n$, as asserted. To prove the final statement of the lemma, we recall that $R_n = K(E_{\rho^{*n+1}})$ by Lemma 2, and thus $R_n$ is certainly the compositum of $F_n$ and $H$. Now $\rho$ is totally ramified in $K(E_{\pi^{*n+1}})$ by the rudiments of Lubin-Tate theory. As $\rho$ does not divide the conductor of $F$ over $K$, it follows that each prime of $F$ above $\rho$ is totally ramified in $F_n$. Since $\rho$ does not divide $g$ by hypothesis, and $H$ is the ray class field modulo $g$ by Lemma 2, we deduce that $F_n \cap H = F$, as required.

3. $p$-Adic logarithmic derivatives

We use the same notation as [4] for the formal groups $\hat{E}$ and $\mathcal{E}$. Thus $\hat{E}$ is the formal group giving the kernel of reduction modulo $\rho$ on $E$, and $\mathcal{E}$ is the Lubin-Tate formal group for which $[\pi](w) = \pi w + w^p$. By Lubin-Tate theory, $\hat{E}$ and $\mathcal{E}$ are isomorphic over the ring $\mathcal{O}_p$ of integers of the completion $K_p$ of $K$ at $\rho$. For a fuller discussion, see §3 of [4].

Choose a fixed algebraic closure $\bar{K}_\rho$ of $K_p$. We can assume that $E_{\pi}$ lies in $\bar{K}_\rho$, and we define the extension $\Phi$ of $K_p$ by

$$\Phi = K_p(E_\pi) = K_p(\mathcal{E}_\pi).$$

Put $G = G(\Phi/K_p)$. Of course, $G$ is endowed with the canonical character $\chi$, with values in $\mathbb{Z}_p^\times$, giving the action of $G$ on $E_{\pi}$, or equivalently, on $\mathcal{E}_\pi$. Thus, if $A$ is any $\mathbb{Z}_p[G]$-module, it has a canonical decomposition

$$A = \bigoplus_{k=1}^{p-1} A^{(k)},$$

where $A^{(k)}$ is the submodule of $A$ on which $G$ acts via the $k$-th power of $\chi$.

Let $u$ be a fixed generator for $\mathcal{E}_\pi$, so that $u$ is a local parameter for $\Phi$. Let $U$ be the group of units of $\Phi$ which are $\equiv 1 \pmod{u}$. For $1 \leq k \leq p-2$, we define homomorphisms

$$\varphi_k : U \to \mathcal{O}_p/\rho$$

(3)
as follows. If $\alpha \in U$, we choose any power series $f(T) = \sum_{k=0}^{\infty} a_k T^k$, with $a_k \in \mathcal{O}_\nu$, such that $f(u) = \alpha$. We then define $\varphi_k(\alpha)$ to be the residue class in $\mathcal{O}_\nu/\varphi$ of the coefficient of $T^k$ in the power series $T(d/dT) \log f(T)$. Since $1 \leq k \leq p - 2$ and the ramification index of $\Phi$ over $K_\nu$ is $p - 1$, it is easy to see that $\varphi_k(\alpha)$ is independent of the choice of $f(T)$, and so is well defined.

**Remark:** In defining $\varphi_k$ in [4], one insisted that the power series $f(T)$ had $a_0 = 1$. It is more convenient for the arguments in §4 to work with power series whose constant term is not necessarily 1. Of course, the two definitions of $\varphi_k$ are the same for $1 \leq k \leq p - 2$. However, one cannot define $\varphi_{p-1}$ by the present method.

In the proof of Theorem 1, we shall only be interested in the case in which $\Phi$ contains no non-trivial $p$-power roots of unity. Recall that, by Lemma 12 of [4], if $p > 5$, then $\Phi$ can contain a non-trivial $p$-th root of unity if and only if $\pi + \tilde{\pi} = 1$. The next lemma is plain from Lemmas 9 and 10 of [4].

**Lemma 4:** Assume that $\Phi$ contains no non-trivial $p$-th root of unity. Let $k$ be an integer with $1 \leq k \leq p - 2$. Then $\varphi_k$ vanishes on $U^{(j)}$ for $j \neq k \text{ mod}(p - 1)$, and $\varphi_k$ induces an isomorphism

$$
\varphi_k : U^{(j)}_0/(U^{(k)}_0)^p \simeq \mathcal{O}_\nu/\varphi.
$$

Now consider our fixed finite abelian extension $F$ of $K$, and $F_0 = F(E_w)$. Let $\mathcal{S}$ be the set of primes of $F_0$ above $\nu$. For each $q \in \mathcal{S}$, let $F_{0,q}$ be the completion of $F_0$ at $q$, and write $U_q$ for the units in $F_{0,q}$ which are $\equiv 1 \text{ mod } q$. Put

(4)
$$
\mathcal{U} = \prod_{q \in \mathcal{S}} U_q.
$$

Now assume that $\nu$ splits completely in $F$. Thus, for each $q \in \mathcal{S}$, there exists an isomorphism $\tau_q: F_{0,q} \simeq \Phi$, which preserves the valuations of both fields. Composing this isomorphism with the map $\varphi_k$ given by (3), we obtain a homomorphism

(5)
$$
\varphi_{q,k} : U_q \rightarrow \mathcal{O}_\nu/\varphi \quad (1 \leq k \leq p - 2).
$$

We define

(6)
$$
\varphi_{F,k} : \mathcal{U} \rightarrow \prod_{q \in \mathcal{S}} (\mathcal{O}_\nu/\varphi)
$$
to be the product of the homomorphisms (5) over all $q \in \mathcal{S}$. Plainly $G = G(F_0/F) = G(\Phi/K_p)$ acts on (4), because it acts on each of the $U_q$ in the natural way. The next lemma is now plain from Lemma 4.

**Lemma 5:** Assume that $\Phi$ contains no non-trivial $p$-th root of unity, and that $\Phi$ splits completely in $F$. Let $k$ be an integer with $1 \leq k \leq p - 2$. Then $\varphi_{F,k}$ vanishes on $\mathcal{U}^{(j)}$ for $j \neq k \mod(p - 1)$, and $\varphi_{F,k}$ induces an isomorphism

$$\varphi_{F,k} : \mathcal{U}^{(k)}/(\mathcal{U}^{(k)})^p \rightarrow \prod_{q \in \mathcal{S}} (\mathcal{O}_q/p).$$

Put $d = [F : K]$. In practice, we shall use the following immediate consequence of Lemma 5.

**Corollary 6:** Under the same hypotheses as Lemma 5, let $A$ be any $\mathbb{Z}_p[G]$-submodule of $\mathcal{U}$. Then, for each integer $k$ with $1 \leq k \leq p - 2$, the eigenspace $((\mathcal{U}/A)^{(k)} \neq 0$ if and only if $\varphi_{F,k}(A)$ has dimension less than $d$ over the field $\mathcal{O}_q/p$.

### 4. Elliptic units

As in [4], a vital role in the proof of Theorem 1 is played by the elliptic units of Robert [6]. We begin by briefly recalling the definition of these elliptic units. Let $\mathcal{S}$ be the set consisting of all pairs $(A, \mathcal{N})$, where $A = \{a_j : j \in J\}$ and $\mathcal{N} = \{n_j : j \in J\}$, here $J$ is an arbitrary finite index set, the $a_j$ are integral ideals of $K$ prime to $S$ and $p$, and the $n_j$ are rational integers satisfying $\Sigma_{j \in J} n_j(Na_j - 1) = 0$. Given such a pair $(A, \mathcal{N})$, we put

$$\Theta(z, A, \mathcal{N}) = \prod_{j \in J} \Theta(z, a_j)^{n_j},$$

where $\Theta(z, a_j)$ is as defined at the beginning of §4 of [4]. Recall that $f_n = \mathcal{O}_n^{p+1}$ is the conductor of $F_n = F(E^{p+1})$ over $K$. As before, let $\mathcal{R}_n$ be the ray class field of $K$ modulo $f_n$. If $\rho_n$ is an arbitrary primitive $f_n$-division point of $L$, Robert [6] has shown that $\Theta(\rho_n, A, \mathcal{N})$ is a unit of the field $\mathcal{R}_n$. Moreover, as $(A, \mathcal{N})$ ranges over $\mathcal{S}$, the $\Theta(\rho_n, A, \mathcal{N})$ form a subgroup of the group of units of $\mathcal{R}_n$. We denote this subgroup by $\mathfrak{C}_n$, and call it the group of elliptic units of $\mathcal{R}_n$ (note that Robert's definition of the group of elliptic units is different from ours). A
similar argument to that given in the proof of Lemma 20 of [4] shows that \( c_n \) is stable under the action of the Galois group of \( R_n \) over \( K \), and is independent of the choice of the particular primitive \( f_n \)-division point \( p_n \). Finally, we define the elliptic units \( C_n \) of \( F_n = F(E_n) \) to be the group consisting of the norms from \( R_n \) to \( F_n \) of all units in \( C_n \). For simplicity, we often write \( C \) for \( C_0 \).

Let \( \rho = \Omega/g \), where \( q = (g) \). Here \( L = \Omega \mathfrak{O} \) is the period lattice of \( \varphi(z) \). As above, let \( R_0 \) be the ray class field of \( K \) modulo \( f_0 = g \varphi \). Lemma 3 tells us that we have the diagram of fields:

\[
\begin{array}{ccc}
R_0 &=& HF_0 \\
H &=& K(E_\varphi) \\
F &=& H \cap F_0 \\
F_0 &=& F(E_\varphi) \\
K & & \end{array}
\]

If \( L \) is any finite abelian extension of \( K \), and \( \mathfrak{c} \) is an integral ideal of \( K \) prime to the conductor of \( L/K \), we write \( (\mathfrak{c}, L/K) \) for the Artin symbol of \( \mathfrak{c} \) for the extension \( L/K \). We now choose and fix a set \( B \) of integral ideals of \( K \), which are prime to \( f_0 \), and which are such that \( \{(b, R_0/K) : b \in B\} \) is precisely the Galois group of \( R_0/F_0 \). It is then plain from (7) that the restrictions of the \( (b, R_0/K) \), \( b \in B \), to \( H \) is precisely the Galois group of \( H/F \).

If \( \mathfrak{a} \) is an arbitrary integral ideal of \( K \) prime to \( S \) and \( p \), we define:

\[
\Lambda(z, \mathfrak{a}) = \prod_{\mathfrak{b} \in B} \Theta(z + \psi(\mathfrak{b})p, \mathfrak{a}).
\]

**Lemma 7:** \( \Lambda(z, \mathfrak{a}) \) is a rational function of \( \varphi(z) \) and \( \varphi'(z) \) with coefficients in \( F \).

**Proof:** This is entirely similar to the first part of the proof of Lemma 21 of [4], and so we omit it.

It is now convenient to introduce some notation, which will be used repeatedly in this section. Let \( \mathcal{G} \) denote the Galois group of \( F \) over \( K \). If \( \mathfrak{c} \) is an integral ideal of \( K \) prime to the conductor of \( F/K \), we write \( \sigma_\mathfrak{c} \) for the Artin symbol \( (\mathfrak{c}, F/K) \). Finally, if \( \sigma \in \mathcal{G} \) and \( R(z) \) is a rational function of \( \varphi(z) \), \( \varphi'(z) \) with coefficients in \( F \), then \( R_\sigma(z) \) will denote the rational function of \( \varphi(z) \), \( \varphi'(z) \), which is obtained by letting \( \sigma \) act on the coefficients of \( R(z) \).
Let \( k \) be an integer \( \geq 1 \). Recall that \( \psi \) denotes the Grössencharacter of \( E \). For each \( \sigma \in \mathfrak{G} \), we introduce the partial Hecke \( L \)-function

\[
\zeta_F(\sigma, k; s) = \sum_{\substack{(a, a) = 1 \\ \sigma_a = \sigma}} \frac{\psi^k(a)}{(Na)^s},
\]

where the summation is over all integral ideals \( a \) of \( K \), prime to \( \mathfrak{a} \), such that the Artin symbol \( \sigma_a \) is equal to \( \sigma \). It can be shown that \( \zeta_F(\sigma, k; s) \) can be analytically continued over the whole complex plane. Let \( \zeta_F(\sigma, k) \) denote the value of \( \zeta_F(\sigma, k; s) \) at \( s = k \).

**Lemma 8:** For each \( \sigma \in \mathfrak{G} \), we have

\[
z \frac{d}{dz} \log \Lambda_\sigma(z, a) = \sum_{k=1}^{\infty} c_k(a, \sigma) z^k,
\]

where

\[
c_k(a, \sigma) = 12(-1)^{k-1} \rho^{-k}(Na \zeta_F(\sigma, k) - \psi^k(a) \zeta_F(\sigma \sigma_a, k)) \quad (k = 1, 2, \ldots).
\]

**Proof:** Let \( \mathfrak{c} \) be an integral ideal of \( K \), prime to \( \mathfrak{a} \), such that \( \sigma = \sigma_{\mathfrak{c}} \). By the definition of the Grössencharacter \( \psi \) in [7], we have

\[
\xi(\psi(b) \rho^{(c, H/K)} = \xi(\psi(b \mathfrak{c}) \rho).
\]

It follows easily from the expression for \( \Theta(z + \psi(b) \rho, a) \) as a rational function of \( \varphi(z), \varphi'(z) \), with coefficients in \( H \) (see (23) of [4]), that

\[
\Lambda_\sigma(z, a) = \prod_{b \in B} \Theta(z + \psi(b \mathfrak{c}) \rho, a).
\]

If \( \mathcal{L} \) is any lattice in the complex plane, let \( \xi(z, \mathcal{L}) \) and \( \varphi(z, \mathcal{L}) \) be the Weierstrass zeta and \( \varphi \)-functions of \( \mathcal{L} \). Define

\[
\Omega(z, \mathcal{L}) = z \frac{d}{dz} \log \left( \prod_{b \in B} \vartheta(z + \psi(b \mathfrak{c}) \rho, \mathcal{L}) \right).
\]

Then (cf. the proof of Lemma 21 of [4]) \( \Omega(z, \mathcal{L}) \) has the power series expansion \( \sum_{k=1}^{\infty} d_k(\mathcal{L}) z^k \), where \( \eta = \psi(\mathfrak{c}) \rho \) and

\begin{align*}
(8) \quad & d_1(\mathcal{L}) = 12 \sum_{b \in B} (\xi(\psi(b) \eta, \mathcal{L}) - s_2(\mathcal{L}) \psi(b) \eta), \\
(9) \quad & d_2(\mathcal{L}) = -12 \sum_{b \in B} (\varphi(\psi(b) \eta, \mathcal{L}) + s_2(\mathcal{L})), \\
(10) \quad & d_k(\mathcal{L}) = -12 \sum_{b \in B} \varphi^{(k-2)}(\psi(b) \eta, \mathcal{L}))/k! \quad (k \geq 3).
\end{align*}
Thus we must show that $c_k(a, \sigma)$, as defined in Lemma 8, satisfies

(11) \[ c_k(a, \sigma) = N \lambda_k d_k(L) - \delta_k(a^{-1}L) \quad (k \geq 1). \]

As in [4], we put $\lambda_k = 12(-1)^{k+1} \rho^k$. We write $\mathcal{B}$ for a fixed set of generators of the ideals in $B$. Also, we let $\gamma$ denote a fixed generator of the ideal $a$, and $c$ a fixed generator of $c$. The argument now breaks up into three cases. Much of the reasoning is similar to that in the proof of Lemma 21 of [4], so that we refer there for details from time to time.

**Case 1.** We suppose that $k \geq 3$. Since

\[ \varphi^{(k-2)}(z, \mathcal{L}) = (-1)^k(k-1)! \sum_{\omega \in \mathcal{L}} (z - \omega)^{-k} \quad (k \geq 3), \]

we conclude easily from (10) that

\[ d_k(L) = \lambda_k \sum_{b \in \mathcal{B}} \sum_{a \in a} (\psi(bc) - \alpha)^{-k}. \]

We now write $\psi(bc) = \epsilon(bc)bc$, where $b$ is the generator of $b$ in $\mathcal{B}$, and $\epsilon(bc)$ is a root of unity in $K$, and argue in exactly the same way as in Case 1 of the proof of Lemma 21 in [4]. In this way, it follows that

\[ d_k(L) = \lambda_k \sum_{b \in \mathcal{B}} \sum_{a \in a} \overline{\psi}(b(b - \alpha))N(b(b - \alpha)^{-k}, \]

where $N$ denotes the norm from $K$ to $Q$. Let $W$ denote the group of roots of unity in $K$. Since the Grössencharacter $\psi$ is defined modulo $\mathfrak{g}$, the natural map of $W$ into $(\mathcal{O}/\mathfrak{g})^\times$ is plainly injective. Now, as $H$ is the ray class field modulo $\mathfrak{g}$ by Lemma 2, we can identify the Galois group of $H$ over $K$ with $(\mathcal{O}/\mathfrak{g})^\times W$ via the Artin map. Since the Artin symbol of $c = (c)$ for $F/K$ is equal to $\sigma$, it is therefore clear that $\{\mu bc : \mu \in W, b \in \mathcal{B}\}$ is a complete set of representatives of those elements in $(\mathcal{O}/\mathfrak{g})^\times$, whose Artin symbol has restriction to $F$ equal to $\sigma$. In other words,

\[ \{\mu bc - \alpha : \mu \in W, b \in \mathcal{B}, \alpha \in \mathfrak{g}\} \]

is the set of all algebraic integers in $K$, prime to $\mathfrak{g}$, such that the Artin symbol for $F/K$ of the associated principal ideal is equal to $\sigma$. Since
we can plainly rewrite the above expression for \( d_k(L) \) as

\[
d_k(L) = \frac{\lambda_k}{w_k} \sum_{\mu \in W} \sum_{b \in B} \sum_{a \in g} \bar{\psi}^k((\mu bc - \alpha)) N(\mu bc - \alpha)^{-k},
\]

where \( w_k \) denotes the number of roots of unity in \( K \), it follows that

\[
d_k(L) = \lambda_k \xi_F(\sigma, k).
\]

Now consider \( d_k(a^{-1}L) \). Recalling that \( a = (\gamma) \), it follows from (10) that

\[
d_k(a^{-1}L) = \lambda_k \gamma^k \sum_{b \in B} \sum_{a \in g} (\gamma \psi(bc) - \alpha)^{-k}.
\]

Substitute \( \gamma = \psi(a) \epsilon^{-1}(\gamma) \) for the first occurrence of \( \gamma \) on the right hand side of this equation. Again arguing in the same way as in Case 1 of the proof of Lemma 21 in [4], we obtain

\[
d_k(a^{-1}L) = \lambda_k \psi^k(a) \sum_{b \in B} \sum_{a \in g} \bar{\psi}^k((\gamma bc - \alpha)) N(\gamma bc - \alpha)^{-k}.
\]

Now

\[
\{\mu \gamma bc - \alpha : \mu \in W, b \in B, \alpha \in g\}
\]

is the set of all algebraic integers in \( K \), prime to \( g \), such that the Artin symbol for \( F/K \) of the associated principal ideal is equal to \( \sigma \sigma_4 \). Thus

\[
d_k(a^{-1}L) = \lambda_k \psi^k(a) \xi_F(\sigma \sigma_4, k).
\]

We have therefore proven (11) in this case.

**Case 2.** We assume that \( k = 2 \). Now, for any lattice \( \mathcal{L} \),

\[
\varphi(z, \mathcal{L}) = \lim_{s \to 0} \sum_{\omega \in \mathcal{L}} (z - \omega)^{-2s} |z - \omega|^{-2s} - s_2(L),
\]

where \( s_2(L) \) is as defined at the beginning of §4 of [4]. Taking \( \mathcal{L} = L \), we deduce from (9) that

\[
d_2(L) = \lambda_2 \lim_{s \to 0} \sum_{b \in B} \sum_{a \in g} (\psi(bc) - \alpha)^{-2s} |\psi(bc) - \alpha|^{-2s}.
\]
Arguing as in the previous case, we obtain $d_2(L) = \lambda_2 \zeta_F(\sigma, 2)$. Similarly, $d_2(a^{-1}L) = \lambda_2 \psi(a) \zeta_F(\sigma a, 2)$, and so we obtain (11) in this case.

Case 3. We assume that $k = 1$. If $\mathcal{L}$ is any lattice, let $H(s, z, \mathcal{L})$ denote the analytic continuation in $s$ of the series

\[ \sum_{\alpha \in \mathcal{L}} (\bar{z} + \bar{\alpha})|z + \alpha|^{-2s} \]

(this series converges for $R(s) > 3/2$). Then, as is shown in case 3 of the proof of Lemma 21 of [4], we have

\[ \zeta(z, \mathcal{L}) - zs_{2}(\mathcal{L}) = H(1, z, \mathcal{L}) + \bar{z}g(\mathcal{L}), \]

where $g(\mathcal{L})$ is defined in the same proof. First take $\mathcal{L} = L$. It follows from (8) that

\[ d_1(L) = \lambda_1 \lim_{s \to 1} \sum_{b \in B} \sum_{a \in \mathcal{A}} \frac{\tilde{\psi}(bc) + \tilde{\alpha}}{|\psi(bc) + \alpha|^{2s}} + rg(L), \]

where $r = \sum_{b \in B} (\tilde{\psi}(bc)\tilde{\rho})$ (here, by the limit as $s \to 1$, we mean the value of the analytic continuation at $s = 1$). As before, we deduce easily that

\[ d_1(L) = \lambda_1 \zeta_F(\sigma, 1) + rg(L). \]

Next take $\mathcal{L} = \gamma^{-1}L$. Then

\[ d_1(a^{-1}L) = \lambda_1 \lim_{s \to 1} \sum_{b \in B} \sum_{a \in \mathcal{A}} \frac{\tilde{\psi}(bc) + \tilde{\alpha}}{|\gamma\psi(bc) + \alpha|^{2s}} + rg(\gamma^{-1}L). \]

Taking the factor $\gamma^{-1}$ out of each $\alpha$, and recalling that $g(\gamma^{-1}L) = Nag(L)$, we conclude that

\[ d_1(a^{-1}L) = \lambda_1 \gamma \lim_{s \to 1} \sum_{b \in B} \sum_{a \in \mathcal{A}} \frac{\tilde{\gamma}\tilde{\psi}(bc) + \tilde{\alpha}}{|\gamma\psi(bc) + \alpha|^{2s}} + rNag(L). \]

We now argue in the same way as in case 1 to deduce that

\[ d_1(a^{-1}L) = \lambda_1 \psi(a) \zeta_F(\sigma a, 1) + rNag(L). \]

Combining these two expressions for $d_1(L)$ and $d_1(a^{-1}L)$, we see that (11) is true for $k = 1$. This completes the proof of Lemma 8.
COROLLARY 9: For each integer \( k \geq 1 \), and each \( \sigma \in \mathcal{G} \), \( \Omega^{-k} \zeta_F(\sigma, k) \) belongs to \( F \). Moreover, if \( \tau \in \mathcal{G} \), then \( (\Omega^{-k} \zeta_F(\sigma, k))^{\tau} = \Omega^{-k} \zeta_F(\tau \sigma, k) \).

PROOF: The first assertion is plain from Lemmas 7 and 8, on taking \( a \neq 1 \) to be an integral ideal of \( K \), prime to \( S \) and \( p \), such that \( \sigma_a = 1 \). The second assertion follows similarly, on noting that \( c_k(a, \sigma)^{\tau} = c_k(a, \tau \sigma) \) for all \( k \geq 1 \) because \( \Lambda_c(z, a)^{\tau} = \Lambda_{\sigma}(z, a) \). Here \( \Lambda_c(z, a)^{\tau} \) denotes the rational function of \( \varphi(z) \) and \( \varphi'(z) \), with coefficients in \( F \), which is obtained by letting \( \tau \) act on the coefficients of \( \Lambda_c(z, a) \).

Let \( \psi_F \) denote the Grössencharacter of \( F \), which is obtained by composing \( \psi \) with the norm map from \( F \) to \( K \). Plainly \( \psi_F \) is unramified outside \( \mathfrak{q} \). Thus, for each integer \( k \geq 1 \), we can define

\[
L_F(\psi_F^k, s) = \prod_{(\mathfrak{B}, \mathfrak{d})=1} (1 - \psi_F^k(\mathfrak{B})(N\mathfrak{B})^{-s})^{-1},
\]

the product being taken over all primes \( \mathfrak{B} \) of \( F \) which do not divide \( \mathfrak{q} \). Of course, \( L_F(\psi_F^k, s) \) will not, in general, be a primitive Hecke \( L \)-function, but this will not be important in the proof of Theorem 1.

Let \( \hat{G} \) denote the group of all homomorphisms from \( G \) into the group of non-zero complex numbers. If \( \theta \in \hat{G} \), we associate with it the complex \( L \)-function

\[
L_F(\psi_F^k \theta, s) = \sum_{\sigma \in \hat{G}} \theta(\sigma) \zeta_F(\sigma, k; s).
\]

One verifies immediately that we have the product decomposition

\[
L_F(\psi_F^k, s) = \prod_{\theta \in \hat{G}} L_F(\psi_F^k \theta, s).
\]

The next lemma gives the basic rationality properties of the value of \( L_F(\psi_F^k, s) \) at \( s = k \).

LEMMA 10: For each integer \( k \geq 1 \), \( \Omega^{-kd} L_F(\psi_F^k, k) \) belongs to \( F \), and the ideal that it generates is fixed by the action of \( \mathcal{G} \).

PROOF: By (12) and the first assertion of Corollary 9, we see that \( \nu_k = \Omega^{-kd} L_F(\psi_F^k, k) \) belongs to \( M \), where \( M \) is the field obtained by adjoining to \( F \) the values of all \( \theta \in \hat{G} \). But, again by (12), it is clear that \( \nu_k \) is fixed by the Galois group of \( M \) over \( F \), and so belongs to \( F \). Now take \( \tau \) to be any element of \( \mathcal{G} \), and let \( \tau_1 \) be an element of \( G(M/K) \) whose restriction to \( F \) is \( \tau \). The second assertion of Corol-
lary 9 implies that

\[(13) \quad \Omega^{-k} L_F(\tilde{\psi}^k \theta, k)^{\tau_1} = \theta^{\tau_1} (\tau^{-1}) \Omega^{-k} L_F(\tilde{\psi}^k \theta^{\tau_1}, k),\]

whence it is plain from (12) that the ideal in \( F \) generated by \( \nu_k \) is fixed by \( \mathcal{G} \).

**Remark:** If \( \mathcal{G} \) has no quadratic characters, (12) and (13) show that \( \Omega^{-kd} L_F(\tilde{\psi}^k, k) \) is actually fixed by \( \mathcal{G} \), and so belongs to \( K \).

We now investigate the integrality properties of the numbers in Corollary 9 and Lemma 10. Let \( \mathfrak{B} \) be any prime of \( F \) lying above \( \wp \), \( F_{\mathfrak{B}} \) the completion of \( F \) at \( \mathfrak{B} \), and \( \mathcal{O}_{\mathfrak{B}} \) the ring of integers of \( F_{\mathfrak{B}} \). We can view \( \Lambda_\sigma(z, a) \) as being a rational function of \( \wp(z) \) and \( \wp'(z) \) with coefficients in \( F_{\mathfrak{B}} \), via the canonical inclusion of \( F \) in \( F_{\mathfrak{B}} \). Hence we can expand \( \Lambda_\sigma(z, a) \) in terms of the parameter \( t = -2\wp(z)/\wp'(z) \) of the formal group \( \hat{E} \).

**Lemma 11:** Let \( \mathfrak{B} \) be any prime of \( F \) above \( \wp \). In terms of the parameter \( t = -2\wp(z)/\wp'(z) \), \( \Lambda_\sigma(z, a) \) has an expansion

\[\Lambda_\sigma(z, a) = \sum_{k=0}^{\infty} h_{k,\sigma}(a, \mathfrak{B}) t^k,\]

whose coefficients all belong to \( \mathcal{O}_{\mathfrak{B}} \), and where \( h_{0,\sigma}(a, \mathfrak{B}) \) is a unit in \( \mathcal{O}_{\mathfrak{B}} \).

**Proof:** This is the same as the proof of Lemma 23 of [4] (on recalling that \( (\mathfrak{p}, \wp) = 1 \) by hypothesis), and so we omit the details.

**Lemma 12:** Let \( k \) be an integer with \( 1 \leq k \leq p - 1 \). Then (i) for \( \sigma \in \mathfrak{G} \), \( \Omega^{-kd} \xi_k(\sigma, k) \) is integral at each prime of \( F \) above \( \wp \), and (ii) \( \Omega^{-kd} L_F(\tilde{\psi}^k, k) \) is integral at each prime of \( F \) above \( \wp \).

**Proof:** In view of (12), it is plain that (ii) is a consequence of (i). We now proceed to deduce (i) from the previous lemma. Let \( w \) be the parameter of the Lubin-Tate formal group \( \mathfrak{B} \) such that \([\pi](w) = \pi w + w^p \) (cf. §3 of [4]). Fix a prime \( \mathfrak{B} \) of \( F \) above \( \wp \). For the moment, take \( a \) to be an arbitrary integral ideal of \( K \), prime to \( S \) and \( p \). Since \( t \) can be written as a power series in \( w \) with coefficients in \( \mathcal{O}_w \), it follows from Lemma 11 that \( \Lambda_\sigma(z, a) \) can be expanded as a power series in \( w \), say \( f(w) \), with coefficients in \( \mathcal{O}_w \), and whose constant term \( f(0) \) is a unit in \( \mathcal{O}_w \). Moreover, since \( z = w + \sum_{i=2}^{\infty} a_i w^i \), where \( a_i = 0 \) unless
\[ i \equiv 1 \mod (p - 1) \] (cf. Lemma 7 of [4]), the coefficients of \( z^k \) and \( w^k \) \((0 \leq k \leq p - 1)\) in the \( z \)-expansion of \( \Lambda_\sigma(z, a) \) and in \( f(w) \) are plainly equal. It follows that the coefficients of \( z^k \) and \( w^k \) \((1 \leq k \leq p - 1)\) in the \( z \)-expansion of \( z(d/dz) \log \Lambda_\sigma(z, a) \) and in \( w(d/dw) \log f(w) \) are also equal. But the coefficients of this latter series lie in \( \mathcal{O}_\mathbb{Q} \), because the constant term \( f(0) \) of \( f(w) \) is a unit in \( \mathcal{O}_\mathbb{Q} \). We conclude from Lemma 8 that

\[
(14) \quad \Omega^{-k}(N_a \xi_F(\sigma, k) - \psi^k(a) \xi_F(\sigma a, k))
\]

is integral at \( \mathfrak{p} \) for \( 1 \leq k \leq p - 1 \). We now make a special choice of the ideal \( a \). Let \( e \) denote a generator of the ideal \((12g) \cap \mathbb{Z}\). Choose \( n \) to be a rational integer, prime to \( p \), such that \( 1 + ne \pi \) is not divisible by \( \mathfrak{p} \), and take \( a = (1 + ne \pi) \). Then \( Na \equiv 1 \mod p \). Also \( \sigma_0 = 1 \) because the conductor of \( F/K \) divides \( e \), and \( \psi^k(a) = (1 + en \pi)^k \equiv 1 \mod p \), because the conductor of \( \psi \) divides \( e \). Thus \( N_a \psi^k(a) \) is a unit at \( \mathfrak{p} \), and so assertion (i) follows from (14). This completes the proof of Lemma 12.

We now prove a technical lemma, which establishes the existence of \( d \) pairs \((A, N)\) in \( \mathcal{J} \), with properties which will be needed later in this section. To simplify the statement of the lemma, we choose a fixed numbering of the elements of \( \mathcal{J} \), say \( \sigma_1, \ldots, \sigma_d \), with \( \sigma_1 = 1 \).

**Lemma 13:** Let \( k \) be an integer with \( 1 \leq k \leq p - 2 \). Then there exist \( d \) pairs \((A^{(h)}, N^{(h)})\) \( \in \mathcal{J} \), where

\[
A^{(h)} = \{a_1^{(h)}, a_2^{(h)}\}, \quad N^{(h)} = \{n_1^{(h)}, n_2^{(h)}\} \quad (1 \leq h \leq d),
\]

with the following properties. Firstly, \( \psi^k(a_2^{(h)}) \not\equiv 1 \mod p \). Secondly, for \( 1 \leq h \leq d \), we have (i) \( \psi^k(a_1^{(h)}) \equiv 1 \mod p \), (ii) \( \sigma_{a_1^{(h)}} = 1 \), (iii) \( \sigma_{a_2^{(h)}} = \sigma_{a_1^{(h)}}^{-1} \), and (iv) \( n_2^{(h)} \) is prime to \( p \).

**Proof:** Let \( e \) denote a generator of the ideal \((12g) \cap \mathbb{Z} \), and let \( \beta \mod p \) be a generator of \((\mathcal{O}/\mathfrak{p})^\times \). First consider the case \( h = 1 \). Let \( n \) be a rational integer, prime to \( p \), such that \( 1 + ne \pi \) is prime to \( \mathfrak{p} \), and take \( a_1^{(1)} = (1 + en \pi) \). Choose \( a_2^{(1)} = (a_1^{(1)})^{(e \pi)} \), where \( a_1^{(1)} \) is an algebraic integer in \( K \) satisfying \( a_2^{(1)} = 1 \mod e\pi \), and \( a_2^{(1)} = \beta \mod \pi \). Let \( n_1^{(1)} = Na_2^{(1)} - 1 \) and \( n_2^{(1)} = -(Na_1^{(1)} - 1) \), so that \( n_2^{(1)} \) is prime to \( p \) because \( (p, ne) = 1 \). Moreover, as the conductor of \( \psi \) divides \( e \), we have \( \psi^k(a_1^{(1)}) \equiv 1 \mod p \), and \( \psi^k(a_2^{(1)}) \equiv \beta^k \not\equiv 1 \mod p \). Finally, both ideals are prime to \( S \) and \( p \) by construction, and \( \sigma_{a_1^{(1)}} = \sigma_{a_2^{(1)}} = 1 \) because the conductor of \( F \) over \( K \) also divides \( e \). This completes the case \( h = 1 \).
For $h > 1$, again choose $a_1^{(h)} = (1 + ne\pi)$ and $n_2^{(h)} = - (Na_1^{(h)} - 1)$. Take $a_2^{(h)}$ to be an integral ideal of $K$, prime to $S$ and $p$, such that $\sigma a_2^{(h)} = \sigma a_1^{(h)}$, and let $n_1^{(h)} = Na_2^{(h)} - 1$. The proof of the lemma is now complete.

So far in this section, we have made no hypothesis on the decomposition of $p$ in the extension $F/K$, other than requiring that $p$ does not ramify in $F/K$. We now suppose, until further notice, that $p$ splits completely in $F$. We use the notation of the last part of §13. Thus $\mathcal{S}$ will denote the set of prime of $F_0 = F(E_\pi)$ above $p$, and $\mathfrak{A}$ will again be given by (4). Let

$$i : F_0 \to \prod_{q \in \mathcal{S}} F_{0,q}$$

be the canonical embedding of $F_0$ in the product of its completions at the primes $q$ in $\mathcal{S}$. Recall that $C$ denotes the group of elliptic units of $F_0$, as defined at the beginning of this section. We write $\mathbb{C}$ for the subgroup of $C$ consisting of all elements which are $\equiv 1 \mod q$ for each $q \in \mathcal{S}$. Let $\overline{i(\mathbb{C})}$ be the closure of $i(\mathbb{C})$ in the $p$-adic topology. Our aim is to compute, for $1 \leq k \leq p - 2$, the image of $\overline{i(\mathbb{C})}$ under the homomorphism $\varphi_{F,k}$ given by (6).

Recall that $\Phi$ is the field $K_\rho(E_\pi)$, which lies inside our fixed algebraic closure of $K_\rho$. Since $\rho$ splits completely in $F$ by hypothesis, the completion of $F_0$ at each $q$ in $\mathcal{S}$ is plainly topologically isomorphic to $\Phi$. To simplify notation, we adopt the following convention. We fix one embedding of $F_0$ in $\Phi$, and view this embedding as simply being an inclusion. This amounts to choosing one fixed prime in $\mathcal{S}$, which we denote by $q$. Let $\Omega$ denote the Galois group of $F_0$ over $K(E_\pi)$. Since $\rho$ is totally ramified in $K(E_\pi)$, and splits completely in $F_0/K(E_\pi)$, the other primes in $\mathcal{S}$ are given precisely by the $q^\sigma$ for $\sigma \in \Omega$, and the embedding of $F_0$ in $\Phi$ corresponding to $q^\sigma$ is given by $\sigma$ itself. With this convention, the map (15) is simply given by

$$i(x) = (x^\sigma)_{\sigma \in \Omega}.$$

Now take $x$ to be any elliptic unit in $\mathbb{C}$. More explicitly, let $\xi(\tau)$ be the point of $E_\pi$ corresponding to our chosen generator $u$ of $\mathcal{E}_\pi$ under our fixed isomorphism from $\hat{E}$ to $\mathcal{E}$. Then, by definition, $x$ will be of the form

$$x = \prod_{\tau \in \hat{E}} \Lambda(\tau, a_i)^{n_i}.$$
for some pair \((A, \mathcal{N})\) belonging to \(\mathscr{S}\). Now \(\Omega = G(F_0/K(E_\sigma))\) is canonically isomorphic to \(\mathscr{G} = G(F/K)\) under the restriction map, and we shall identify these two Galois groups in this way when there is no danger of confusion. Since \(\Omega\) fixes \(E_\sigma\), it is then plain that

\[
x^\sigma = \prod_{\ell \in \mathcal{I}} \Lambda_{\ell}(\tau, a_j)^{v_{\ell}} \quad \text{for} \quad \sigma \in \Omega,
\]

where \(\Lambda_{\ell}(z, a_j)\) is as defined just after Lemma 7.

**Lemma 14:** Let \(x\) be the elliptic unit in \(\mathfrak{O}\) given by (17). Then, for each integer \(k\) with \(1 \leq k \leq p - 2\), we have

\[
\phi_{F,k}(i(x)) = \left( \lambda_k \sum_{j \in \mathcal{J}} n_j(Na_j^k \zeta_{\ell}(\sigma, k) - \psi^k(a_j) \zeta_{\ell}(\sigma a_p, k)) \mod q^\sigma \right)_{\sigma \in \Omega},
\]

where \(\lambda_k = 12(-1)^{k-1}p^{-k}\).

**Proof:** We can obtain a power series \(f_\sigma(w)\), with coefficients in \(\mathfrak{O}_\mathfrak{p}\), such that \(f_\sigma(u) = x^\sigma\) in the following manner. Let \(w\) be the parameter of the Lubin-Tate formal group \(\mathfrak{G}\), and expand the rational function of \(\varphi(z)\) and \(\varphi'(z)\), with coefficients in \(F\), given by

\[
(18) \quad \prod_{\ell \in \mathcal{I}} \Lambda_{\ell}(z, a_j)^{v_{\ell}}
\]

as a formal power series in \(w\). Denote the power series obtained in this way by \(f_\sigma(w)\). By lemma 11 and the fact that \(t\) can be written as a power series in \(w\) with coefficients in \(\mathfrak{O}_\mathfrak{p}\), we conclude that \(f_\sigma(w)\) does indeed have coefficients in \(\mathfrak{O}_\mathfrak{p}\). It is then plain that \(x^\sigma = f_\sigma(u)\). Moreover, as \(z = w + \sum_{i=2}^{\infty} a_i w^i\), where \(a_i = 0\) unless \(i \equiv 1 \mod(p - 1)\) (cf. Lemma 7 of [4]), we see that the coefficients of \(z^k\) and \(w^k\) \((0 \leq k \leq p - 1)\) in the series expansions of (18) in terms of \(z\) and \(w\) must be equal. Thus the conclusion of the lemma is now clear from Lemma 8 and the definition of \(\phi_{F,k}\).

We now come to the first main result of this section. Since the elliptic units of \(F_0\) are stable under the action of the Galois group of \(F_0\) over \(K\) (cf. Lemma 20 of [4]), it follows, in particular, that \(\mathfrak{i}(\mathfrak{G})\) is a \(\mathbb{Z}_p[G]\)-submodule of \(\mathfrak{U}\), where \(G = G(F_0/F)\). We can therefore take the canonical decomposition (2) of \(\mathfrak{U}/\mathfrak{i}(\mathfrak{G})\). We follow the terminology of [4] and say that \(p\) is anomalous for \(E\) if \(\pi + \pi = 1\).
THEOREM 14: Assume that $p$ is a prime number $>5$ satisfying (i) $p$ does not belong to the finite exceptional set $S$, (ii) $p$ splits in $K$, say $(p) = \mathfrak{p}\mathfrak{q}$, (iii) $\mathfrak{p}$ splits completely in $F|K$, and (iv) $p$ is not anomalous for $E$. Let $\mathcal{E}$ be the group of elliptic units of $\mathcal{F} = F(E_{\mathfrak{P}})$, which are $\equiv 1 \mod q$ for each $q \in \mathcal{S}$. Then, for each integer $k$ with $1 \leq k \leq p - 2$, the eigenspace $(\mathfrak{U}/i(\mathcal{E}))^{(k)}$ is non-trivial if and only if $\Omega^{-kd}L_{\mathcal{F}}(\overline{\psi}_{\mathcal{F}}, k) \equiv 0 \mod q$ for each $q \in \mathcal{S}$.

REMARK: By Lemma 10, $\Omega^{-kd}L_{\mathcal{F}}(\overline{\psi}_{\mathcal{F}}, k) \equiv 0 \mod q$ for one prime $q$ in $\mathcal{S}$ if and only if the same congruence is valid for all $q$ in $\mathcal{S}$.

PROOF: We adopt the same convention as before, in which we have fixed one prime $q$ in $\mathcal{S}$, and view $\mathcal{F}_0$ as being contained in $\Phi$. We make use of the following formal identity in the group ring $F[\mathcal{G}]$, which is very reminiscent of computations with Stickelberger elements in cyclotomic fields. For each $\sigma \in \mathcal{G}$, put

$$\zeta_{\mathcal{F}}^{\Phi}(\sigma, k) = \lambda_k \zeta_{\mathcal{F}}(\sigma, k).$$

By Corollary 9, $\zeta_{\mathcal{F}}^{\Phi}(\sigma, k)$ belongs to $F$. Write

$$\alpha = \sum_{\sigma \in \mathcal{G}} \zeta_{\mathcal{F}}^{\Phi}(\sigma, k)\sigma^{-1}. \quad (19)$$

Then, for each integral ideal $a$ of $K$ which is prime to $q$, we plainly have

$$Na - \psi_k(a)\sigma_a\alpha = \sum_{\sigma \in \mathcal{G}} \delta_k(\sigma, a)\sigma^{-1}, \quad (20)$$

where

$$\delta_k(\sigma, a) = Na\zeta_{\mathcal{F}}^{\Phi}(\sigma, k) - \psi_k(a)\zeta_{\mathcal{F}}^{\Phi}(\sigma_a, k). \quad (21)$$

By Corollary 6, the eigenspace $(\mathfrak{U}/i(\mathcal{E}))^{(k)}$ will be trivial if and only if $\varphi_{\mathcal{F}k}(I(\mathcal{E}))$ has dimension $d$ over the finite field $F_p$ with $p$ elements. This suggests that we study the image under $\varphi_{\mathcal{F}k}$ of any $d$ elements of $i(\mathcal{E})$. Suppose therefore that $(A^{(h)}, \mathcal{N}^{(h)})$ $(1 \leq h \leq d)$ are any $d$ elements of $\mathcal{S}$. Let $x_h$, given by (17), be the elliptic unit corresponding to $(A^{(h)}, \mathcal{N}^{(h)})$. We assume that $x_1, \ldots, x_d$ belong to $\mathcal{E}$. Write

$$A^{(h)} = \{a^{(h)}_j : j \in J_h\}, \quad \mathcal{N}^{(h)} = \{n^{(h)}_j : j \in J_h\},$$

where
and
\[ \gamma_h = \sum_{j \in J_h} n_j^{(h)}(Na_j^{(h)} - \psi^k(a_j^{(h)})\sigma_j^{(h)}). \]

For \( \sigma \in \mathcal{G} \) and \( 1 \leq h \leq d \), we define
\[ b_{ho} = \sum_{j \in J_h} n_j^{(h)}\delta_h(\sigma, a_j^{(h)}), \]
where \( \delta_h(\sigma, a_j^{(h)}) \) is given by (21). It is then plain from (20) that we have the identity
\[ \gamma_h \sigma = \sum_{\sigma \in \mathcal{G}} b_{ho} \sigma^{-1} \quad (1 \leq h \leq d). \]

We let \( \Xi \) denote the \( d \times d \)-determinant form from the \( b_{ho} \) \( (h = 1, \ldots, d, \sigma \in \mathcal{G}) \).

By Lemma 14, the determinant of the \( d \) vectors
\[ \varphi_{\mathcal{G}, k}(i(x_h)) \quad (1 \leq h \leq d) \]
is equal to \( \Xi \mod q \). We now proceed to compute \( \Xi \). To this end, let \( \hat{\mathcal{G}} \) be the group of homomorphisms from \( \mathcal{G} \) to the multiplicative group of non-zero complex numbers. Let \( \sigma_1 = 1, \sigma_2, \ldots, \sigma_d \) denote the distinct elements of \( \mathcal{G} \), and \( \chi_1 = 1, \chi_2, \ldots, \chi_d \) the distinct elements of \( \hat{\mathcal{G}} \). Write \( \Gamma \) and \( \Sigma \) for the \( d \times d \)-determinants formed from the \( \chi_i(\gamma_h) \), \( \chi_i(\sigma_h^{-1}) \) \( (1 \leq i, h \leq d) \), respectively. Applying each of the \( \chi_i \) to the equation (22), we conclude that
\[ \left( \prod_{i=1}^{d} \chi_i(\alpha) \right) \Gamma = \Sigma \Xi. \]

We now make two observations. Put \( L_{\mathcal{E}}(\tilde{\psi}_{F}, k) = \lambda_F^d L_F(\tilde{\psi}_{F}, k) \). Then it is plain from (12) and (19) that
\[ \prod_{i=1}^{d} \chi_i(\alpha) = L_{\mathcal{E}}(\tilde{\psi}_{F}, k). \]

Secondly, \( \Sigma \neq 0 \) and \( \Gamma/\Sigma \) is an algebraic integer in \( K \). The former assertion is clear. To prove the latter one, we note that we can write
\[ \gamma_h = \sum_{\sigma \in \mathcal{G}} e_{ho} \sigma^{-1}, \]
where the $e_{hr}$ are algebraic integers in $K$. Applying each of the $\chi_i$ to (25), it follows that $\Gamma = A\Sigma$, where $A$ is the $d \times d$-determinant formed from the $e_{hr}$. Since $\Sigma$ is obviously an algebraic integer in $K$, it follows that the same is true for $\Sigma = \Gamma/A$.

We can now complete the proof of Theorem 14. Suppose first that $L_F(\psi_F, k) \equiv 0 \mod q$. Then we conclude from (23), (24) and the above remarks that $\mathcal{E} \equiv 0 \mod q$ for all choices of the $d$ pairs $(A^{(h)}, \mathcal{N}^{(h)})$ in $\mathfrak{F}$. Thus $\varphi_{F,k}(i(\mathfrak{O}))$ has dimension strictly less than $d$ over $F_p$, and hence $(\mathcal{U}/i(\mathfrak{O}))^{(h)} \neq 0$. Conversely, assume that $L_F(\psi_F^{(h)}, k) \not\equiv 0 \mod q$. Then it follows from (23) and (24) that $\mathcal{E} \not\equiv 0 \mod q$ only if we can choose the $d$ pairs $(A^{(h)}, \mathcal{N}^{(h)})$ such that the determinant $A$ defined above is not congruent to $0$ modulo $p$. But this is always possible. Indeed, make the choice of the $d$ pairs $(A^{(h)}, \mathcal{N}^{(h)})$ specified in Lemma 13. Note that, by multiplying each of the $n_1^{(h)}, n_2^{(h)} (1 \leq h \leq d)$ by $p - 1$ (which changes none of the other conditions in Lemma 13), we can certainly assume that the corresponding elliptic units lie in $\mathfrak{O}$. Using the relation $\sum_{i=1}^{d} n_1^{(h)}(N_{A_i^{(h)}} - 1) = 0$ and the fact that $\psi^{(h)}(a_1^{(h)}) \equiv 1 \mod \varphi$, we conclude that

$$\gamma_h = n_1^{(h)} - n_2^{(h)}\psi^{(h)}(a_1^{(h)})\sigma_h^{(1)} \mod \varphi \quad (1 \leq h \leq d);$$

here the congruence mod $\varphi$ means that we have taken the coefficients in the group ring mod $\varphi$. It is now trivial to verify from the other conditions of Lemma 13 that $A \not\equiv 0 \mod \varphi$. This completes the proof of Theorem 14.

**Lemma 15:** There are infinitely many rational primes $p$ satisfying conditions (i), (ii), (iii), and (iv) of Theorem 14.

**Proof:** As before, let $H = K(E_g)$. Applying Cebotarev’s density theorem to a Galois extension of $\mathfrak{O}$ containing $H$, we conclude that there are infinitely many rational primes $p$ which split completely in $H$. We claim that any rational prime $p$, not in $S$, which splits completely in $H$, satisfies (i), (ii), (iii) and (iv). The only part which is not obvious is that such a $p$ satisfies (iv). Take such a $p$, and let $(p) = \varphi \overline{\varphi}$ be its factorization in $K$. Since $\varphi$ splits completely in $H$, the Artin symbol $(\varphi, H/K)$ fixes $E_g$. On the other hand, as $\psi(\varphi) = \pi$, Shimura’s reciprocity law gives $\xi(\varphi)(\rho, H/K) = \xi(\pi \rho)$ for each $\rho \in E_g$. Thus we must have $\pi = 1 \mod g$. Now, if $p$ were anomalous, it would follow that $\pi \overline{\pi} = (\pi - 1)(\overline{\pi} - 1)$, and this is clearly impossible because $p$ was prime to $g$ by hypothesis. This completes the proof.

We now begin the proof of the second main result of this section.
As before, let $F_n = F(E_{\pi^{n+1}})$. Since $\varphi$ is totally ramified in $K(E_{\pi^{n+1}})$, it is clear that each prime of $F$ above $\varphi$ is totally ramified in $F_n$. Write $\mathcal{S}_n$ for the set of primes of $F_n$ above $\varphi$. Let $C_n$ be the group of elliptic units of $F_n$, as defined at the beginning of this section, and let $\mathcal{C}_n$ be the subgroup of $C_n$ consisting of all elements which are $\equiv 1 \mod q$ for each $q \in \mathcal{S}_n$. If $m \geq n$, we write $N_{m,n}$ for the norm map from $F_m$ to $F_n$. The next lemma, which is, in essence, one of the main results of [6], is valid without any hypothesis on the decomposition of $\varphi$ in $F$.

**Lemma 16:** For each $m \geq n \geq 0$, we have $N_{m,n}(\mathcal{C}_m) = \mathcal{C}_n$.

**Proof:** Recall that $\mathfrak{f}_n = \mathfrak{p}^{n+1}$ is the conductor of $F_n$ over $K$, by Lemma 3. Let $f_n$ denote a generator of the ideal $\mathfrak{f}_n \cap \mathbb{Z}$, and let $g_n$ be the largest divisor of $f_n$ such that the $g_n$-th roots of unity lie in $F_n$. We claim that $g_n = g_0$ for all $n \geq 0$, and that $g_0$ is prime to $\varphi$. Indeed, $F_n$ can contain no non-trivial $\varphi$-power roots of unity, because $\varphi$ does not divide the conductor of $F_n/K$. Moreover, since $F_n/F_0$ is totally ramified at the primes above $\varphi$, it follows that $F_n$ and $F_0$ have the same group of roots of unity for all $n \geq 0$. Let $D$ be the group of $g_0$-th roots of unity in $F_0$. Robert (cf. [6], p. 43) has defined $\Omega_{F_n}$ to be the group $DC_n$. Moreover, since $\mathfrak{f}_0$ divides $\mathfrak{f}_n$ and $\mathfrak{f}_0$ and $\mathfrak{f}_n$ are divisible by the same primes, it is shown in [6] (cf. Proposition 17, p. 43) that $N_{m,n}(\Omega_{F_m})D = \Omega_{F_n}$. Since the order of $D$ is prime to $\varphi$ (and hence no element of $D$ is $\equiv 1 \mod q$ for $q \in \mathcal{S}_n$), it follows immediately that $N_{m,n}(\mathcal{C}_m) = \mathcal{C}_n$. This completes the proof.

For each integer $n \geq 0$, let $\Phi_n = K_\varphi(E_{\pi^{n+1}})$, and let $\varphi_n$ be the maximal ideal of $\Phi_n$. Write $U_n$ for the units of $\Phi_n$ which are $\equiv 1 \mod \varphi_n$, and $U'_n$ for the subgroup of $U_n$ consisting of all elements with norm 1 to $K_\varphi$. Plainly

\[(U'_n)^{(k)} = U_n^{(k)} \quad \text{for} \quad k \not\equiv 0 \mod (p-1).\]

If $m > n$, we also write $N_{m,n}$ for the norm map from $\Phi_m$ to $\Phi_n$.

**Lemma 17:** Suppose that $k \not\equiv 0 \mod (p-1)$. If $m \geq n$, then the norm map from $U_m^{(k)}$ to $U_n^{(k)}$ is surjective, and its kernel is equal to $(U_m^{(k)})^{1-\tau}$, where $\tau$ is a generator of $G(\Phi_m/\Phi_n)$.

**Proof:** The norm map from $U'_m$ to $U'_n$ is surjective, because $U'_n$ consists of those elements of $U_n$ which are norms from $\Phi_m$ for all $m \geq n$ (cf. Lemma 8 of [4]). Thus the first assertion is plain from (26). As for the second, let $V_m$ denote the kernel of the norm map from $U_m$
to $U_n$. Since $\Phi_m/\Phi_n$ is a totally ramified cyclic extension of degree $p^{m-n}$, a standard computation (cf. [5], p. 188) shows that

$$[V_m : U_m^{1-r}] = [V_m^{(0)} : U_m^{(0)(1-r)}] = p^{m-n}. $$

Hence $[V_m^{(k)} : U_m^{(k)(1-r)}] = 1$ for all $k \not\equiv 0 \mod(p-1)$, as required.

The following elementary lemma is certainly well known, but we have been unable to find a suitable reference.

**Lemma 18:** Let $A$ be a cyclic group of prime order $p \neq 2$, operating on a finitely generated $\mathbb{Z}_p$-module $M$. Let $\tau$ be a generator of $A$. If $M = (\tau - 1)M$, then $M = 0$.

**Proof:** Since $\tau^p = 1$ and $p$ is odd, it is clear that

$$ (\tau - 1)^p \in p\mathbb{Z}[A], $$

where $\mathbb{Z}[A]$ is the group ring of $A$ with coefficients in $\mathbb{Z}$. Let $N$ be the torsion submodule of $M$, so that $M/N$ is a free $\mathbb{Z}_p$-module of finite rank with $(\tau - 1)(M/N) = (M/N)$. But this shows that $(\tau - 1)^p$ is surjective on $M/N$, and this is impossible by (27) unless $M/N = 0$. Hence we can suppose that $M$ is a finite abelian $p$-group. But again (27) implies that $M = 0$ if $(\tau - 1)M = M$. This completes the proof.

For each $q \in \mathcal{S}_n$, let $F_{n,q}$ be the completion of $F_n$ at $q$, and again let $i$ be the canonical inclusion of $F_n$ in $\prod_{q \in \mathcal{S}_n} F_{n,q}$. Write $U_{n,q}$ for the units in $F_{n,q}$ which are $\equiv 1 \mod q$, and put

$$ \mathcal{U}_n = \prod_{q \in \mathcal{S}_n} U_{n,q}. $$

Thus, in terms of our earlier notation, $\mathcal{U}_0 = \mathcal{U}$ and $\mathcal{C}_0 = \mathcal{C}$.

**Theorem 19:** Let $p$ be a prime number satisfying (i) $p$ does not belong to $S$, (ii) $p$ splits in $K$, $(p) = p\mathfrak{p}$, and (iii) $p$ splits completely in $F$. Let $k$ be an integer with $1 \leq k \leq p-2$. Let $m, n$ be any two integers $\geq 0$, with $m > n$. Then $(\mathcal{U}_m/i((\mathcal{C}_m))^{(k)}) \neq 0$ if and only if $(\mathcal{U}_n/i((\mathcal{C}_n))^{(k)}) \neq 0$.

**Proof:** Since $p$ splits completely in $F$, we can identify $F_{n,q}$, for each $q \in \mathcal{S}_n$, with the field $\Phi_n$, and $U_{n,q}$ with $U_n$. Let $N_{m,n} : \mathcal{U}_m \to \mathcal{U}_n$ be the map given by the product of the local norms from $\Phi_m$ to $\Phi_n$ at each $q \in \mathcal{S}_n$. Suppose now that $1 \leq k \leq p-2$. Put $A_n = \mathcal{U}_n^{(k)}/i((\mathcal{C}_n)^{(k)})$. It
follows from the first part of Lemma 17 that the norm map from \( U_n^{(k)} \) to \( U_n^{(k)} \) is surjective, whence the induced map from \( A_m^{(k)} \) to \( A_n^{(k)} \) is also surjective. Thus it is clear that \( A_m^{(k)} = 0 \) implies that \( A_n^{(k)} = 0 \). To prove the converse, we note that Lemmas 16 and 17 together imply that the kernel of the norm map from \( A_m^{(k)} \) to \( A_n^{(k)} \) is \( (A_m^{(k)})^{1-\tau} \), where \( \tau \) is a generator of the Galois group of \( F_m \) over \( F_n \). Suppose now that \( A_m^{(k)} = 0 \). Since \( A_n^{(k+1)} \) is a finitely generated \( \mathbb{Z}_p \)-module, we conclude from Lemma 18 that \( A_n^{(k+1)} = 0 \). Repeating the argument a finite number of times, it follows that \( A_m^{(k)} = 0 \) for all \( m \geq n \). This completes the proof.

5. Proof of Theorem 1

We can now complete the proof of Theorem 1 in an entirely similar fashion to the proof of Theorem 1 in [4]. If \( N \) is an abelian extension of \( F_n \), which is Galois over \( F \), then \( G_n = G(F_n/F) \) operates on \( X = G(N/F_n) \) via inner automorphisms in the usual way. In particular, \( G = G(F_0/F) \) operates on \( X \), because we can identify \( G \) with a subgroup of \( G_n \). Thus, if \( N \) is a \( p \)-extension of \( F_n \), we can take the canonical decomposition (2) of \( X \) into eigenspaces for the action of \( G \).

As before, let \( \mathcal{S}_n \) be the set of primes of \( F_n \) over \( p \). Let \( M_n \) denote the maximal abelian \( p \)-extension of \( F_n \), which is unramified outside \( \mathcal{S}_n \), and let \( L_n \) be the \( p \)-Hilbert class field of \( F_n \). Let \( \mathcal{U}_n \) be defined by (28), that is, \( \mathcal{U}_n \) is the product of the local units \( =1 \) in the completions of \( F_n \) at the primes \( q \in \mathcal{S}_n \). Write \( N_{F_n/K} : \mathcal{U}_n \to K_p \) for the map given by the product of the local norms at all \( q \in \mathcal{S}_n \). We denote the kernel of \( N_{F_n/K} \) by \( \mathcal{U}_n' \). Plainly

\[
(29) \quad \mathcal{U}_n^{(k)} = (\mathcal{U}_n')^{(k)} \quad \text{whenever} \quad k \not\equiv 0 \mod(p - 1).
\]

As is explained in detail in [3], global class field theory gives the following explicit description of \( G(M_n/L_n F_n) \) as a \( G_n \)-module, where \( F_n = \bigcup_{n \geq 0} F_n \). Let \( E_n \) be the group of all global units of \( F_n \) which are \( =1 \mod q \) for each \( q \in \mathcal{S}_n \). Let \( i(E_n) \) be the closure of \( i(E_n) \) in \( \mathcal{U}_n \) in the \( p \)-adic topology.

**Theorem 20:** For each \( n \geq 0 \), \( \mathcal{U}_n / i(E_n) \) is isomorphic as a \( G_n \)-module, via the Artin map, to \( G(M_n/L_n F_n) \).

Suppose now that there does exist a point \( P \) in \( E(F) \) of infinite
order. Take \( p \) to be a rational prime satisfying (i) \( p \) does not belong to \( S \), (ii) \( p \) splits in \( K \), \( (p) = \wp \tilde{\wp} \), and (iii) \( p \) splits completely in \( F \). As before, let \( \pi = \psi(p) \). For each \( n \geq 0 \), choose \( Q_n \) in \( E(\tilde{F}) \) such that
\[
\pi^{n+1}Q_n = P,
\]
and form the extension \( H_n = F_n(Q_n) \). Thus \( H_n/F_n \) is a cyclic extension of degree dividing \( p^{n+1} \), and as \( P \) lies in \( E(F) \), one verifies easily that
\[
(30) \quad x^\sigma = \chi(\sigma)x \quad \text{for all} \quad x \in G(H_n/F_n) \quad \text{and} \quad \sigma \in G.
\]
An entirely similar argument to that given in Lemma 33 of [4] shows that \( H_n/F_n \) is unramified outside \( \mathcal{S}_n \). Finally, as \( \wp \) splits completely in \( \tilde{F} \), the local arguments in Theorem 11 and Lemma 35 of [4] again show that the extension \( H_nF_n/F_n \) is non-trivial and ramified for all sufficiently large \( n \).

Assume now that \( n \) is so large that \( H_nF_n/F_n \) is non-trivial and ramified. Hence the extension \( H_n\mathcal{L}_nF_n/F_n \) is non-trivial. As this extension lies inside \( M_n \), we conclude from (29), (30) and Theorem 20 that
\[
(31) \quad (\mathcal{U}_n| i(E_n))^{(1)} \neq 0.
\]
As before, let \( \mathcal{C}_n \) be the group of elliptic units of \( F_n \), which are \( \equiv 1 \mod q \) for each \( q \in \mathcal{S}_n \). As \( \mathcal{C}_n \subset E_n \), it follows that \( (\mathcal{U}_n| i(\mathcal{C}_n))^{(1)} \neq 0 \). Therefore, by Theorem 19, \( (\mathcal{U}_0| i(\mathcal{C}_0))^{(1)} \neq 0 \). Assume, in addition, that \( p > 5 \) and is not anomalous for \( E \). Theorem 14 then implies that
\[
\Omega^{-d}L_F(\tilde{\psi}_F, 1) \equiv 0 \mod q \quad \text{for each} \quad q \in \mathcal{S}_n.
\]
But, by Lemma 15, there certainly are infinitely many rational primes \( p \) satisfying the conditions we have imposed on \( p \). Thus \( \Omega^{-d}L_F(\tilde{\psi}_F, 1) \) is divisible by infinitely many distinct prime ideals of \( F \), and so must be equal to \( 0 \). Since the Hasse-Weil zeta function of \( E \) over \( F \) is equal to \( L_F(\psi_F, s)L_F(\tilde{\psi}_F, s) \), up to finitely many Euler factors which do not vanish at \( s = 1 \) (cf. Theorem 7.42 of [7]), this completes the proof of Theorem 1.

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(Oblatum 27-I-1977) Department of Pure Mathematics and Mathematical Statistics
University of Cambridge
16 Mill Lane
Cambridge, England