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Smooth and admissible representations of 
p-adic unipotent groups

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§1. Introduction

A representation $\pi$ of a totally disconnected group $G$ on a complex vector space $V$ is said to be smooth if for each $v \in V$ the mapping

$$x \mapsto \pi(x)v \quad (x \in G)$$

is locally constant. $\pi$ is called admissible if in addition the following condition is satisfied: For any open subgroup $K$ of $G$, the space of vectors $v \in V$ left fixed by $\pi(K)$ is finite-dimensional. An admissible representation is said to be pre-unitary if $V$ carries a $\pi(G)$-invariant scalar product.

These representations play an important role in the harmonic analysis on reductive $p$-adic groups [6]. The aim of this paper is to emphasize their importance in harmonic analysis on unipotent $p$-adic groups. Let $\mathcal{O}$ be a $p$-adic field of characteristic zero. $G$ will denote a connected unipotent algebraic group, defined over $\mathcal{O}$ and $G$ its subgroup of $\mathcal{O}$-rational points. Let $\mathfrak{g}$ be the Lie algebra of $G$ and $\mathfrak{g}$ its subalgebra of $\mathcal{O}$-points. $G$ is a totally disconnected group. We show:

(i) any irreducible smooth representation of $G$ is admissible,
(ii) any irreducible admissible representation of $G$ is pre-unitary.

Jacquet [7] has shown that (i) holds for reductive $p$-adic groups $G$. Actually, we make use of a remarkable lemma from [7]. The main tool for the proof of (i) and (ii) is the interference of so-called supercuspidal representations, which are known to play a decisive role in the representation theory of reductive groups [6]. We apply some results of Casselman concerning these representations [3], which originally were only stated for $GL(2)$. For the proof, which is by
induction on \( \dim G \), one has to go to the three-dimensional \( p \)-adic Heisenberg group. A new version of von Neumann's theorem ([11], Ch. 2) is needed to complete the induction. All this is to be found in sections 2, 3, 4 and 5.

Section 6 is concerned with the Kirillov construction of irreducible unitary representations of \( G \), which is standard now. In the next section we discuss the character formula, following Pukanszky [12]. As a byproduct we obtain a homogeneity property for the distribution, defined by a \( G \)-orbit \( O \) in \( \mathcal{G}' \): if \( \dim O = 2m \), then

\[
\int_O \phi(tv) \, dv = |t|^{-m} \int_O \phi(v) \, dv \quad (\phi \in C_c^\infty(\mathcal{G}'))
\]

for all \( t \in \Omega, \ t \neq 0 \). Similar results are true for nilpotent orbits of reductive \( G \) in \( \mathcal{G} \) [2]; there they form a substantial help in proving that the formal degrees of supercuspidal representations are integers, provided Haar measures are suitably normalized. Let \( Z \) denote the center of \( G \).

Section 8 deals with square-integrable representations mod \( Z \) of \( G \). Moore and Wolf [10] have discussed them for real unipotent groups. The main results still hold for \( p \)-adic groups.

Let \( \pi \) be an irreducible square-integrable representation mod \( Z \) of \( G \). For any open compact subgroup \( K \) of \( G \), let \( m(\pi, 1) \) denote the multiplicity of the trivial representation of \( K \) in the restriction of \( \pi \) to \( K \). Normalize Haar measures on \( G \) and \( Z \) in such a way that \( \text{vol}(K) = \text{vol}(K \cap Z) = 1 \). Choose Haar measure on \( G/Z \) accordingly. Then, according to a general theorem ([5], Theorem 2) one has:

\[
m(\pi, 1) \leq \frac{1}{d(\pi)}, \quad \text{where } d(\pi) \text{ is the formal degree of } \pi.
\]

Now assume in addition \( K \) to be a lattice subgroup of \( G \): \( L = \log K \) is a lattice in \( \mathcal{G} \). Moreover, let \( m(\pi, 1) > 0 \). Then we have equality:

\[
m(\pi, 1) = \frac{1}{d(\pi)}.
\]

This is proved in section 9.

In section 10 we relate our results to earlier work of C.C. Moore [9] on these multiplicities, involving numbers of \( K \)-orbits. We conclude with an example in section 11.
§2. Smooth representations

We call a Hausdorff space $X$ a totally disconnected (t.d.) space if it satisfies the following condition: Given a point $x \in X$ and a neighborhood $U$ of $x$ in $X$, there exists an open and compact subset $\omega$ of $X$ such that $x \in \omega \subseteq U$. Clearly a t.d. space is locally compact.

Let $X$ be a t.d. space and $S$ a set. A mapping $f: X \to S$ is said to be smooth if it is locally constant. Let $V$ be a complex vector space. We write $C_\infty^\omega(X, V)$ for the space of all smooth functions $f: X \to V$ and $C_\infty^\omega(X, V)$ for the subspace of those $f$ which have compact support. If $V = \mathbb{C}$ we simply write $C_\infty^\omega(X)$ and $C_\infty^\omega(X)$ respectively. One can identify $C_\infty^\omega(X, V)$ with $C_\infty^\omega(X) \otimes V$ by means of the mapping $i: C_\infty^\omega(X) \otimes V \to C_\infty^\omega(X, V)$ defined as follows: If $f \in C_\infty^\omega(X)$ and $v \in V$, then $i(f \otimes v)$ is the function $x \mapsto f(x)v$ ($x \in X$) from $X$ to $V$.

Let $G$ be a t.d. group, i.e. a topological group whose underlying space is a t.d. space. It is known that $G$ has arbitrarily small open compact subgroups. By a representation of $G$ on $V$, we mean a map $\pi: G \to \text{End}(V)$ such that $\pi(1) = 1$ and $\pi(xy) = \pi(x)\pi(y)$ ($x, y \in G$). A vector $v \in V$ is called $\pi$-smooth if the mapping $x \mapsto \pi(x)v$ of $G$ into $V$ is smooth.

Let $V_\infty$ be the subspace of all $\pi$-smooth vectors. Then $V_\infty$ is $\pi(G)$-stable. Let $\pi_\infty$ denote the restriction of $\pi$ on $V_\infty$. $\pi$ is said to be a smooth representation if $V = V_\infty$. Of course $\pi_\infty$ is always smooth.

We call a smooth representation $\pi$ on $V$ irreducible if $V$ has no non-trivial $\pi(G)$-invariant subspaces.

Let $\pi$ be a representation of $G$ on the complex vector space $V$. $\pi$ is called admissible if

(i) $\pi$ is smooth,

(ii) for any open subgroup $K$ of $G$, the space of vectors $v \in V$ which are left fixed by $\pi(K)$, is finite-dimensional.

An admissible representation $\pi$ of $G$ on $V$ is called pre-unitary if $V$ carries a $\pi(G)$-invariant scalar product. Let $\mathcal{H}$ be the completion of $V$ with respect to the norm, defined by the scalar product. Then $\pi$ extends to a continuous unitary representation $\rho$ of $G$ on $\mathcal{H}$ such that $V = \mathcal{H}_\infty$ and $\pi = \rho_\infty$. It is well-known that $\pi$ is irreducible if and only if $\rho$ is topologically irreducible. Note that $V$ is dense in $\mathcal{H}$.

Let $\pi$ be a smooth representation of $G$ on $V$ and $V'$ the (algebraic) dual of $V$. Then the dual representation $\pi'$ of $G$ on $V'$ is given by

$$\langle v, \pi'(x)\lambda \rangle = \langle \pi(x^{-1})v, \lambda \rangle \quad (x \in G, \lambda \in V', v \in V).$$
Put $\tilde{V} = (V')_\pi$ and $\tilde{\pi} = (\pi')_\sigma$. Then $\tilde{\pi}$ is a smooth representation which is called contragredient to $\pi$. It is easily checked that $\pi$ is admissible if and only if $\tilde{\pi}$ is.

Let $H$ be a closed subgroup of $G$ and $\sigma$ a smooth representation of $H$ on $W$. Then we define a smooth representation $\pi = \text{ind}_{H \backslash G} \sigma$ as follows: Let $V$ denote the space of all smooth functions $f : G \to W$ such that

1. $f(hx) = \sigma(h)f(x)$ \quad ($h \in H, x \in G$),
2. $\text{Supp} f$ is compact mod $H$.

Then $\pi$ is the representation of $G$ on $V$ given by

$$\pi(y)f(x) = f(xy) \quad (x, y \in G, f \in V).$$

Let $\pi_1$, $\pi_2$ be two smooth representations of $G$ on $V_1$ and $V_2$ respectively. We say that $\pi_1$ is equivalent to $\pi_2$ if there is a linear bijection $T : V_1 \to V_2$ such that $\pi_2(x)T = T\pi_1(x)$ for all $x \in G$.

§3. Smooth and admissible representations of the three-dimensional $p$-adic Heisenberg group

Let $\Omega$ be a $p$-adic field, i.e. a locally compact non-discrete field with a discrete valuation. There is an absolute value on $\Omega$, denoted $|\cdot|$, which we assume to be normalized in the following way. Let $dx$ be an additive Haar measure on $\Omega$. Then $d(ax) = |a|dx$ ($a \in \Omega^*$). Let $\mathcal{O}$ be the ring of integers: $\mathcal{O} = \{x \in \Omega : |x| \leq 1\}$; $\mathcal{O}$ is a local ring with unique maximal ideal $P$, given by $P = \{x \in \Omega : |x| < 1\}$. The residue-class field $\mathcal{O}/P$ has finitely many, say $q$, elements. $P$ is a principal ideal with generator $w$. So $P = \pi\mathcal{O}$, $|w| = q^{-1}$. Put $P^n = \pi^n\mathcal{O}$ ($n \in \mathbb{Z}$).

Since $P^n$ is a compact subgroup of the additive group of $\Omega$ and $\Omega = \bigcup_n P^n$, any additive character of $\Omega$ is unitary. Let $G = H_3$ be the 3-dimensional Heisenberg group over $\Omega$:

$$G = \left\{ [x, y, z] = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} ; \ x, y, z \in \Omega \right\}.$$ 

$G$ is a t.d. group. The group multiplication is given by:

$$[x, y, z][x', y', z'] = [x + x', y + y', z + z' + xy'].$$
THEOREM 1: (1) Each irreducible smooth representation \( \pi \) of \( H_3 \) is admissible; (2) Each irreducible admissible representation \( \pi \) of \( H_3 \) is pre-unitary.

We make use of the following result of Jacquet [7].

LEMMA 1: Let \( H \) be a group and \( \rho \) an (algebraically) irreducible representation of \( H \) on a complex vector space \( V \) of at most denumerable dimensions. Then every operator \( A \) which commutes with \( \rho(H) \) is a scalar.

Let \( V \) be the space of \( \pi \). Let \( v \in V \), \( v \neq 0 \) and \( K = \{ g \in G : \pi(g)v = v \} \). Then \( K \) is open and \( G/K \) is denumerable. Since \( V = \text{span}\{\pi(g)v : g \in G/K\} \), the lemma applies. \( Z = \{[0, 0, z] : z \in \Omega \} \) is the center of \( G \). Therefore, there exists an additive (unitary) character \( \psi_\pi \) of \( \Omega \) such that \( \pi([0, 0, z]) = \psi_\pi(z)I \) \((z \in \Omega)\), where \( I \) is the identity in \( \text{End}(V) \). We have two cases:

(a) \( \psi_\pi = 1 \). Then \( \pi \) actually is a representation of \( G/Z = \Omega^2 \) which is (again by the lemma) one-dimensional and, as observed above, unitary.

(b) \( \psi_\pi \neq 1 \). Fix \( w \in V \), \( w \neq 0 \). For any \( v \in V \), put \( c_v(g) = \langle \pi(g)v, w \rangle \) \((g \in G)\). The mapping \( v \mapsto c_v \) is a linear injection of \( V \) into the space of smooth functions \( f \) on \( G \), satisfying

\[
f([x, y, z]) = \psi_\pi(z)f([x, y, 0]).
\]

Let \( K \) be a (small) open compact subgroup of \( G \) such that \( \pi(k)w = w \) for all \( k \in K \). Call \( V_K = \{ v \in V : \pi(k)v = v \text{ for all } k \in K \} \). Then \( f = c_v \) satisfies

\[
f(gk) = f(gk) = f(g) \quad (g \in G; \ k \in K)
\]

for all \( v \in V_K \).

Write \( g = [x, y, 0] \), \( k = [x', y', 0] \). Then

\[
f([x, y, 0]) = f([x + x', y + y', xy']) = f([x + x', y + y', x'y]).
\]

Hence

\[
f([x, y, 0]) = f([x + x', y, 0]) = f([x, y + y', 0]).
\]

Therefore \( f([x + x', y + y', 0]) = f([x, y, 0]) \) for all \( x, y \in \Omega \) and \( x', y' \).
small (only depending on $K$, not on the particular choice of $v \in V_K$). Moreover:

$$f([x, y, 0]) = f([x, y, 0])\psi_\pi(xy') = f([x, y, 0])\psi_\pi(x'y)$$

for $x'$, $y'$ as above. Since $\psi_\pi \neq 1$, $f([x, y, 0]) = 0$ for $x$ or $y$ large enough (only depending on $K$, not on the particular choice of $v \in V_K$). Since $f([x, y, z]) = \psi_\pi(z)f([x, y, 0])$, $f$ is completely determined by the values $f([x, y, 0])$, $(x, y \in \Omega)$. Consequently, $\dim V_K = \dim\{c_v : v \in V_K\} < \infty$. Part (1) of the theorem is now evident. To prove part (2) it suffices to take the following scalar product on $V$:

$$(v, v') = \int_\Omega \int_\Omega c_v([x, y, 0])\overline{c_v([x, y, 0])} \, dx \, dy \quad (v, v' \in V).$$

**Remark:** It is clear that the same observations remain true for the higher dimensional $p$-adic Heisenberg groups.

### §4. Supercuspidal representations

$G$ is a t.d. group and $\pi$ a smooth representation of $G$ on $V$. By a matrix coefficient of $\pi$, we mean a function on $G$ of the form

$$x \mapsto \langle \pi(x)v, \tilde{v} \rangle \quad (x \in G)$$

where $v$ and $\tilde{v}$ are fixed elements in $V$ and $\tilde{V}$ respectively. Let $Z$ denote the center of $G$. We call $\pi$ a supercuspidal representation if each matrix coefficient of $\pi$ has compact support modulo $Z$. The proof of Theorem 1 emphasizes the significance of this kind of representations. Actually, one has the following lemma.

**Lemma 2:** Let $\pi$ be a smooth representation of $H_3$ such that $\pi([0, 0, z]) = \psi_\pi(z)I$ $(z \in \Omega)$ for some non-trivial additive character $\psi_\pi$ of $\Omega$. Then $\pi$ is a supercuspidal representation.

Assume, from now on, $G$ to satisfy the second axiom of countability. Let $\pi$ be an irreducible smooth representation of $G$ on $V$. Then by Lemma 1, there is a character $\lambda_\pi$ of $Z$ such that $\pi(z) = \lambda_\pi(z)I$ $(z \in Z)$.

**Lemma 3:** Let $\pi$ be an irreducible, admissible and supercuspidal representation of $G$ on $V$. Assume $\lambda_\pi$ unitary. Then $\pi$ is pre-unitary.
and one has the following orthogonality relations: There exists a positive constant \( d_\pi \) (the formal degree of \( \pi \)), only depending on the choice of Haar measure \( d\hat{g} \) on \( G/Z \) such that

\[
\int_{G/Z} \langle \pi(g)u, \tilde{u} \rangle \langle \pi(g^{-1})v, \tilde{v} \rangle \, d\hat{g} = d_\pi^{-1} \langle u, \tilde{u} \rangle \langle v, \tilde{v} \rangle
\]

for all \( u, v \in V \), \( \tilde{u}, \tilde{v} \in \tilde{V} \).

To make \( \pi \) pre-unitary, choose any \( w \in \tilde{V} \), \( w \neq 0 \) and define the following \( G \)-invariant scalar product on \( V \):

\[
(v, v') = \int_{G/Z} \langle \pi(g)v, w \rangle \overline{\langle \pi(g)v', w \rangle} \, d\hat{g}.
\]

\( \pi \) extends to an irreducible unitary representation on the completion \( \mathcal{H} \) of \( V \) such that \( \mathcal{H}_\pi = V \). The orthogonality relations now follow easily from those for irreducible unitary supercuspidal representations ([5], Theorem 1).

The following theorem is due to Casselman ([3], Theorem 1.6).

**Theorem 2:** Let \( \rho \) be an irreducible, admissible and supercuspidal representation of \( G \) on \( W \) such that \( \rho(z) = \lambda(z)I \) \((z \in Z)\), where \( \lambda \) is a unitary character of \( Z \). Let \( \pi \) be any smooth representation of \( G \) on \( V \) such that \( \pi(z) = \lambda(z)I \) \((z \in Z)\). Given a \( G \)-morphism \( f \neq 0 \) from \( \pi \) to \( \rho \), there exists a \( G \)-morphism splitting \( f \).

**Proof:** Let \( S_\lambda(G) \) denote the space of smooth functions \( h \) on \( G \) with compact support mod \( Z \) such that \( h(xz) = h(x)\lambda(z^{-1}) \) \((x \in G, z \in Z)\). \( S_\lambda(G) \) is a \( G \)-module, \( G \) acting by left translation. Fix \( \tilde{w}_0 \in \tilde{W} \), \( \tilde{\tilde{w}}_0 \neq 0 \). The mapping \( F: W \to S_\lambda(G) \), defined by

\[
F(w)(x) = \langle \rho(x^{-1})w, \tilde{w}_0 \rangle \quad (w \in W, x \in G)
\]

is a \( G \)-morphism. Choose \( w_0 \in W \) and \( v_0 \in V \) such that \( \langle w_0, \tilde{\tilde{w}}_0 \rangle = d_\rho \), \( f(v_0) = w_0 \). By \( P \) we denote the \( G \)-morphism from \( S_\lambda(G) \) to \( V \) given by

\[
P(h) = \int_{G/Z} h(x)\pi(x)v_0 \, d\hat{x} \quad (h \in S_\lambda(G)).
\]

Then \( P \circ F \) is the \( G \)-morphism, splitting \( f \).
\[ \langle f \circ P \circ F(w), \tilde{w} \rangle = \int_{G/\mathbb{Z}} \langle \rho(x^{-1})w, \tilde{w}_0 \rangle (f(x)v_0, \tilde{w}) \, d\tilde{x} \]
\[ = \int_{G/\mathbb{Z}} \langle \rho(x^{-1})w, \tilde{w}_0 \rangle (\rho(x)w_0, \tilde{w}) \, d\tilde{x} \]
\[ = d_{\rho}^{-1}(w_0, \tilde{w}_0) \langle w, \tilde{w} \rangle \quad \text{(by Lemma 3)} \]
\[ = \langle w, \tilde{w} \rangle \quad \text{for all } \tilde{w} \in \tilde{W}. \]

Hence \( f \circ P \circ F(w) = w \) for all \( w \in W \).

Let us now turn back to \( H_3 \). The irreducible unitary representations of \( H_3 \) are well-known (cf. [11]). Their restrictions to the space of smooth vectors are admissible. Keeping in mind Theorem 1, we have therefore the following list of irreducible admissible representations of \( H_3 \). Let \( \chi_0 \) denote any non-trivial additive character of \( \Omega \). Then:

(a) One-dimensional representations \( \rho_{\mu,\nu} \, (\mu, \nu \in \Omega) \), trivial on \( \mathbb{Z} \);
\[ \rho_{\mu,\nu} ((x, y, z)) = \chi_0(\mu x + \nu y). \]

(b) Supercuspidal representations \( \rho_\lambda \, (\lambda \in \Omega^*) \), non-trivial on \( \mathbb{Z} \), on the space \( C_c^\infty(\Omega) \);
\[ \rho_\lambda ((x, y, z)) f(t) = \chi_0(\lambda(z + ty)) f(t + x) \quad (f \in C_c^\infty(\Omega)). \]

We have the following analogue of the famous theorem of von Neumann for \( H_3 \) ([11], Ch. 2).

**Theorem 3:** Let \( \pi \) be a smooth representation of \( H_3 \) such that \( \pi((0, 0, z)) = \chi_0(\lambda z)I \quad (z \in \Omega) \) for some \( \lambda \neq 0 \). Then \( \pi \) is the (algebraic) direct sum of irreducible representations equivalent to \( \rho_\lambda \).

**Proof:** Let \( V \) be the space of \( \pi \). Due to Theorem 1, every irreducible subrepresentation of \( \pi \) is equivalent to \( \rho_\lambda \). By Lemma 2, \( \pi \) is a supercuspidal representation. We shall prove the following: Given any \( G \)-invariant subspace \( W \) of \( V \), \( W \neq V \), there exists an irreducible subspace \( U \) of \( V \) such that \( U \cap W = (0) \). An easy application of Zorn’s Lemma then yields the theorem.

Let \( W \) be a proper \( G \)-invariant subspace of \( V \). Put \( \tilde{V} = V/W \). \( \tilde{V} \) is a \( G \)-module; the action of \( G \) is a smooth and supercuspidal representation of \( G \). Let \( \tilde{v}_0 \in \tilde{V} \), \( \tilde{v}_0 \neq 0 \). The \( G \)-module \( \tilde{V}_0 \) generated by \( \tilde{v}_0 \) contains a maximal proper \( G \)-module. Therefore \( \tilde{V}_0 \) has an irreducible quotient, which is also supercuspidal, and admissible by Theorem 1. By Theorem 2, \( \tilde{V}_0 \) and hence \( \tilde{V} \), even has an irreducible subspace, say \( \tilde{V}_1 \), on which \( G \) acts as an admissible, supercuspidal representation. Let \( V_1/W \) be its pre-image in \( V \). Then \( V_1/W \) is a \( G \)-invariant subspace of \( V \) and the canonical map from \( V \) to \( \tilde{V} \) induces a
non-zero $G$-morphism from $V_1 + W$ to $\tilde{V}_1$. Again Theorem 2 implies the existence of an irreducible subspace $U$ of $V$ such that $U \cap W = (0)$, $U + W = V_1 + W$. This concludes the proof of Theorem 3.

§5. Smooth and admissible representations of unipotent $p$-adic groups

Let $\Omega$ be a $p$-adic field of characteristic zero. By $G$ we mean a connected algebraic group, defined over $\Omega$, consisting of unipotent elements, with Lie algebra $\mathfrak{g}$. Let $G, G$ be the sets of $\Omega$-points of $G, G$ respectively. We have the $\Omega$-isomorphism of algebraic varieties $\exp: G \to G$, which map $G$ onto $G$. Let ‘log’ denote its inverse. We shall call $G$ a unipotent $p$-adic group and say that $G$ is its Lie algebra.

Let $Z$ be the center of $G$, its Lie algebra $\mathfrak{z}$. One has $\exp \mathfrak{z} = Z$. More generally: the exponential of a subalgebra of $G$ is a unipotent $p$-adic subgroup of $G$, the exponential of an ideal in $G$ is a normal subgroup of $G$.

Let $G$ be a unipotent $p$-adic group.

THEOREM 4: Each irreducible smooth representation $\pi$ of $G$ is admissible and pre-unitary.

PROOF: We use induction on $\dim G$. Lemma 1 is the main source to prove the theorem in case $\dim G = 1$. Assume $\dim G > 1$. Fix any non-trivial character $\chi_0$ of $\Omega$. By Lemma 1 there exists a (unitary) character $\lambda\pi$ of $Z$ such that $\pi(z) = \lambda\pi(z)I$ for all $z \in Z$. $\lambda\pi \circ \exp$ is an additive character of $\mathfrak{z}$, hence $\lambda\pi \circ \exp = \chi_0 \circ f$ for some $f \in \mathfrak{z}'$. $\ker(f)$ is a subalgebra of $\mathfrak{z}$, $\exp(\ker f) = \ker(\lambda\pi)$ therefore a unipotent $p$-adic subgroup of $Z$ of codimension at most one. If $\dim Z > 1$ or $\dim Z = 1$ and $\lambda\pi = 1$, $\pi$ actually reduces to an irreducible representation $\pi_0$ of $G_0 = G/\ker \lambda\pi$. But $\dim G_0 < \dim G$. The theorem follows from the induction hypotheses.

It remains to consider the case: $\dim Z = 1$ and $\lambda\pi \neq 1$. We will first show the existence of a unipotent $p$-adic subgroup $G_1$ of codimension one in $G$ and an irreducible smooth representation $\pi_1$ of $G_1$ such that $\pi$ is equivalent to $\ind_{G_1 \uparrow \downarrow G} \pi_1$.

Let $Y_0 \in G$ be such that, $[Y_0, G] \supset \mathfrak{z}$, $Y_0 \notin \mathfrak{z}$. Put $G_1 = \{ U : [U, Y_0] = 0 \}$. $G_1$ is an ideal in $G$ of codimension 1. Choose $X_0 \notin G_1$ and define $Z_0 = [X_0, Y_0]$. Observe $Z_0 \in \mathfrak{z}$, $Z_0 \neq 0$. Then $\{ X_0, Y_0, Z_0 \}$ is a basis for a 3-dimensional subalgebra of $G$ isomorphic to the Lie algebra of $H_3$. Let $S$ denote the subgroup of $G$ corresponding to this subalgebra and write, as usual,

\[ [x, y, z] = \exp yY_0 \cdot \exp xX_0 \cdot \exp zZ_0 \quad (x, y, z \in \Omega) \]
We can choose $\lambda \in \Omega$, $\lambda \neq 0$ with the following property:

$$\lambda_s([0, 0, z]) = \chi_0(\lambda z) \quad (z \in \Omega).$$

Let us assume, for the moment, that $\pi$ is an irreducible smooth representation of $G$ on $V$. By Theorem 3, the restriction of $\pi$ to $S$ is a direct sum of irreducible representations of $S$, all equivalent to the representation $\rho_s$ of $S$ in $C_c^\infty(\Omega)$ given by

$$\rho_s([x, y, z])f(t) = \chi_0(\lambda(z + ty))f(t + x) \quad (f \in C_c^\infty(\Omega)).$$

So $V = \bigoplus_{i \in I} V_i^\lambda$ for some index-set $I$, each $V_i^\lambda$ being isomorphic to $C_c^\infty(\Omega)$. We may regard $I$ as a t.d. space in the obvious way. Then we have

$$V = C_c^\infty(I, C_c^\infty(\Omega)) = C_c^\infty(I) \otimes C_c^\infty(\Omega) = C_c^\infty(\Omega, W),$$

where $W = C_c^\infty(I)$. Moreover, with these identifications,

$$\pi([x, y, z])f(t) = \chi_0(\lambda(z + ty))f(t + x) \quad (f \in C_c^\infty(\Omega, W)).$$

Let $G_1$ denote the unipotent $p$-adic subgroup of $G$ with Lie algebra $\mathfrak{g}_1$. $G_1$ is a closed normal subgroup of $G$ and $G = G_1 \cdot (\exp tX_0)_{t \in \Omega}$ (semi-direct product). Since $Y_0$ is in the center of $\mathfrak{g}_1$, $\pi(G_1)$ and $\pi(\exp yY_0)$ ($y \in \Omega$) commute. Recall

$$\pi(\exp yY_0)f(t) = \chi_0(\lambda ty)f(t) \quad (y, t \in \Omega; f \in C_c^\infty(\Omega, W)).$$

Our aim now is to prove the following lemma.

**Lemma 4:** For each $t \in \Omega$, there exists a smooth representation $g_1 \mapsto \pi(g_1, t)$ of $G_1$ on $W$ such that

(a) $(\pi(g_1)f)(t) = \pi(g_1, t) \cdot f(t)$ for all $f \in C_c^\infty(\Omega, W)$, $g_1 \in G_1$ and $t \in \Omega$;

(b) $\pi(g_1, t + t_0) = \pi(\exp t_0X_0 \cdot g_1 \cdot \exp(-t_0X_0), t)$ for all $t, t_0 \in \Omega$, $g_1 \in G_1$.

Obviously, this lemma implies $\pi = \text{ind}_{G_1 \uparrow G} \pi_1$ where $\pi_1$ is given by $\pi_1(g_1) = \pi(g_1, 0)$ ($g_1 \in G_1$). The irreducibility of $\pi$ yields the irreducibility of $\pi_1$.

To prove the lemma, we start with a linear map $A : C_c^\infty(\Omega, W) \rightarrow$
commuting with all operators \( \pi(\exp y_0) \) \((y \in \Omega)\). Thus:

\[
\{A(\chi_0(y \cdot) f(\cdot))(t) = \chi_0(ty)(Af)(t)
\]

for all \( t, y \in \Omega \) and \( f \in C_c^\omega(\Omega, W) \).

Since \( C_c^\omega(\Omega) \) is closed under Fourier transformation, we can easily establish the following: Given \( \phi \in C_c^\omega(\Omega) \) and an open compact subset \( K \) of \( \Omega \), there exists an integer \( m > 0 \), \( \lambda_1, \ldots, \lambda_m \in \mathbb{C} \) and \( y_1, \ldots, y_m \in \Omega \) such that

\[
\phi(t) = \sum_{i=1}^{m} \lambda_i \chi_0(y_i t) \quad (t \in K).
\]

For \( \phi \in C_c^\omega(\Omega) \) let \( L_\phi \) denote the linear map \( C_c^\omega(\Omega, W) \to C_c^\omega(\Omega, W) \) given by \( L_\phi(f)(t) = \phi(t)f(t) \) \((f \in C_c^\omega(\Omega, W))\). Then, putting \( K = \text{Supp} f \cup \text{Supp} Af \), we obtain:

\[
\{A(L_\phi f)(t) = A \left( \sum_{i=1}^{m} \lambda_i \chi_0(y_i t)f(\cdot) \right)(t)
\]

\[
= \sum_{i=1}^{m} \lambda_i \chi_0(y_i t)Af(t) = \{L_\phi(Af)(t) = \{L_\phi(Af)(t)
\]

\((t \in \Omega, f \in C_c^\omega(\Omega, W))\). Hence \( AL_\phi = L_\phi A \) for every \( \phi \in C_c^\omega(\Omega) \). In particular we have: \( \pi(g_1)L_\phi = L_\phi \pi(g_1) \) for all \( g_1 \in G_1, \phi \in C_c^\omega(\Omega) \). Let \( \psi_n \) denote the characteristic function of \( P^n \). In addition, put \( L_n \phi(s) = \phi(s - t) \) \((s, t \in \Omega, \phi \) any function on \( \Omega)\). Define:

\[
\pi(g_1, t)w = \pi(g_1)(L_n \psi_n \otimes w)(t) \quad (g_1 \in G_1, t \in \Omega, w \in W).
\]

Here, as usual, \( L_n \psi_n \otimes w \) is identified with the function \( s \to L_n \psi_n(s) \cdot w \) \((s \in \Omega)\). \( \pi(g_1, t) \) is well-defined: assuming \( n' \leq n \), we obtain

\[
\pi(g_1)(L_n \psi_n \otimes w)(t) = \pi(g_1)(L_n \psi_{n'} \cdot L_n \psi_n \otimes w)(t).
\]

But this equals, by the above result,

\[
L_n \psi_n(t)\pi(g_1)(L_n \psi_{n'} \otimes w)(t) = \pi(g_1)(L_n \psi_{n'} \otimes w)(t).
\]

Let us show now that \( \pi(g_1, t) \) satisfies condition (a) of Lemma 4. Fix \( f \in C_c^\omega(\Omega, W) \) and determine integers \( m, n > 0 \), \( t_1, \ldots, t_m \in \Omega \) and
\( w_1, \ldots, w_m \in W \) such that

\[
f = \sum_{i=1}^{m} L_{\psi_n} \otimes w_i.
\]

Then

\[
\pi(g_1)f(t) = \pi(g_1) \left( \sum_{i=1}^{m} L_{\psi_n} \otimes w_i \right)(t)
\]

\[
= \sum_{i=1}^{m} \pi(g_1)(L_{\psi_n} \otimes w_i)(t)
\]

\[
= \sum_{i=1}^{m} \{ L_{\psi_n} \cdot \pi(g_1)(L_{\psi_n} \otimes w_i) \}(t)
\]

\[
= \sum_{i=1}^{m} \pi(g_1)(L_{\psi_n} \cdot L_{\psi_n} \otimes w_i)(t)
\]

\[
= \sum_{i=1}^{m} \{ L_{\psi_n} \cdot \pi(g_1)(L_{\psi_n} \otimes w_i) \}(t)
\]

\[
= \sum_{i=1}^{m} L_{\psi_n}(t) \cdot \pi(g_1, t)w_i
\]

\[
= \pi(g_1, t) \cdot f(t) \quad (t \in \Omega, g_1 \in G_i).
\]

Condition (b) is also fulfilled. Indeed,

\[
\pi(\exp t_0X \cdot g_1 \cdot \exp -t_0X_0, t)w
\]

\[
= \pi(\exp t_0X_0)\pi(g_1)\pi(\exp -t_0X_0)(L_{\psi_n} \otimes w)(t)
\]

\[
= \pi(g_1)\pi(\exp -t_0X_0)(L_{\psi_n} \otimes w)(t + t_0).
\]

Furthermore,

\[
\pi(\exp -t_0X_0)(L_{\psi_n} \otimes w)(u) = L_{\psi_n} \otimes w(u - t_0)
\]

\[
= L_{t+t_0\psi_n} \otimes w(u) \quad (u \in \Omega).
\]

Hence,

\[
\pi(\exp t_0X \cdot g_1 \cdot \exp -t_0X_0, t)w
\]

\[
= \pi(g_1)(L_{t+t_0\psi_n} \otimes w)(t + t_0) = \pi(g_1, t + t_0)w.
\]

Finally, it is easily checked, that condition (a) forces \( g_1 \mapsto \pi(g_1, t) \) \((g_1 \in G_i)\) to be a smooth representation of \( G_i \) for each \( t \in \Omega \). This concludes the proof of Lemma 4.
**Corollary:** Each irreducible smooth representation of $G$ is monomial.

Let us continue the proof of Theorem 4. By induction we assume that $\pi_1$ is admissible and pre-unitary. Hence $\pi = \text{ind}_{G_1 \times G} \pi_1$ is pre-unitary. Let $K$ be an open subgroup of $G$ and let $V_K$ denote the space of all $f \in C_c^\infty(\Omega)$ such that $\pi(g)f = f$ for all $g \in K$. Let $f \in V_K$. Since

$$\pi(\exp xX_0)f(t) = f(x + t) \quad (x, t \in \Omega),$$

there exists an integer $n > 0$, only depending on $K$, such that $f$ is constant on cosets of $P^n$.

The relation

$$\pi(\exp yY_0)f(t) = \chi_0(\lambda y)f(t) \quad (y, t \in \Omega)$$

implies that $\text{Supp} f \subset P^m$ for some integer $m > 0$, only depending on $K$. Assume $m < n$. Then $P^m = \bigcup_{i=1}^k (t_i + P^n)$ for some $t_1, \ldots, t_k \in \Omega$. Now consider the mapping

$$f \mapsto (f(t_1), \ldots, f(t_k))$$

of $V_K$ into $W^k$. This mapping is linear and injective. Since

$$(\pi(g_1)f(t) = \pi_1(\exp tX_0 \cdot g_1 \cdot \exp -tX_0)f(t) \quad (g_1 \in G_1, t \in \Omega)$$

we obtain that $f(t_i)$ is fixed by $\exp t_i X_0 \cdot (K \cap G_i) \exp(-t_i X_0)$, being an open subgroup of $G_i$ ($i = 1, 2, \ldots, k$). Therefore, each $f(t_i)$ stays in a finite-dimensional subspace of $W$. Consequently $\dim V_K < \infty$.

We have shown that $\pi$ is admissible. This concludes the proof of Theorem 4.

**Remark:** Similar to the proof of Theorem 4 one can easily show that the restriction of an irreducible unitary representation of $G$ to its subspace of smooth vectors is an admissible representation of $G$.

§6. Kirillov’s theory

Let $G$ be as in §5. What remains is to describe the irreducible unitary representations of $G$. This is done by Kirillov [8] for the real groups $G$ and, as observed by Moore [9], the whole machinery works
in the $p$-adic case as well. For completeness and for later purposes, we give the result.

Given $f \in \mathscr{G}$, put $B_f(X, Y) = f([X, Y])$ ($X, Y \in \mathscr{G}$). $B_f$ is an alternating bilinear form on $\mathscr{G}$. A subalgebra $\mathfrak{S}$ of $\mathscr{G}$ which is at the same time a maximal totally isotropic subspace for $B_f$ is called a polarization at $f$. Polarizations at $f$ exist ([4], 1.12.10). They coincide with the subalgebra's $\mathfrak{S} \subset \mathscr{G}$ which are maximal with respect to the property that $\mathfrak{S}$ is a totally isotropic subspace for $B_f$ (cf. [8], Lemma 5.2, which carries over to the $p$-adic case with absolutely no change). Let $\mathfrak{S}$ be any subalgebra of $\mathscr{G}$ which is a totally isotropic subspace for $B_f : f \in \mathfrak{S}$, $\mathfrak{S} = 0$. Put $H = \exp \mathfrak{S}$. We may define a character $\chi_f$ of $H$ by the formula:

$$\chi_f(\exp X) = \chi_0(f(X)) \quad (X \in \mathfrak{S}).$$

Let $\rho(f, \mathfrak{S}, G)$ denote the unitary representation of $G$ induced by $\chi_f$.

**Theorem 5 ([8], [9]):**

(i) $\rho(f, \mathfrak{S}, G)$ is irreducible if and only if $\mathfrak{S}$ is a polarization at $f$,

(ii) each irreducible unitary representation of $G$ is of the form $\rho(f, \mathfrak{S}, G)$,

(iii) $\rho(f_1, \mathfrak{S}_1, G)$ and $\rho(f_2, \mathfrak{S}_2, G)$ are unitarily equivalent if and only if $f_1$ and $f_2$ are in the same $G$-orbit in $\mathscr{G}$.

§7. The character formula

The main reference for this section is [12]. $G$ acts on $\mathscr{G}$ by $Ad$ and hence on $\mathscr{G}$ by the contragredient representation. It is well-known (and can be proved similar to the real case) that all $G$-orbits in $\mathscr{G}$ are closed.

Let us fix a non-trivial (unitary) character $\chi_0$ of the additive group of $\Omega$.

We shall choose a Haar measure $dg$ on $G$ and a translation invariant measure $dX$ on $\mathscr{G}$ such that $dg = \exp(dX)$.

Let $f \in \mathscr{G}$, $\mathfrak{S}$ a polarization at $f$ and $O$ the orbit of $f$ in $\mathscr{G}$. Put $\pi = \rho(f, \mathfrak{S}, G)$. Given $\psi \in C_c^\infty(G)$, we know that $\pi(\psi)$ is an operator of finite rank (§5, Remark). Put $\psi_f(X) = \psi(\exp X)$ ($X \in \mathscr{G}$). Then $\psi_f \in C_c^\infty(\mathscr{G})$. The Fourier transform of $\psi_f$ is defined by:

$\chi_0$ is (as usual) a fixed non-trivial additive character of $\Omega$.\footnote{Here $\chi_0$ is (as usual) a fixed non-trivial additive character of $\Omega$.}
\[ \hat{\psi}_t(X') = \int_{\mathcal{G}} \psi_1(X) \chi_0((X, X')) \, dX \quad (X' \in \mathcal{G'}). \]

Observe that \( \hat{\psi}_t \in C_c^\infty(\mathcal{G}'). \)

**Theorem 6:** There exists a unique positive \( G \)-invariant measure \( dv \) on \( O \) such that for all \( \psi \in C_c^\infty(G) \):

\[ \operatorname{tr} \pi(\psi) = \int_O \hat{\psi}_t(v) \, dv. \]

Note that the right-hand side is finite, because \( dv \) is also a measure on \( \mathcal{G}' \), since \( O \) is closed in \( \mathcal{G}' \).

Pukanszky’s proof of ([12], Lemma 2),\(^2\) goes over to our situation with no substantial change. Observe that each \( \psi \in C_c^\infty(G) \) is a linear combination of functions of the form \( \phi \ast \hat{\phi} \) (\( \phi \in C_c^\infty(G) \)) where \( \hat{\phi} \) is given by \( \hat{\phi}(g) = \phi(g^{-1}) \) (\( g \in G \)). The algorithm to determine \( dv \) (given \( dg \) and \( dX \) such that \( dg = \exp(dX) \)) is similar to that given by Pukanszky:

1. Put \( K = \exp \mathcal{H}, \Gamma = K \backslash G \). Choose invariant measures \( dk \) and \( d\gamma \) on \( K \) and \( \Gamma \) respectively such that \( dg = dk \, d\gamma \).
2. Choose a translation invariant measure \( dH \) on \( \mathcal{H} \) such that \( dk = \exp(dH) \).
3. Let \( dX' \) and \( dH' \) denote the dual measures of \( dX \) and \( dH \) respectively.
4. Let \( \mathcal{H}^\perp = \{ X' \in \mathcal{G}': \langle \mathcal{H}, X' \rangle = 0 \} \). Take \( dH^\perp \) on \( \mathcal{H}^\perp \) such that \( dX' = dH' \cdot dH^\perp \).
5. Let \( S \) be the stabilizer of \( f \) in \( G \). Then \( S \subset K \). Choose \( d\lambda \) on \( S \backslash K \) such that \( d\lambda \) is the inverse-image of \( dH^\perp \) under the bijection

\[ Sk \mapsto k^{-1} \cdot f \quad (k \in K) \]

of \( S \backslash K \) onto \( f + H^\perp \).
6. Finally, put \( dv = \text{image of } d\lambda \, d\gamma \) under the bijective mapping \( Sg \mapsto g^{-1} \cdot f \) (\( g \in G \)) of \( S \backslash G \) onto \( O \).

The invariant measure \( dv \) depends on the choice of the character \( \chi_0 \).

Taking instead of \( \chi_0 \) the character \( x \mapsto \chi_0(tx) \) for some \( t \in \Omega, t \neq 0 \), we obtain, by applying the above algorithm, the following homogeneity

\(^2\) Part (d) of his proof has to be omitted here.
property for \( dv \):

**COROLLARY:** Let \( O \) be a \( G \)-orbit in \( \mathcal{G}' \) of dimension \( 2m \). Then

\[
\int_{O} \phi(tv) \, dv = |t|^{-m} \int_{O} \phi(v) \, dv
\]

for all \( \phi \in C_c^{\infty}(\mathcal{G}') \) and all \( t \in \Omega, t \neq 0 \).

Observe that we may choose in the corollary \( dv \) to be any \( G \)-invariant positive measure on \( O \).

Let \( O \) be as above. \( O \) carries a canonical measure \( \mu \), which is constructed as follows. For any \( p \in O \), define \( \alpha_p : G \to O \) by \( \alpha_p(a) = a \cdot p \) (\( a \in G \)). The kernel of the differential \( \beta_p \) of \( \alpha_p \), \( \beta_p : \mathcal{G} \to T_p \) (\( T_p = \) tangent space to \( O \) in \( p \)) coincides with the radical of the alternating bilinear form \( B_p \) on \( \mathcal{G} \). Let \( \text{Stab}_G(p) \) be the stabilizer of \( p \) in \( G \). Then also, \( \ker \beta_p = \text{Lie algebra of Stab}_G(p) \). Hence \( B_p \) induces a non-degenerate alternating bilinear form \( \omega_p \) on \( T_p \). In this way a 2-form \( \omega \) is defined on \( O \). One easily checks that \( \omega \) is \( G \)-invariant (cf. [12] for the real case). Let \( d = 2m \) be the dimension of \( O \). Assume \( d > 0 \). Then \( \mu \) is given by \( \mu = |(1/2^m m!) A^m \omega| \).

**THEOREM 7:** Let us fix the character \( \chi_0 \) of \( O \) in such a way that \( \chi_0 = 1 \) on \( O \), \( \chi_0 \neq 1 \) on \( P^{-1} \). Let \( O \) be any \( G \)-orbit in \( \mathcal{G}' \) of positive dimension. Then the invariant measure \( dv \) and the canonical measure \( \mu \) on \( O \) coincide.

The proof is essentially the same as in the real case ([12], Theorem).

§8. Square-integrable representations mod \( Z \)

Let \( G \) and \( Z \) be as in §5. An irreducible unitary representation \( \pi \) of \( G \) on \( \mathcal{H} \) is called square-integrable mod \( Z \) if there exist \( \xi, \eta \in \mathcal{H} - (0) \) such that

\[
\int_{G/Z} |\langle \pi(x)\xi, \eta \rangle|^2 \, dx < \infty.
\]

Such representations are extensively discussed by C.C. Moore and J. Wolf for real unipotent groups [10]. For \( p \)-adic unipotent groups, see [13]: the restriction of \( \pi \) to the space \( \mathcal{H}_\infty \) of \( \pi \)-smooth vectors is a
supercuspidal representation. Our main goal is to find a closed formula for the multiplicity of the trivial representation of well-chosen open and compact subgroups $K$ of $G$ in the restriction of $\pi$ to $K$.

Let $f \in \mathcal{G}$. By $O_f$ we denote the $G$-orbit of $f$ in $\mathcal{G}$ and by $\pi_f$ an irreducible unitary representation of $G$, corresponding to $f$ (more precisely: to $O_f$) by Kirillov's theory (§6). Let $\mathcal{H}_f$ denote the space of $\pi_f$. Then we have, similar to ([10], Theorem 1):

**Theorem 8:** The following four statements are equivalent:

(i) $\pi_f$ is square-integrable mod $Z$,

(ii) $\dim O_f = \dim G/Z$,

(iii) $O_f = f + \mathcal{X}$,

(iv) $B_f$ is a non-degenerate bilinear form on $\mathcal{G}/\mathcal{X}$.

Here $\mathcal{X} = \{X' \in \mathcal{G}; (X', \mathcal{X}) = 0\}$.

Now assume $\pi_f$ to be square-integrable mod $Z$. The orbit $O_f$ carries the canonical measure $\mu$. We shall define another $G$-invariant measure $\nu$ on $O_f$. Let us fix a $G$-invariant differential form $\omega$ on $\mathcal{G}/\mathcal{X}$ of maximal degree. Let $\sigma$ denote the adjoint representation of $G$ on $\mathcal{G}$ and let $\rho$ be the representation of $G$ contragredient to $\sigma$. Fix $p \in O_f$. We have $\text{Stab}_G(p) = Z$ and $g \mapsto \rho(g)h$ is an isomorphism of $G/Z$ onto $O_f$. Call $\beta_p$ the differential of this map at $e$; $\beta_p : \mathcal{G}/\mathcal{X} \to T_h$. Define

$$\omega_p(\beta_p(X_1), \ldots, \beta_p(X_n)) = \omega(X_1, \ldots, X_n)$$

$$\quad (n = \dim \mathcal{G}/\mathcal{X}; X_1, \ldots, X_n \in \mathcal{G}/\mathcal{X}).$$

In this way we get a $n$-form $\omega'$ on $O_f$. We claim that $\omega'$ is $G$-invariant:

$$\omega_p(\beta_p(X_1), \ldots, \beta_p(X_n)) = \omega_q(d\rho_p(a)\beta_p(X_1), \ldots, d\rho_p(a)\beta_p(X_n))$$

if $p, q \in O_f$, $q = \rho(a)p$ (\(X_1, \ldots, X_n \in \mathcal{G}/\mathcal{X}\)). This is a simple exercise:

$$\omega_q(d\rho_p(a)\beta_p(X_1), \ldots, d\rho_p(a)\beta_p(X_n)) = \omega_q(\beta_q(\sigma(a)X_1), \ldots, \beta_q(\sigma(a)X_n))$$

$$= \omega(\sigma(a)X_1, \ldots, \sigma(a)X_n) = \omega(X_1, \ldots, X_n) = \omega_p(\beta_p(X_1), \ldots, \beta_p(X_n)).$$

Call $\nu$ the measure on $O_f$ corresponding to $\omega'$; $\nu$ is uniquely determined by the choice of the volume form $\omega$ on $\mathcal{G}/\mathcal{X}$. Let $|P(f)|$ denote

---

3 Here isomorphism is meant in the sense of algebraic geometry.
the constant relating \( \mu \) and \( \nu \): \( \mu = |P(f)|\nu. \) The volume form \( \omega \) fixes, on the other hand, a Haar measure \( d\gamma \) on \( G/Z \). It is obvious that \( \nu \) is the image of \( d\gamma \) under the mapping \( g \mapsto \rho(g)f \) of \( G/Z \) onto \( O_f \). From the definition of \( \nu \) we see that the same is true for the mapping \( g \mapsto \rho(g)h \) of \( G/Z \) onto \( O_h \), for any \( h \in O_f \).

Let us denote by \( d(\pi_f) \) the formal degree of \( \pi_f \):

\[
\int_{G/Z} |\langle \pi_f(g)\xi, \xi \rangle|^2 d\gamma = d(\pi_f)^{-1}\langle \xi, \xi \rangle \quad (\xi \in \mathcal{H}).
\]

**Theorem 9:** \( d(\pi_f) \) is a positive real number, which satisfies the following identity: \( d(\pi_f) = |P(f)| \).

This is proved exactly the same way as in the real case ([10], Theorem 4).

§9. Multiplicities

Let \( G \) be as usual, \( f \in \mathcal{G}' \) such that \( \pi_f \) is square-integrable mod \( Z \). Let \( K \) be an open and compact subgroup of \( G \). We shall call \( K \) a lattice subgroup if \( L = \log K \) is a lattice in \( \mathcal{G} \), i.e. an open and compact, \( \mathcal{O} \)-submodule of \( \mathcal{G} \).

**Theorem 10:** Let \( K \) be a lattice subgroup of \( G \), \( L = \log K \). Normalize Haar measures \( d\gamma \) on \( G \) and \( dz \) on \( Z \) such that \( \int_K d\gamma = \int_{K \cap \mathcal{O}} dz = 1 \). Choose a Haar measure \( d\gamma \) on \( G/Z \) such that \( d\gamma = dz d\gamma \). Then the trivial representation of \( K \) occurs in the restriction of \( \pi_f \) to \( K \) if and only if \( f(L \cap \mathcal{L}) \subset \mathcal{O} \); moreover, its multiplicity \( m(\pi_f, 1) \) is \( 1/d(\pi_f) \).

The proof of Theorem 10 is rather long and proceeds by a careful induction on \( \dim G \). The theorem is obvious if \( \dim G = 1 \). So assume \( \dim G = n > 1 \). Put \( \mathcal{L}^0 = \ker f \cap \mathcal{L} \) and \( Z^0 = \exp \mathcal{L}^0 \). We have two cases:

1. \( \mathcal{L}^0 \neq 0 \). Replace \( \mathcal{G} \) by \( \mathcal{G}/\mathcal{L}^0 \) and \( G \) by \( G/Z^0 \). The center of \( G/Z^0 \) is \( Z/Z^0 \) (cf [13], proof of Theorem, (i)). Replace also \( K \) by \( K^0 = KZ^0/Z^0 \). \( K^0 \) is a lattice subgroup of \( G/Z^0 \); \( \log K^0 = L/L \cap \mathcal{L}^0 \). Let \( f^0 \), \( \pi_f^0 \) be the pull down of \( f \), \( \pi_f \) to \( \mathcal{G}/\mathcal{L}^0 \) and \( G/Z^0 \) respectively. It is well-known that \( \pi_f^0 \) is equivalent to \( \pi_f^0 \). Hence \( m(\pi_f, 1) = m(\pi_f^0, 1) \).

\( P(f) \) actually is the Pfaffian of the canonical differential form, defining \( \mu \), relative to \( \omega \) ([1], §5, no. 2).
Furthermore, $f(L \cap Z) = f_0(L^0 \cap Z/\mathcal{Z}^0)$. Normalizing the Haar measures on $G/Z^0$, $Z/Z^0$ and $G/Z^0/Z/Z^0$ as prescribed in the theorem, one obtains $d(\pi_f) = d(\pi_{f_0})$. The assertion for $G$ now follows immediately from the result for $G/Z^0$, which is of smaller dimension.

2. dim $\mathcal{Z} = 1$ and $f \neq 0$ on $\mathcal{Z}$. $L \cap \mathcal{Z}$ is a lattice of rank one. Let $Z$ be a generator of $L \cap \mathcal{Z}$. Choose $X \in \mathcal{Z}$ such that $[X, \mathcal{Z}] \subset \mathcal{Z}$. Put $\mathcal{G}_0 = \{U : [U, X] = 0\}$. $\mathcal{G}_0$ is an ideal in $\mathcal{Z}$ of codimension one with center $\mathcal{Z}_0 = \mathcal{Z} + (X)$ (cf. [13], p. 149). $\mathcal{Z}_0 \cap L$ is a lattice of rank two; $\mathcal{Z}_0 \cap L/\mathcal{Z} \cap L$ is a lattice of rank one. We may assume that $X$ is chosen in such a way that $X \mod(\mathcal{Z} \cap L)$ generates $\mathcal{Z}_0 \cap L/\mathcal{Z} \cap L$. Then obviously,

$$\mathcal{Z}_0 \cap L = \mathcal{O}X + \mathcal{Z} \cap L = \mathcal{O}X + \mathcal{O}Z.$$ 

Since $L/L \cap \mathcal{G}_0$ is a lattice of rank one, we can choose $Y \in L$, $Y \in \mathcal{G}_0$ such that $L = \mathcal{O}Y + L \cap \mathcal{G}_0$. Put $G_0 = \exp \mathcal{G}_0$, $G_1 = \exp(sY)$ for $s \in \Omega$. Then $G = G_0 \cdot G_1$ and $G_0 \cap G_1 = \{e\}$.

Now choose a basis $Z, X, e_1, \ldots, e_{n-3}$ of $\mathcal{G}_0$ such that $L \cap \mathcal{G}_0 = \mathcal{O}Z + \mathcal{O}X + \mathcal{O}e_1 + \cdots + \mathcal{O}e_{n-3}$ and such that $e_1, \ldots, e_{n-3}$ is a supplementary basis of $\mathcal{Z}_0$ in the sense of Pukanszky ([12], section 3). One easily checks that this is possible. Given $X_0 \in \mathcal{G}_0$, write

$$X_0 = zZ + tX + t_1e_1 + \cdots + t_{n-3}e_{n-3}$$

and choose $(z, t, t_1, \ldots, t_{n-3})$ as coordinates of the second kind on $G_0$. Then $d\mathcal{G}_0 = dz \cdot dt \cdot dt_1 \cdots dt_{n-3}$ is a Haar measure on $G_0$ and $ds \cdot d\mathcal{G}_0$ is a Haar measure on $G$. Moreover, if $Z_0 = \exp \mathcal{Z}_0$, $K_0 = K \cap G_0$, we now have:

$$\text{vol}(K) = \text{vol}(K_0) = \text{vol}(K \cap Z) = \text{vol}(K_0 \cap Z_0) = 1 \cdot 5$$

Let $f_0$ denote the restriction of $f$ to $\mathcal{G}_0$. It is part of the Kirillov theory that $\pi_f$ is equivalent to $\text{ind}_{G_0}^{G} \pi_{f_0}$. Moreover, $\pi_{f_0}$ is square-integrable mod $Z_0$ ([13], p. 149). We need a relation between $d(\pi_f)$ and $d(\pi_{f_0})$. The Haar measures on $G/Z$ and $G_0/Z_0$ should be chosen as prescribed in the theorem. The following lemma is proved by computations, similar to those given in ([13], Section 5).

**Lemma 5:** Let $r = f[X, Y]$. Furthermore, put for any $s \in \Omega$, $f_s(X_0) = f(Ad(exp-sY)X_0)$ $(X_0 \in \mathcal{G}_0)$ and $\pi_s = \pi_f$. Then $\pi_s$ is square-
integrable mod $Z_0$ and

$$d(\pi_s) = \frac{1}{|r|} d(\pi_f)$$

for all $s \in \Omega$.

**Proof:** The space $\mathcal{H}_f$ of $\pi_f$ may be identified with $L^2(\Omega, \mathcal{H}_f)$. Fix a smooth vector $v \in \mathcal{H}_f$, $v \neq 0$. Choose $\psi \in C_c^\infty(\Omega)$, $\psi \neq 0$ and put $\psi_v(x) = \psi(x)v$ ($x \in \Omega$).

Then $\psi_v \in \mathcal{H}_f$. Furthermore, the computations in ([13], Section 5), show

$$\int_{G/Z} |(\pi_f g) \psi_v, \psi_v|^2 dg$$

and

$$\frac{1}{|r|} \int_\Omega \int_\Omega |\psi(s + s_1) \bar{\psi}(s)|^2 \left\{ \int_{G_0/Z_0} |(\pi_s(g_0) v, v)|^2 dg_0 \right\} ds ds_1.$$

Moreover,

$$\int_{G_0/Z_0} |(\pi_s(g_0) v, v)|^2 dg_0$$

$$= \int_{G_0/Z_0} |(\pi_0(\exp s \cdot g_0 \cdot \exp - s \cdot Y) v, v)|^2 dg_0$$

$$= \int_{G_0/Z_0} |(\pi_0(g_0) v_1 v)|^2 |\det_{g_0(Z_0)} Ad(\exp - s \cdot Y)| dg_0$$

$$= \int_{G_0/Z_0} |(\pi_0(g_0) v, v)|^2 dg_0 \quad \text{for all } s \in \Omega.$$}

Hence, $\pi_s$ is square-integrable mod $Z_0$ and $d(\pi_s) = d(\pi_0)$ for all $s \in \Omega$. In addition:

$$\langle \psi_v, \psi_v \rangle d(\pi_f)^{-1} = \frac{1}{|r|} \langle v, v \rangle \langle \psi, \psi \rangle d(\pi_0)^{-1},$$

or $d(\pi_0) = \frac{1}{|r|} d(\pi_f)$.

This completes the proof of the lemma.

Let $\phi, \phi_0$ denote the characteristic functions of $K, K_0$ respectively. Given $\psi \in L^2(\Omega, \mathcal{H}_f)$, we have
Hence, by a p-adic analogue of Mercer's theorem,

\[ \text{tr } \pi_f(\phi) = \int_\Omega \text{tr} \left\{ \int_{G_0} \phi(g_0) \pi_{b_0}(\exp s \cdot g_0 \cdot \exp -s \cdot \exp s \cdot g_0) \psi(s) \, ds \right\} \, ds. \]

So, we obtain the following relation:

\[ \text{tr } \pi_f(\phi) = \int_\Omega \text{tr } \pi_s(\phi_0) \, ds. \]

Equivalently:

**Lemma 6:** \( m(\pi_f, 1) = \int_\Omega m(\pi_s, 1) \, ds. \)

Now assume \( m(\pi_f, 1) > 0 \). Then \( m(\pi_s, 1) > 0 \) for some \( s \in \Omega \). By induction, \( f_s(L_0 \cap \mathcal{Z}_0) \subset \mathcal{O} \), where \( L_0 = L \cap G_0 \). Hence

\[ f(L \cap \mathcal{Z}) = f_s(L \cap \mathcal{Z}) \subset f_s(L_0 \cap \mathcal{Z}_0) \subset \mathcal{O}. \]

Conversely, assume \( f(L \cap \mathcal{Z}) \subset \mathcal{O} \). Let \( s \in \Omega \). Then \( f_s(L_0 \cap \mathcal{Z}_0) \subset \mathcal{O} \) if and only if \( f_s(X) \subset \mathcal{O} \). We have:

\[ f_s(X) = f(X) + sf[X, Y] = f(X) + sr. \]

Hence, by induction, \( m(\pi_s, 1) > 0 \) if and only if \( s \in (1/r)(-f(X) + \mathcal{O}) \).

Moreover, again by induction, applying Lemma 5 and 6,

\begin{align*}
m(\pi_f, 1) &= \int_{(1/r)(-f(X) + \mathcal{O})} \frac{1}{d(\pi_s)} \, ds = \frac{|r|}{d(\pi_f)} \text{vol} \left( \frac{1}{r}(-f(X) + \mathcal{O}) \right) \\
&= \frac{|r|}{d(\pi_f)} \cdot \frac{1}{|r|} = \frac{1}{d(\pi_f)}. \end{align*}

This completes the proof of Theorem 10.
§10. Multiplicities and $K$-orbits

Let $K$ be a lattice subgroup of $G$, $L = \log K$. Choose a basis $e_1, \ldots, e_p$ of $\mathcal{H}$ and let $e_{p+1}, \ldots, e_n$ be a supplementary basis of $\mathcal{H}$ such that $L = \sum_{i=1}^n a_i e_i$ ($n = \dim \mathcal{G}$). Choose $(t_1, \ldots, t_n)$ as coordinates on $\mathcal{G}$. Then $(t_1, \ldots, t_n)$ can also be used as coordinates of the second kind on $G$. Similarly $(t_1, \ldots, t_p)$ will denote coordinates on $Z$. Choose corresponding Haar measures on $G$ and $Z$, as usual. Then $\text{vol}(K) = \text{vol}(K \cap Z) = 1$. Moreover, fix a volume form $\omega$ on $\mathcal{G}/\mathcal{H}$ by $\omega = dt_{p+1} \wedge \cdots \wedge dt_n$.

Let $\phi$ denote the characteristic function of $K$. Fix $f \in \mathcal{G}$. To compute $m(\pi_f, 1)$ we can apply the character formula (§7). We obtain:

$$m(\pi_f, 1) = \text{tr} \pi_f(\phi) = \int_{O_f} \hat{\phi}(v) \, d\mu_f(v),$$

where $\mu_f$ is the canonical measure on $O_f$.

Observe that $\hat{\phi}$ is the characteristic function of the lattice $L'$, dual to $L$; $L' = \{l \in \mathcal{G} : l(L) \subseteq \mathcal{O}\}$. Hence $m(\pi_f, 1) = \mu_f$-measure of $L' \cap O_f$. K acts on $L' \cap O_f$; $L' \cap O_f$ is a disjoint union of finitely many, say $l_f$, $K$-orbits.

Now assume $\pi_f$ to be square-integrable mod $Z$. Then we have the measure $\nu$, relative to $\omega$, (§8) on $O_f$. It follows from its construction, that all $K$-orbits in $L' \cap O_f$ have the same $\nu$-measure, namely, one. Since $\mu_f = d(\pi_f)\nu$ (§8), we get:

$$m(\pi_f, 1) = l_f \cdot d(\pi_f).$$

On the other hand, $m(\pi_f, 1) = 1/d(\pi_f)$, provided $m(\pi_f, 1) > 0$ (Theorem 10). So we have the following result:

**Theorem 11:** Let $K$ be a lattice subgroup of $G$, $L = \log K$ and $L' = \{l \in \mathcal{G} : l(L) \subseteq \mathcal{O}\}$. Fix $f \in \mathcal{G}$ and let $O_f$ denote the $G$-orbit of $f$. Let $l_f$ be the number of $K$-orbits in $L'$. Then $m(\pi_f, 1) > 0$ if and only if $l_f > 0$. Moreover, if $\pi_f$ is square-integrable mod $Z$, then $m(\pi_f, 1) = \sqrt{l_f}$.

This theorem is related to work of C.C. Moore [9]. Actually, Moore proves the inequality:

$$m(\pi_f, 1) \leq l_f$$

for all $f \in \mathcal{G}$.
§11. An example

We consider the p-adic Heisenberg group $H_3$, consisting of matrices of the form

\[
\begin{pmatrix}
1 & x & z \\
0 & 1 & y \\
0 & 0 & 1
\end{pmatrix}
\]

where $x, y, z \in \mathbb{Q}_p$, $p \neq 2$. Put

\[K = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z}_p \right\}.\]

$K$ is easily seen to be a lattice subgroup of $H_3$ and

\[\log K = L = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} : x, y, z \in \mathbb{Z}_p \right\}.\]

Choosing Haar measures $dx\,dy\,dz$ on $G$ and $dz$ on the center $Z$ of $H_3$, we have $\text{vol}(K) = \text{vol}(K \cap Z) = 1$. Normalize the Haar measures on $G/Z$ and $\mathcal{G}/\mathcal{Z}$ in the usual way.

Given $f \in \mathcal{G}'$, we shall write $f = \{\alpha, \beta, \gamma\}$ if

\[f \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \alpha x + \beta y + \gamma z \quad (x, y, z, \alpha, \beta, \gamma \in \mathbb{Q}_p)\]

Similar to the real case, we have $|P(f)| = |\gamma|$ ([10]). Put $f_0 = \{0, 0, \lambda\}$, $\lambda \neq 0$. Then $\pi_{f_0}$ is square-integrable mod $Z$ and $d(\pi_{f_0}) = |\lambda|$. The $G$-orbit of $f_0$ consists of all triples

\[\{y\lambda, -x\lambda, \lambda\} \quad (x, y \in \mathbb{Q}_p)\]

Assume $|\lambda| \leq 1$. $L' = \{\{\alpha, \beta, \gamma\} : \alpha, \beta, \gamma \in \mathbb{Z}_p\}$ and

\[L' \cap O_{f_0} = \left\{ \{y\lambda, -x\lambda, \lambda\} : x, y \in \frac{1}{\lambda} \mathbb{Z}_p \right\}.\]
$K$ acts on $L' \cap O_{l_0}$; if

$$k = \begin{pmatrix} 1 & u & w \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix},$$

then

$$k \cdot \{y\lambda, -x\lambda, \lambda\} = \{y\lambda + u\lambda, -x\lambda - v\lambda, \lambda\};$$

therefore $l_{f_0} = 1/|\lambda|^2$.

On the other hand, $\pi_{f_0}$ is given on $L^2(\mathbb{Q}_p)$ by:

$$\pi_{f_0} \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \phi(t) = \chi_0(\lambda(z + ty))\phi(t + x).$$

We have

$$m(\pi_{f_0}, 1) = \dim\{\phi \in C^\infty_c(\mathbb{Q}_p) : \chi_0(\lambda ty)\phi(t + x) = \phi(t)$$

for $t \in \mathbb{Q}_p; \ x, y \in \mathbb{Z}_p\} = \dim\{\phi \in C^\infty_c(\mathbb{Q}_p) : \text{Supp } \phi \subset (1/\lambda)\mathbb{Z}_p, \ \phi \ \text{Z}_p - \text{periodic}\} = 1/|\lambda|.$

Similar computations can be done for the higher dimensional Heisenberg groups.

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