M. BRODMANN

On some results of L. J. Ratliff

Compositio Mathematica, tome 36, nº 1 (1978), p. 83-99
<http://www.numdam.org/item?id=CM_1978__36_1_83_0>

COMPOSITIO MATHEMATICA, Vol. 36, Fasc. 1, 1978, pag. 83–99
Noordhoff International Publishing
Printed in the Netherlands

ON SOME RESULTS OF L.J. RATLIFF

M. Brodmann

1. Introduction

In this paper we shall give generalizations on some of the almost classical results of [8] and [9]. We mainly shall deal with the equivalences of (locally formally) quasiunmixedness, universal catenaricity, second chain condition for prime ideals and the altitude formula for noetherian rings, as they were established by L.J. Ratliff in the cited papers.

To give an idea of the type of generalization which is in our intention let's give the following notions and definitions:

By ring we mean always "commutative", "unitary" and "locally of finite Krull dimension". Homomorphisms of rings as well as modules over rings are understood to be unitary. For a module M over a ring R we denote by $\min_R(M)$ the set of all minimal primes of the annihilator $\mathrm{ann}_R(M)$ of M. By $\dim_R(M)$ we denote the Krull dimension $\dim(R/\mathrm{ann}_R(M))$ of M over R. If a is a proper ideal of R then $\dim_R(R/a)$ is abbreviated by $\dim(a)$. By local and quasilocal we always mean noetherian too. For a semilocal ring we denote by $\hat{R}$ the completion of R with respect to the topology induced by it's Jacobson-radical $J(R)$.

(1.1) DEFINITION: *For a module M over the ring R put*:

$$d_R(M) = \inf\{\dim(p) \mid p \in \min_R(M)\}.$$

If R is semilocal and M of finite type over R put $\hat{d}_R(M) = d_{\hat{R}}(M \otimes_R \hat{R})$. Moreover put $D_R(M) = \dim_R(M) - d_R(M)$ and $\hat{D}_R(M) = D_{\hat{R}}(M \otimes_R \hat{R})$ $(= \dim_R(M) - \hat{d}_R(M))$.

Note that $\hat{D}_R(M) \geq 0$ for any semilocal ring and any finitely generated R-module M, and that $\hat{D}(R)=0$ is equivalent to the quasiunmixedness of R as it was introduced in [6, pg. 124]. So our basic idea may be clear: Instead of the above equivalences we have to compare some bounds, describing the discrepancy from universal catenaricity, the chain condition for prime ideals and the altitude formula with $\hat{D}(R)$, which latter describes the discrepancy from quasiunmixedness.

It should be noted that $\hat{D}(R)$ has been introduced by U. Schweizer in [11] and that some results on it are contained in that paper.

We shall give first some basic results on $\hat{d}(R)$, in particular the generalization of the conservation of quasiunmixedness by passing to residual domains or localities.

Then, in Section 3 we look at lengths of maximal chains of primes describing in this manner the discrepancy from universal catenaricity and the chain condition for prime ideals, and comparing it with $\hat{D}(R)$. We moreover give there a result on lengths of chains of primes in polynomial algebras, which is close to a corresponding result of [11]. The main tool of this section is that one introduced by L.J. Ratliff to prove the equivalences in [8] and [9]: His quadratic integral extensions which push down the lengths of chains of primes.

In Section 4 we deal with another fundamental notion in dimension theory – with degrees of transcendence. We shall there compare $\hat{D}(R)$ with the discrepancy from the altitude formula. Note that this may be done in a very simple way if R is catenary, just in following the traces of the proof of the well known result that in this case universal catenaricity and the altitude formula are equivalent [5, (14.D) or 4, (5.6.1)]. It was again L.J. Ratliff who dropped the condition of catenaricity of R in making a sort of "catenarization": namely an extension R' of R (for a given prime p of R) having a unique prime p' lying over p and such that a chain of primes of maximal length passes through p' [8, (2.13)]. We shall also take up and generalize this idea in (4.9) and (4.10).

Note that there are similarities to the results on the numbers

$$\hat{\delta}(R)=\min\{\dim(\hat{p}) \mid \hat{p}\in \operatorname{ass}(\hat{R})\},$$
$$\hat{\Delta}(R)=\dim(R)-\hat{\delta}(R),$$

(R being semilocal), as they have been introduced and treated in [1] and [2]. But it should be noted that $\hat{d}(R)$ is easier to handle, as (2.4) is not known to have an analogue for $\hat{\Delta}$. So in (2.11) we give connections to the result on $\hat{\Delta}$ concerning the passage to localities. Also in (4.4) $\hat{\delta}$ appears in a statement which gives a more convenient form of [1, (6.1)]. Note that for a semilocal ring R we have $\hat{\delta}(R) \leq \hat{d}(R)$, hence $\hat{\Delta}(R) \geq \hat{D}(R)$. But there is known that it may hold $\hat{\Delta}(R) > \hat{D}(R)$. Indeed in [3] a local domain of dimension 2 is constructed which has an embedded prime in $\hat{R}$, hence $\hat{\Delta}(R) > 0$ but such that it's integral closure is regular. From this regularity it then follows $\hat{D}(R) = 0$ by [8, (3.5)].

As for the notations we use, see [5]. Two further conventions are the following ones: $Q(R) = S^{-1}R$, where S is the set reg(R) of regular elements of R.

$X, X_1, \dots, X_n, \dots$ are always indeterminates.

Finally note that (3.6) and (3.7) are given in [10] in a slightly different, more general form.

2. Basic properties of $\hat{d}(R)$

(2.1) Lemma: *Let R be a semilocal ring. Then*:

(i) $\hat{d}(R) = 0 \Leftrightarrow \min(R) \cap \max(R) \neq \emptyset$.

(ii) $\hat{d}(R) = \min\{\hat{d}(R/p) \mid p \in \min(R)\}$.

(iii) $\hat{d}(R) = \min\{\hat{d}(R_M) \mid m \in \max(R)\}$.

Proof: (i) is immediate by $\min(R) = \{\hat{p} \cap R \mid \hat{p} \in \min(\hat{R})\}$, $\max(R) = \{\hat{m} \cap R \mid \hat{m} \in \max(\hat{R})\}$ and $\max(\hat{R}) = \{m\hat{R} \mid m \in \max(R)\}$.

(ii) is clear by $\min(\hat{R}) = \bigcup_{p\in\min(R)} \min_{\hat{R}}(\hat{R}/p\hat{R})$.

(iii) follows from $\hat{R} \cong \prod_{m\in\max(R)} (R_m)\hat{}$.

(2.2) Lemma: *Let R be semilocal and $x \in J(R)$. Then $d(R/xR) \geq \hat{d}(R) - 1$.*

Proof: Assume first that R is local and let $\hat{p}$ be a minimal prime of $x\hat{R}$ such that $\dim(\hat{p}) = \hat{d}(R/xR)$. Then there is a minimal prime $\hat{q}$ of $\hat{R}$ such that $\hat{q} \subseteq \hat{p}$. As $\mathrm{ht}(\hat{p}) \leq 1$ and as $\hat{R}$ is catenary [5, pg. 211] we get $\hat{d}(R) \leq \dim(\hat{q}) = \dim(\hat{p}) + \mathrm{ht}(\hat{p}/\hat{q}) \leq \hat{d}(R/xR) + 1$. So we have the result in the local case. The general case follows now by (2.1) (iii) and the fact that $\max(R/xR) = \{m/xR \mid m \in \max(R)\}$, which is implied by $x \in J(R)$.

(2.3) Corollary: *Let R be semilocal and $p \in \mathrm{spec}(R)$. Then $\hat{d}(R/p) \geq \hat{d}(R) - \mathrm{ht}(p)$.*

PROOF: (Induction on $\mathrm{ht}(p)$). If $\mathrm{ht}(p) = 0$ the result is clear by (2.1) (ii). If $\mathrm{ht}(p) > 0$ there is nothing to prove if $\hat{d}(R) = 0$. If $\hat{d}(R) > 0$ it follows from (2.1) (i) that there is a parameter $x \in J(R) \cap p$. But then, by (2.2) and the hypothesis of induction $\hat{d}(R/p) = \hat{d}(R/xR/p/xR) \geq \hat{d}(R/xR) - \mathrm{ht}(p/xR) \geq \hat{d}(R) - 1 - (\mathrm{ht}(p) - 1) = \hat{d}(R) - \mathrm{ht}(p)$.

(2.4) COROLLARY: *Let R be semilocal and $p \in \mathrm{spec}(R)$. Then* $\hat{D}(R/p) \leq \hat{D}(R)$.

PROOF: $\hat{D}(R/p) = \dim(R/p) - \hat{d}(R/p) \leq \dim(R) - \mathrm{ht}(p) - \hat{d}(R/p) \leq \dim(R) - \hat{d}(R) = \hat{D}(R)$.

For $\hat{D}(R) = 0$, (2.4) is the well known fact that the quasiunmixedness of a semilocal ring implies the quasiunmixedness of its residual domains (cf. [6, (34.5)]).

Now we look whether it is possible to choose $x \in J(R)$ such that in (2.2) equality holds.

(2.5) LEMMA: *Let R be noetherian and let $p_0, p_1, \ldots, p_n \in \min(R)$. Let $y \in \bigcap_{i=1}^{n} p_i - p_0$ and let $q_1, \ldots, q_r$ be the minimal primes of $yR + p_0$. Choose $x \in R - \bigcup_{i=1}^{r} q_i$ and let q be a minimal prime of $xR + p_0$. Then p_0 is the only minimal prime of R among $p_0, \ldots, p_n$ contained in q.*

PROOF: Assume $p_i \subseteq q$ for $i \neq 0$. Then $y \in q$. As on the other hand $\mathrm{ht}(q/p_0) = 1$, this and $y \notin p_0$ would imply that q is a minimal prime of $yR + p_0$, hence that $q = q_j$ for a conveniently chosen index j. But then $x \in q_j$, contrary to our choice.

(2.6) PROPOSITION: *Let R be semilocal and assume $\hat{d}(R) > 1$. Then for any finite set $P = \{p_1, \ldots, p_s\}$ of* $\mathrm{spec}(R)$ *such that* $\dim(p_i) \geq \hat{d}(R) - 1$ *for $i = 1, \ldots, s$ we have $F_P = J(R) - \bigcup_{k=1}^{s} p_k \neq \emptyset$. Moreover there is such a set P with the following property*:

$$x \in F_P \Rightarrow \hat{d}(R/xR) = \hat{d}(R) - 1.$$

PROOF: The first statement is immediate. For the proof of the second one let $\hat{p}_0, \ldots, \hat{p}_n$ be the minimal primes of $\hat{R}$, $\hat{p}_0$ being such that $\dim(\hat{p}_0) = \hat{d}(R)$. Let $\hat{y} \in \bigcap_{i=1}^{n} \hat{p}_i - \hat{p}_0$ and let $\hat{q}_1, \ldots, \hat{q}_r$ be the minimal primes of $\hat{y}\hat{R} + \hat{p}_0$. Put $P = \{\hat{p}_j \cap R \mid j = 1, \ldots, n\} \cup \{\hat{q}_i \cap R \mid i = 1, \ldots, r\}$. Then, if $p = \hat{p}_j \cap R$ we have $\dim(p) \geq \dim(\hat{p}_j) \geq \hat{d}(R)$. If $p = \hat{q}_i \cap R$ we have $\dim(p) \geq \dim(\hat{q}_i)$. As $\hat{p}_0$ is contained in a unique maximal ideal of $\hat{R}$ and as $\hat{R}$ is catenary we have $\dim(\hat{q}_i) = \dim(\hat{p}_0) - \mathrm{ht}(\hat{q}_i/\hat{p}_0) \geq \hat{d}(R) - 1$. So we have indeed $\dim(p) \geq \hat{d}(R) - 1$ for any $p \in P$.

Let $x \in F_P$. By (2.2) we then have $\hat{d}(R/xR) \geq \hat{d}(R) - 1$. On the other hand let $\hat{q}$ be a minimal prime of $x\hat{R} + \hat{p}_0$. Then by (2.5) $\hat{p}_0$ is the only minimal prime of $\hat{R}$ contained in $\hat{q}$. As $x \notin \hat{p}_0$ this implies that $\mathrm{ht}(\hat{q}) = 1$. But this, together with $x \notin \bigcup_{i=0}^{n} \hat{p}_i$ implies that $\hat{q}$ is a minimal prime of $x\hat{R}$. From this we get $\hat{d}(R/xR) \leq \dim(\hat{q}) \leq \dim(\hat{p}_0) - 1 = \hat{d}(R) - 1$.

Now we are going to look at the behaviour of $\hat{d}$ and $\hat{D}$ under localization. The proofs will follow the traces sketched at [7, pg. 60] for the case $\hat{D}(R) = 0$.

(2.7) LEMMA: *Let $R \subseteq R'$ be semilocal rings such that R is a subspace of R' and such that*

(i) $J(R)R' \subseteq J(R')$, $\dim(J(R)R') = 0$

(ii) *For any $\hat{p} \in \min(\hat{R})$ there is a $\hat{p} \in \min(\hat{R})$ such that $\hat{p} \subseteq \hat{p}' \cap \hat{R}$. Then $\hat{d}(R') \leq \hat{d}(R)$.*

PROOF: Let $\hat{p} \in \min(\hat{R})$ such that $\dim(\hat{p}) = \hat{d}(R)$, $\hat{p}' \in \min(\hat{R}')$ such that $\hat{p} \subseteq \hat{p}' \cap \hat{R}$. Then $\hat{R}'/\hat{p}'$ and $\hat{R}/\hat{p}$ are local and the maximal ideal of $\hat{R}/\hat{p}$ generates in $\hat{R}'/\hat{p}'$ an ideal primary to the maximal ideal of $\hat{R}'/\hat{p}$. Thus $\dim(\hat{p}) \geq \dim(\hat{p}')$.

(2.8) REMARK: The conditions of (2.7) hold if R' is a finite integral extension of R or if R and R' satisfy the transition theorem (cf. [6, pg. 64]). Indeed in these cases $\hat{R}'$ is a finite integral extension of $\hat{R}$ resp. $\hat{R}$ and $\hat{R}'$ satisfy the transition theorem. In particular (ii) is implied by the lying over property of integral extensions resp. the $\hat{R}$-flatness of $\hat{R}'$ (cf. [6, pg. 65]).

(2.9) PROPOSITION: *Let R be semilocal and $p \in \mathrm{spec}(R)$. Then $\hat{d}(R_p) \geq \hat{d}(R) - \dim(R/p)$.*

PROOF: Let $\hat{p}$ be a minimal prime of $p\hat{R}$. Then by [6, (19.1, 2)] R_p and $\hat{R}_{\hat{p}}$ satisfy the theorem of transition and by (2.8) we have only to show that $\hat{d}(\hat{R}_{\hat{p}}) \geq \hat{d}(R) - \dim(p)$. As $\hat{d}(R) = \hat{d}(\hat{R})$, $\dim(p) \geq \dim(\hat{p})$ we only must show $\hat{d}(\hat{R}_{\hat{p}}) \geq \hat{d}(\hat{R}) - \dim(\hat{p})$. So, in making use of (2.1) (ii) we may assume that R is local and complete. Hence we may write $R = A/a$, where A is a regular local ring and a an ideal of A. Let $q \in \mathrm{spec}(A)$ such that $p = q/a$. Then $R_p = A_q/aA_q$. Note that A and A_q are both quasiunmixed, as they are regular. Therefore $\hat{d}(R) =$

$\hat{d}(A/a) = d(A/a)$, $\hat{d}(R_p) = \hat{d}(A_q/aA_q) = d(A_q/aA_q)$. So there is a minimal prime r of a such that $r \subseteq q$ and $\mathrm{ht}(q/r) = \hat{d}(R_p)$. As $\hat{d}(R) \leq \dim(r)$ and as A is catenary we thus get $\hat{d}(R_p) = \mathrm{ht}(q) - \mathrm{ht}(r) = \dim(r) - \dim(q) \geq \hat{d}(R) - \dim(p)$.

(2.10) COROLLARY: *Let R be semilocal and $p \in \mathrm{spec}(R)$. Then $\hat{D}(R_p) \leq \hat{D}(R)$.*

PROOF: Indeed, by (2.9) we have $\hat{D}(R_p) = \mathrm{ht}(p) - \hat{d}(R_p) \leq \mathrm{ht}(p) - \hat{d}(R) + \dim(R/p) \leq \dim(R) - \hat{d}(R)$.

(2.11) REMARK: Note that in the proofs of (2.7, 9, 10) all arguments stay valid if "minimal prime" is replaced by "associated prime" and if $\hat{d}$ is replaced by $\hat{\delta}$ and $\hat{D}$ by $\hat{\Delta}$. So these results are true with $\hat{\delta}$ and $\hat{\Delta}$ instead of $\hat{d}$ resp. $\hat{D}$. So in particular for local domains (2.9) resp. (2.10) with $\hat{\delta}$ and $\hat{\Delta}$ instead of $\hat{d}$ and $\hat{D}$ coincide with [1, (5.6)] resp. [1, (5.7)]. In [1] these results were got in a different way, namely in making use of the quasiregular sequences introduced in that paper.

The last part of this section is devoted to the question what happens to $\hat{d}$ and $\hat{D}$ if we pass from R to an algebra of essentially finite type. If (R, m) is local we denote by $R(X)$ the ring $R[X]_{(m,X)}$. (cf. [2], where this notation was introduced in a more general context).

(2.12) LEMMA: *Let (R, m) be a local ring and $n \in \max(R[X])$ such that $n \cap R = m$. Then there is an injective homomorphism of R-algebras $R(X) \to R[X]_n$, making a finitely generated free $R(X)$-module of $R[X]_n$.*

PROOF: n is of the form (m, f) where $f \in R[X]$ is monic and irreducible modulo m. Inclusion makes obviously a finite and free $R[f]$-module of $R[X]$. As f is monic it does not satisfy any algebraic equation over R and therefore there is a canonical isomorphism $R[X] \xrightarrow{i} R[f]$, sending X to f. This isomorphism sends (m, X) of $R[X]$ to (m, f) of $R[f]$, and moreover n is the only prime of $R[X]$ lying over $mR[f]' + fR[f]$. So composing i by inclusion and localizing at (m, X) we get the wished embedding.

(2.13) LEMMA: *Let (R, m) be local and n a maximal ideal of $R[X]$ lying over m. Then $\hat{d}(R[X]_n) = \hat{d}(R) + 1$.*

PROOF: By (2.12) $R' = R[X]_n$ is a finite and free $R(X)$-module and therefore $\hat{R}'$ is finite and free as $(R(X))^\wedge$-module. The $(R(X))^\wedge$-flatness of $\hat{R}'$ then implies that $\min((R(X))^\wedge = \{\hat{p}' \cap (R(X))^\wedge \mid \hat{p}' \in \min(\hat{R}')\}$. By the integral dependence of $\hat{R}'$ on $(R(X))^\wedge$ we get furthermore $\dim(\hat{p}') = \dim(\hat{p}' \cap R(X))$ and therefore we have $\hat{d}(R') = \hat{d}(R(X))$. So we may choose $R' = R(X)$. But then $\hat{R}' = \hat{R}[[X]]$ and $\min(\hat{R}[[X]]) = \{\hat{p}\hat{R}[[X]] \mid \hat{p} \in \min(\hat{R})\}$ together with $\dim(\hat{p}\hat{R}[[X]]) = \dim(\hat{p}) + 1$ imply the result.

(2.14) LEMMA: *Let (R, m) be a local domain and let $R' = R[a]$ be a simply generated extension domain of R. Let n be a maximal ideal of R' lying over m. Then $\hat{d}(R'_n) \geq \hat{d}(R)$.*

PROOF: Write $R' = R[X]/p$, $p \in \operatorname{spec}(R[X])$. Then $\operatorname{ht}(p) \leq 1$. Let q be the maximal ideal of $R[X]$ for which $q/p = n$. Then $q \cap R = m$, $R'_n = R[X]_q/pR[X]_q$ and $\operatorname{ht}(pR[X]_n) \leq 1$ imply by (2.13) and (2.3) that $\hat{d}(R'_n) \geq \hat{d}(R[X]_q) - 1 = \hat{d}(R)$.

(2.15) COROLLARY: *Let $R \subseteq R'$ be semilocal domains such that R' is a finite integral extension of R. Then $\hat{d}(R') = \hat{d}(R)$.*

PROOF: $\hat{d}(R') \leq \hat{d}(R)$ is clear by (2.7) and (2.8). To prove $\hat{d}(R') \geq \hat{d}(R)$ we may restrict ourselves to the case $R' = R[a]$, in making induction on the number of generators of R'. But then (2.1) (ii) implies that there is a $n \in \max(R')$ such that $\hat{d}(R') = \hat{d}(R'_n)$ and as $m = n \cap R \in \max(R)$ we get by (2.1) (ii) and (2.14): $\hat{d}(R'_n) \geq \hat{d}(R_m) \geq \hat{d}(R)$.

(2.16) PROPOSITION: *Let $R \subseteq R'$ be local domains such that R' is essentially of finite type over R. Then we have $\hat{D}(R') \leq \hat{D}(R)$.*

PROOF: Without loss of generality we may assume that R' is a localization of a simply generated domain $R[a]$. So we may write $R' \cong R[X]_q/pR[X]_q$, where $p, q \in \operatorname{spec}(R[X])$ such that $p \subseteq q$. Then by (2.10) and (2.13) we have $\hat{D}(R[X]_q) \leq \hat{D}(R)$, and the result is clear by (2.4).

(2.16) is a generalization of the well known result that a locality over a quasi-unmixed domain is again quasiunmixed.

Note moreover that (2.16) is true for $\hat{\Delta}$ instead of $\hat{D}$ [1, (6.1)]. A proof of this may not be got in replacing $\hat{d}$ resp. $\hat{D}$ by $\hat{\delta}$ resp. $\hat{\Delta}$ in all results occuring in the proof of (2.16), as (2.4) is not shown for $\hat{\Delta}$ instead of $\hat{D}$.

3. Chains of primes

In this section we give some connections between $\hat{d}(R)$ and the lengths of chains of primes in some extensions of R. The results are generalizations of some of those in [8] and [9].

Let $p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_l$ be a chain of primes of the ring R. Then l is called the length of the chain. The above chain is called maximal if there is no $p \in \operatorname{spec}(R)$ such that one of the following relations holds:

$$P_i \subsetneq p_0, \quad p_i \subsetneq p \subsetneq p_{i+1}, \quad p_1 \subsetneq p.$$

Note that any ring of locally finite dimension has finite maximal chains of primes.

(3.1) DEFINITION: *Let $c(R)$ be the minimum of lengths of maximal chains of primes of R and put $C(R) = \dim(R) - c(R)$.*

The following is immediately clear:

(3.2) LEMMA: (i) $c(R) = 0 \Leftrightarrow \min(R) \cap \max(R) \neq \emptyset$.
(ii) $c(R) = \min\{c(R_m) \mid m \in \max(R)\}$.
(iii) $c(R) = \min\{c(R/p) \mid p \in \min(R)\}$.
(iv) $c(R) \leq c(R/p) + c(R_p)$ $(p \in \operatorname{spec}(R))$.

(3.3) LEMMA: *If R' is an integral extension of R we have $c(R') \leq c(R)$, hence $C(R') \geq C(R)$.*

PROOF: Let $p_0 \subsetneq \cdots \subsetneq p_l$ be a maximal chain of primes of R. Then, by the first theorem of Cohen–Seidenberg there is a chain of primes of R', $p_0' \subsetneq \cdots \subsetneq p_l'$ such that $p_i = p_i' \cap R$ $(i = 0, \ldots, l)$ and this chain is maximal.

(3.4) LEMMA: *Let R be semilocal. Then $c(R) \geq \hat{d}(R)$, hence $C(R) \leq \hat{D}(R)$.*

PROOF: As $\hat{R}$ is catenary it suffices to prove $c(\hat{R}) \leq c(R)$. Let $p_0 \subset \cdots \subset p_l$ be a maximal chain of primes. Then, as $p_l\hat{R} \neq \hat{R}$ there is a chain of primes $\hat{p}_0 \subseteq \cdots \subseteq \hat{p}_l$ of $\hat{R}$ such that $\hat{p}_i$ is a minimal prime of $p_i\hat{R}$ for $i = 0, \ldots, l$. As $R \to \hat{R}$ has the going-down property we then have $\hat{p}_i \cap R = p_i$ $(i = 0, \ldots, l)$, and as furthermore $\hat{p}_l = p_l\hat{R} \in \max(\hat{R})$ and $\hat{p}_0 \in \min(\hat{R})$, the constructed chain is maximal.

The main tool of this section is the following result, due to L.J. Ratliff.

(3.5) LEMMA: *Let R be a local domain with $\hat{d}(R) = 1$. Then there is a $y \in Q(R)$, satisfying a quadratic integral equation over R such that $c(R[y]) = 1$.*

Indeed, if $\dim(R) = 1$ choose $y = 1$. If $\dim(R) > 1$ choose c, $b \in R$ as it is done in [8, (2.17)] and put $y = c/b$. Then it is verified in the proof of [8, (3.1)] and in [8, (3.2) (v)] that y has the requested properties.

From this we get:

(3.6) COROLLARY: *Let R be a local domain. Then there is a simply generated extension domain $R[y]$ of R such that y satisfies a quadratic integral equation over R and such that $c(R[y]) = \hat{d}(R)$. If $c(R) > \hat{d}(R)$ we moreover may assume that $R[y]$ is not local.*

PROOF: (Induction on $\hat{d}(R)$). The case $\hat{d}(R) = 0$ being trivial let $\hat{d}(R) = 1$. Then choose y as in (3.5). As $c(R[y]) = 1$ $R[y]$ may not be local if $c(R) > 1$. So assume $\hat{d}(R) > 1$. Then by (2.6) there is an $x \in R - (0)$ such that $\hat{d}(R/xR) = \hat{d}(R) - 1$, and by (2.1) (ii) there is a minimal prime p of xR such that $\hat{d}(R/p) = \hat{d}(R) - 1$. As $\mathrm{ht}(p) = 1$ we get by (3.2) (iv) that $c(R/p) \geq c(R) - 1$ and $c(R) > \hat{d}(R)$ implies $c(R/p) > \hat{d}(R/p)$. Now by the hypothesis of induction applied to R/p there is a $\tilde{p} \in \mathrm{spec}(R[X])$ lying over p, containing a monic polynomial f of at most degree 2 and such that $c(R[X]/\tilde{p}) = \hat{d}(R) - 1$; and if $c(R) > \hat{d}(R)$ we may assume that $R[X]/\tilde{p}$ is not local. Choose a minimal prime $\tilde{q}$ of $fR(X)$ such that $\tilde{q} \subseteq \tilde{p}$. Then, as f is monic we have $\tilde{q} \cap R = (0)$ and $R[X]/\tilde{q} = R[y]$ is a simply generated extension domain of R such that $f(y) = 0$. But now $\mathrm{ht}(p) = 1$ implies that $\mathrm{ht}(\tilde{p}/\tilde{q}) = 1$, and by (3.2) (iv) we get $c(R[y]) \leq c(R[X]/\tilde{p}) + 1 \leq \hat{d}(R)$. On the other hand by (3.4) and (2.15) we have $c(R[y]) \geq \hat{d}(R[y]) = \hat{d}(R)$.

(3.7) COROLLARY: *Let (R, m) be a local ring and let n be a maximal ideal of $R[X]$ lying over m. Then $c(R[X]_n) = \hat{d}(R) + 1$.*

PROOF: By (2.12), (2.13), (3.3) and (3.4) it suffices to prove $c(R(X)) \leq \hat{d}(R) + 1$. By (2.1) (ii) there is a $p \in \min(R)$ such that $\hat{d}(R) = \hat{d}(R/p)$, and by (3.2) (iii) we may replace R by R/p, hence assume that R is a domain. If $c(R) = \hat{d}(R)$ $R \cong R(X)/XR(X)$ and (3.2) (iv) imply $c(R(X)) \leq \hat{d}(R) + 1$. If $c(R) > \hat{d}(R)$ we find by (3.6) a $\tilde{q} \in \mathrm{spec}(R[X])$, lying over (0), containing a monic polynomial f of degree 2 such that $R[y] = R[X]/\tilde{q}$ is not local and satisfies $c(R[y]) =$

$\hat{d}(R)$. By (3.2) (iii) we then find an $n \in \max(R[y])$ such that $\hat{d}(R) = c(R[y]_n)$. As f is monic of degree 2 and as $R[y]$ is not local there is a $e \in R$ such that n is of the form $(m, X+e)/\tilde{q}$. As $\mathrm{ht}(\tilde{q}) = 1$ we thus get by (3.2) (iv) that $c(R[X]_{(m,X+e)} \leq \hat{d}(R)+1$. Moreover we have $R[X]_{(m,X+e)} = R[X+e]_{(m,X+e)}) \cong R(X)$, which implies the result.

(3.8) Definition: *For $R[X_1, \dots, X_r]$ let's write R_r. R_0 stands then for R.*

(3.9) Lemma: *Let R be noetherian and let n be a maximal ideal of R_r. Then:*

(i) *$R/n \cap R$ is a semilocal domain of at most dimension 1.*

(ii) *If $n \cap R$ is maximal, then so is $n \cap R_i$ for $i = 1, \dots, r$.*

Proof: (i) Put $A = R/n \cap R$ and let x_i be the image of X_i in R_r/n. Then we may assume without loss of generality that x_i is not algebraic over $A[x_1, \dots, x_{i-1}]$ for $i = 1, \dots, s$, and that x_j is algebraic over $B = A[x_1, \dots, x_s]$ for $j = s+1, \dots, r$. So there is a $b \in B-(0)$ such that $x_j b$ is integral over B for $j = s+1, \dots, r$, and as R_r/n is a field we may write it as $B[bx_{s+1}, \dots, bx_n]\ [1/b]$. As $B[bx_{s+1}, \dots, bx_n] = C$ is noetherian the fact that $C[1/b]$ is a field implies that C is a semilocal domain of at most dimension 1. As C integral over B, the same holds for B. In particular we get $s = 0$, hence $B = A$ and by the above the result is shown.

(ii) By the above R_r/n is algebraic over A, hence integral dependent on it as $n \cap R$ is maximal. So $R_i/n \cap R_i$ is integral dependent on A too, and therefore a field.

From this we get:

(3.10) Proposition: *Let (R, m) be local and let n be a maximal ideal of R_r, $(r > 0)$. Then*

(i) $c((R_r)_n) \geq r + \hat{d}(R) - 1$,

(ii) $c((R_r)_n) = r + \hat{d}(R)$, if $n \cap R = m$,

Proof: (ii) is clear by induction on r from (3.9) (ii), (2.13) and (3.7).
(i) follows from (ii), (3.9) (i) and (2.9) in replacing R by $R_{(n \cap R)}$.

Now the following is clear as $\dim(R_r) = \dim(R) + r$ and by (3.2)(ii).

(3.11) Corollary: *For a local ring R we have $C(R_r) \leq \hat{D}(R) + 1$ if $r > 0$.*

Note that (3.10) (i) is proved in [11] in a different way and that there moreover a class K of local rings is introduced for which the following holds: $R \in K$ implies that there is an $r_0 \in N$ such that $c((R_r)_n) = r + \hat{\delta}(R)$ for any $r \geq r_0$ and any maximal ideal n of R_r lying over the maximal ideal of R (s. [11, Satz 6]). Thus, by (3.10) (ii) for this class K we must have: $R \in K \Rightarrow \hat{d}(R) = \hat{\delta}(R)$.

Finally, to deal with integral extensions, let's give the following:

(3.12) LEMMA: *Let $R \subset R'$ be domains, such that R is semilocal and such that R' is integral over R. Then $c(R') \geq \hat{d}(R)$, hence $C(R') \leq \hat{D}(R)$.*

PROOF: (Induction on $c(R')$). If $c(R') = 0$ (3.2) (i) implies that R', hence R is a field. So we have $\hat{d}(R) = 0$. To treat the case $c(R') > 0$ note that by (3.3) we may assume that R' contains the integral closure $\bar{R}$ of R in $Q(R)$. Let $(0) \subsetneq p'_1 \subsetneq \cdots \subsetneq p'_{c(R'Q)}$ be a maximal chain of primes of R' and put $p = p'_1 \cap \bar{R}$. Then, as $\bar{R}$ is normal, we have $\mathrm{ht}(\bar{p}) = 1$. Put $p = \bar{p} \cap R$. Then by [6, (33.10)] there are only finitely many primes of $\bar{R}$ lying over p. Therefore we find a finite integral extension R'' of R in $\bar{R}$ such that $\bar{p}$ is the unique prime of $\bar{R}$ lying over $p'' = \bar{p} \cap R''$. But then we must have $\mathrm{ht}(p'') = \mathrm{ht}(\bar{p}) = 1$, and (2.3) implies that $\hat{d}(R''/p'') \geq \hat{d}(R'') - 1$. On the other hand by the hypothesis of induction applied to the extension $R''/p'' \to R'/p'_1$ we get $c(R') - 1 = c(R'/p'_1) \geq \hat{d}(R''/p'') \geq \hat{d}(R'') - 1 = \hat{d}(R) - 1$, where the last equality is a consequence of (2.15).

(3.13) REMARK: (3.12) applied in the case $\hat{D}(R) = 0$ gives the equivalence of quasiunmixedness and the second chain condition for a local domain. This result has been proved by L.J. Ratliff [8].

To leave the local case let's introduce the following:

(3.14) DEFINITION: For a ring R, put:

(i) $\mu C(R) = \max\{C((R/p)_q) \mid p, q \in \mathrm{spec}(R), p \subseteq q\}$. If R is noetherian, put:

(ii) $\mu\hat{D}(R) = \max\{\hat{D}((R/p)_q) \mid p, q \in \mathrm{spec}(R), p \subseteq q\}$.

(3.15) REMARK: If a is an ideal of R, and if $S \subseteq R$ is multiplicatively closed, we have:

$$\mu C(R) \geq \mu C(S^{-1}(R/a)), \quad \mu\hat{D}(R) \geq \mu\hat{D}(S^{-1}(R/a)).$$

Moreover we have the following equivalences:

$$\mu C(R) = 0 \Leftrightarrow R \text{ is catenary.}$$

$\mu\hat{D}(R) = 0 \Leftrightarrow R$ is locally formally catenary; (s. [9, (2.4)]).

(3.16) DEFINITION: Let R' be an R-algebra; put:
(i) $C_R(R') = \max(C((R'/p')_q) \mid p' \in \operatorname{spec}(R'), q \in \operatorname{spec}(R), p' \cap R \subseteq q\}$.

(3.17) REMARK: If a' is an ideal of R' and $S \subseteq R$ is multiplicatively closed, then:

$$C_R(R') \leq C_R(S^{-1}(R'/a')).$$

Moreover R satisfies the chain condition for prime ideals iff $C_R(R') = 0$ for any integral R-algebra R', [s. [6, pg. 122]).

(3.18) LEMMA: *If R' is integral over R we have $\mu C(R') \leq C_R(R')$.*

PROOF: Let p', $q' \in \operatorname{spec}(R')$ such that $p' \subseteq q'$ and $C((R'/p')_{q'}) = \mu C(R')$. Put $p = p' \cap R$, $q = q' \cap R$. Then $(R'/p')_{q'}$ is a localization of $(R'/q')_q$ and we have $\dim((R'/p')_q) \geq \dim((R'/p')_{q'})$. On the other hand $(R'/p')_q$ is an integral extension of $(R/p)_q$, and therefore $(q'/p')_q$ is a maximal ideal of $(R'/p')_q$. Thus by (3.2) (ii) we have $c((R'/p')_q \leq c((R'/p')_{q'})$, and it follows:

$$C((R'/p')_{q'}) \leq \dim((R'/p')_q) - c((R'/p')_q) \leq C((R'/p')_q) \leq C_R(R').$$

(3.19) THEOREM: *Let R' be an algebra over the noetherian ring R. Then*:

(i) *$C_R(R') \leq \mu\hat{D}(R)$ if R' is integral over R. Moreover there is a simply generated extension $R[y]$ of R, y satisfying a quadratic integral equation over R such that we get equality in putting $R' = R[y]$.*
(ii) *$\mu C(R') \leq \mu\hat{D}(R)$ if R' is of essentially finite type over R, and for $R' = R_r$, $r > 0$, we get equality.*

PROOF: (i) Let $p' \in \operatorname{spec}(R')$, $q \in \operatorname{spec}(R)$ such that $p' \cap R \subseteq q$ and $C((R'/p')_q) = C_R(R')$. Then, as $(R'/p')_q$ is an integral extension of the local domain $(R/p' \cap R)_q$ we get by (3.12) $C_R(R') \leq \hat{D}((R/p' \cap R)_q) \leq \mu\hat{D}(R)$.

To prove the second half of (i) let p, $q \in \operatorname{spec}(R)$ such that $p \subseteq q$ and $\mu\hat{D}(R) = \hat{D}((R/p)_q)$. Then by (3.6) we find an extension domain $\bar{R}[\bar{y}]$ of $\bar{R} = (R/p)_q$ such that $C(\bar{R}[\bar{y}]) = \mu\hat{D}(R)$ and such that for convenient r_1, $r_2 \in R$, $s \in R - q$ we have $(\bar{y})^2 + s^{-1}r_1\bar{y} + s^{-1}r_2 = 0$. As s becomes a unity of $\bar{R}$ we may replace $\bar{y}$ by sy, hence assume $s = 1$.

But then $\bar{R}[\bar{y}]$ is a residual domain of $R_q[y] = R_q[X]/(X^2 + r_1X + r_2)$, and we get $\mu\hat{D}(R) \leq C_R(R[y])$.

(ii) Let $p', q' \in \operatorname{spec}(R')$ such that $p' \subseteq q'$ and $\mu C(R') = C((R'/p')_{q'})$. Put $p = p' \cap R$, $q = q' \cap R$. Then, by (3.5) and (2.16) we get $\mu C(R') \leq \hat{D}((R'/p')_{q'}) \leq \hat{D}((R/p)_q) \leq \mu\hat{D}(R)$.

As for the second half of the statement we may restrict ourselves to the case $n = 1$. Let p, q, $\bar{R}$ be as in the proof of (i). Then by (3.7) and (3.15) we get $\hat{D}(R) = \hat{D}(\bar{R}) = C(\bar{R}[X]_{(q\bar{R}[X]+X\bar{R}[X])}) \leq \mu C(R_1)$.

From (3.19) we get by (3.15) and (3.17) that for a noetherian ring R the following properties are equivalent:

(3.20) (i) *R is locally formally catenary*
(ii) *R is universally catenary*
(iii) *R_r is catenary for a natural r*
(iv) *R satisfies the chain condition for prime ideals.*

This equivalence has been proved by L.J. Ratliff in [9].

4. Degrees of transcendence

Let $R \subseteq R'$ be domains. Then by $\operatorname{tr}(R':R)$ we denote the degree of transcendence of $Q(R')$ over $Q(R)$.

Furthermore let f be a function which assigns to each semilocal domain R an integer $f(R)$. Look at the following conditions on f:

(4.1) *For any local domain (R, m):*

(i) *If $R' = R[a]$ is a simply generated extension domain of R and if $m' \in \max(R')$ is such that $m' \cap R = m$, then $f(R) \leq f(R'_{m'})$, if a is algebraic over R.*

(ii) *If in* (i) *a is not algebraic over R, then $f(R) + 1 = f(R'_{m'})$.*

(iii) $p \in \operatorname{spec}(R)$, $\dim(p) = 1 \Rightarrow f(R_p) \geq f(R) - 1$.

Now we get a first result:

(4.2) Lemma: *Assume that f satisfies the conditions* (4.1) *and let $(R, m) \subseteq (R', m')$ be local domains such that R' is of essentially finite type over and such that $m' \cap R = m$. Then* $\operatorname{tr}(R':R) - \operatorname{tr}(R'/m' : R/M) \leq f(R') - f(R)$.

Proof: We find $a_1, \ldots, a_n \in R'$ such that $R' = R[a_1, \ldots, a_n]_p$, where $p \in \operatorname{spec}(R[a_1, \ldots, a_n])$ is conveniently chosen and lies over m. It is easy to see by induction on n, that we may restrict ourselves to the case $n = 1$. Put $a_1 = a$. Then $p \cap R = m$ implies obviously that $\dim(p) = \operatorname{tr}(R'/m' : R/m)$, and therefore the result is clear by the conditions (4.1).

(4.3) Remark: If f satisfies (4.1) with the opposite inequality sign in (i) and (iii), then under the hypotheses of (4.2) we have $\operatorname{tr}(R' : R) - \operatorname{tr}(R'/m' : R/m) \geq f(R') - f(R)$.

Indeed, all the arguments which prove (4.2) stay valid if one passes everywhere to the opposite inequality sign.

(4.4) Corollary: *Let* $(R, m) \subseteq (R', m')$ *be as in* (4.2). *Then the following inequalities hold*:

(i) $\operatorname{tr}(R' : R) - \operatorname{tr}(R'/m' : R/m) \leq \hat{d}(R') - \hat{d}(R)$.

(ii) $\operatorname{tr}(R' : R) - \operatorname{tr}(R'/m' : R/m) \leq \delta(R') - \delta(R)$.

(iii) $\operatorname{tr}(R' : R) - \operatorname{tr}(R'/m' : R/m) \geq \dim(R') - \dim(R)$.

Proof: (i) by (2.14), (2.13) and (2.9) $\hat{d}$ satisfies the conditions (4.1).

(ii) That $\hat{\delta}$ satisfies the conditions (4.1) (i) and (iii) is shown in the proof of [1, 6.1]). For the validity of (4.1) (iii) see (2.11) or [1, (5.6)].

(iii) It is well known that dim satisfies the conditions (4.1) with the opposite inequality sign in (i) and (iii).

(4.5) Definition: *Let* (R, m) *be a local domain and let* (R', m') *be a dominant* R*-locality.* (*This means that* R, R' *are as in* (4.2)). *Then put* $t(R, R') = \operatorname{tr}(R' : R) - \operatorname{tr}(R'/m' : R/m) + \dim(R) - \dim(R')$.

(4.6) Lemma: *Let* (R, m) *be a local domain and let* (R', m') *a dominant* R*-locality.*

Then: $0 \leq t(R, R') \leq \hat{D}(R)$.

Proof: $0 \leq t(R, R')$ is a restatement of (4.4) (iii). As $\hat{d}(R') < \dim(R')$ and by (4.4) (i) we get $t(R, R') \leq \operatorname{tr}(R' : R) - \operatorname{tr}(R'/m' : R/m) + \dim(R) - \hat{d}(R') \leq \dim(R) - \hat{d}(R)$.

(4.7) Lemma: *Let* (R, m) *be a local domain,* (R', m') *a dominant* R*-locality and* (R'', m'') *a dominant* R'*-locality. Then we have*: $t(R, R'') = t(R, R') + t(R', R'')$.

Proof: Indeed $\operatorname{tr}(R'' : R) - \operatorname{tr}(R''/m'' : R/m) + \dim(R) - \dim(R'') =$

$\operatorname{tr}(R'':R') + \operatorname{tr}(R':R) - \operatorname{tr}(R''/m'':R'/m') - \operatorname{tr}(R'/m':R/m) + \dim(R) - \dim(R') + \dim(R') - \dim(R'') = t(R, R') + t(R', R'')$.

(4.8) REMARK: Let R be noetherian, $p \in \operatorname{spec}(R)$ and assume that $x_1, \dots, x_k$ is a system of parameters of p such that p is the only minimal prime of $\Sigma_{i=1}^k x_i R$. Then for any $q \in \operatorname{spec}(R)$ such that $q \supseteq p$ we have $\operatorname{ht}(q) = \operatorname{ht}(p) + \operatorname{ht}(q/p)$.

Indeed, by localizing at q one may assume that (R, q) is local. But then the result is clear as $\operatorname{ht}(q/p) = \dim(\Sigma_{i=1}^k x_i R) = \dim(R) - k = \operatorname{ht}(q) - \operatorname{ht}(p)$.

(4.9) LEMMA: *Let R be noetherian, $p \in \operatorname{spec}(R)$ such that p contains a regular element of R. Then there is an R-algebra R' of finite type and a $p' \in \operatorname{spec}(R')$ such that:*

(i) $R' \subseteq Q(R)$

(ii) $p' \cap R = p$; $R'_{p'} = R_p$, $R'/p' = R/p$

(iii) *For any $q' \in \operatorname{spec}(R')$ such that $p' \subseteq q'$ we have* $\operatorname{ht}(q') = \operatorname{ht}(q'/p') + \operatorname{ht}(p')$.

PROOF: (This will be the construction of the proof of [8, (2.13)]). Indeed, let $c_1, \dots, c_h$ ($h = \operatorname{ht}(p)$) be a system of parameters of p and let $q \cap q_1 \cap \cdots \cap q_n$ be an irredundant primary decomposition of $c = \Sigma_{i=1}^h c_i R$ such that q is p-primary. By our hypothesis we may choose $c_1 \in \operatorname{reg}(R)$. But then it is clear that $q_1 \cap \cdots \cap q_n \not\subseteq p \cup \operatorname{ass}(R)$ and we therefore find a regular $b \in q_1 \cap \cdots \cap q_n - p$. But then $q = (c : b^j)_R$ for any $j \in N$. Now put $y_i = c_i/b$ $(i = 1, \dots, j)$, $R' = R[y_1, \dots, y_h]$, $y = \Sigma_{i=1}^h y_i R'$.

Then obviously $q \subseteq y \cap R$. As on the other hand for any $t \in y$ there is a $j \in N$ such that $b^j t \in q$ we get $q = y \cap R$. This implies $R'/y = R/q$. Choose $p' \in \operatorname{spec}(R')$ such that $p'/y = p/q$. Then y is p'-primary and (ii) holds as $R' \subseteq R_p$.

But then we have $h = \operatorname{ht}(p) = \operatorname{ht}(p')$ and $y_1, \dots, y_h$ is a system of parameters of p'. Thus (iii) follows by (4.8).

(4.10) COROLLARY: *Let $p_0 \subsetneq \cdots \subsetneq p_l$ be a maximal chain of primes of a noetherian domain R. Then there is an R-algebra S of finite type and a maximal chain of primes $q_0 \subsetneq \cdots \subsetneq q_1$ of S such that:*

(i) $S \subseteq Q(R)$

(ii) $p_i = q_i \cap R$ $(i = 0, \dots, l)$

(iii) $R/p_l = S/q_l$, $(R/p_i)_{p_{i+1}} = (S/q_i)_{q_{i+1}}$ $(i = 0, \dots, l-1)$

(iv) $\operatorname{ht}(q_l) = l$.

PROOF: Obviously we have $\mathrm{ht}(p_i)+1 \leq \mathrm{ht}(p_{i+1})$, (*), for $i = 0, \ldots, l-1$. For $i=0$ we even have equality in (*). Let k be the maximal integer such that for $i \leq k$ we have equality in (*). We prove the result by induction on $l-k$.

If $l-k=1$ we obviously have $\mathrm{ht}(p_l)=l$, thus we may choose $S=R$. If $l-k>1$ put $p_{k+1}=p$ and choose R', p' as in (4.9). Then by (4.9) (ii) we find a maximal chain of primes $p'_0 \subsetneq \cdots \subsetneq p'_{k+1} = p' \subsetneq \cdots \subsetneq \cdot p'_l$ of R' such that $p_i = p'_i \cap R$ $(i=0,\ldots,l)$, $\mathrm{ht}(p'_i)=\mathrm{ht}(p_i)$ for $i = 0, \ldots, k+1$, $R/p_l = R'/p'_l$ and $(R/p_i)_{p_{i+1}} = (R'/p'_i)_{p'_{i+1}}$. In particular it follows from this that $\mathrm{ht}(p'_i)+1 = \mathrm{ht}(p'_{i+1})$ for $i \leq k$, and by (4.9) (iii) we get $\mathrm{ht}(p'_{k+1})+1 = \mathrm{ht}(p'_{k+1}) + \mathrm{ht}(p'_{k+2}/p'_{k+1}) = \mathrm{ht}(p'_{k+2})$. So we may apply the hypothesis of induction to R', $p'_0 \subsetneq \cdots \subsetneq \cdot p'_l$, and the result is obvious.

In choosing $p_0 \subsetneq \cdots \subsetneq p_l$ of minimal length $(l = c(R))$ we thus get:

(4.11) COROLLARY: *Let (R, m) be a local domain. Then there is a dominant R-locality (R', m') in $Q(R)$ such that $t(R, R') = C(R)$.*

(4.12) DEFINITION: *Let R' be an algebra of essentially finite type over the noetherian ring R. Then put*:

$$T_R(R') = \max\{t((R/p' \cap R)_{q' \cap R}, (R'/p')_{q'}) \mid p', q' \in \mathrm{spec}(R'),\ p' \subseteq q'\}$$

(4.13) REMARK: If a' is an ideal of R' and if $S \subseteq R'$ is multiplicatively closed we obviously have $T_R(S^{-1}(R'/a')) \leq T_R(R')$. Moreover $T_R(R')=0$ if the dimension formula holds between R and R'. (cf. [5, (14.C)]). $T_R(R')=0$ for any R' of essentially finite type is equivalent to "R satisfies the dimension (or altitude-) formula (s. [6, pg. 129], [9, (2.2)]).

(4.14) LEMMA: *Let R be noetherian and let $R \to R'$, $R' \to R''$ be algebras of essentially finite type. Then $T_R(R'') \geq T_{R'}(R'')$.*

PROOF: Let p'', $q'' \in \mathrm{spec}(R'')$ such that $p'' \subseteq q''$ and $T_{R'}(R'') = t((R'/R' \cap p'')_{q'' \cap R'}, (R''/p'')_{q''})$. But then by (4.6) and (4.7) it is clear that $T_R(R'') \geq t(R/p'' \cap R)_{q'' \cap R}, (R''/p'')_{q''}) \geq T_{R'}(R'')$.

(4.15) THEOREM: *Let R be noetherian. Then*:

(i) *$T_R(R') \leq \mu\hat{D}(R)$ for any R-algebra R' of essentially finite type.*

(ii) *There is an $n_0 \in N$ such that for $R' = R_n$ we have equality in* (i) *if $n \geq n_0$.*

PROOF: (i) Choose p', $q' \in \operatorname{spec}(R')$ such that $p' \subseteq q'$ and $T_R(R') = t((R/p' \cap R)_{q' \cap R}, (R'/p')_{q'})$. Then by (4.6) we have $T_R(R') \leq \hat{D}((R/p' \cap R)_{q' \cap R}) \leq \mu\hat{D}(R)$.

(ii) Let p, $q \in \operatorname{spec}(R_1)$ such that $p \subseteq q$ and $C((R_1/q)_p) = \mu C(R_1)$. Then by (3.19) we have $C((R_1/q)_p) = \mu\hat{D}(R)$. So by (4.11) there is an $m_0 \in N$ such that $\mu\hat{D}(R) \leq T_{(R_1/q)p}(((R_1/q)_p)_{m_0})$. But from this we get $\mu\hat{D}(R) \leq T_R(R_{m+1})$ for $m \geq m_0$ by (4.13) and (4.14). Thus the statement is clear by (i).

In view of (4.13) we get in particular that for a noetherian ring R: "*R satisfies the altitude formula*" is equivalent to each of the statements of (3.20).

So (3.19) and (4.15) generalize [9, (2.6)].

REFERENCES

[1] M. BRODMANN: Über die minimale Dimension der assoziierten Primideale der Komplettion eines lokalen Intergritätsbereiches. *Comment. Math. Helvetici 50* (1975) 219–232.

[2] M. BRODMANN: A macaulayfication of unmixed domains, *J. of Alg. 44* (1977) 221–234.

[3] D. FERRAND–M. RAYNAUD, Fibres formelles d'un anneau local noethérien, *Ann. Sci. ENS 3* (1970) 295–311.

[4] A. GROTHENDIECK, *Eléments de géométrie algebrique IV, Seconde partie*, Publ. IHES, Paris 1964.

[5] H. MATSUMURA, *Commutative Algebra*, Lecture notes series (W.A. Benjamin, New York, 1970).

[6] M. NAGATA, *Local rings* (Interscience publishers, New York 1962).

[7] M. NAGATA, On the chain problem of prime ideals, *Nagoya Math. 10* (1956) 51–64.

[8] L.J. RATLIFF JR., On quasi-unmixed local domains, the altitude formula, and the chain condition for prime ideals (I), *Amer. of Math. 91* (1969) 508–528.

[9] L.J. RATLIFF JR., On quasi-unmixed domains, the altitude formula and the chain condition for prime ideals (II), *Amer. of Math. 92* (1970) 99–144.

[10] L.J. RATLIFF JR., S. MCADAM, *Maximal chains of prime ideals in integral extension domains*, preprint (1976).

[11] U. SCHWEIZER, *Primidealketten*, Dissertation, Basel (1972).

(Oblatum 11–X–1976 & 2–III–1977)

Ecole Polytechnique Fédérale
de Lausanne
Département de Mathématiques
1007 Lausanne, Switzerland